Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-15
DOI:10.22034/cmde.2025.67851.3244

A multi-interval continuous collocation method for Volterra integral equations of the third kind

Ali Davari*
Department of Mathematics, Khansar Campus, University of Isfahan, Iran.

Abstract

This paper considers Volterra integral equations (VIEs) of the third kind. We present a novel multi-interval collocation-based approach for the numerical solution of this type of equations. The proposed method is particularly effective for VIEs when implemented in continuous piecewise polynomial spaces, leveraging its ability to perform localized approximations on subintervals. We also analyze the solvability of the proposed continuous collocation schemes and establish its convergence properties. Numerical experiments are included to demonstrate the spectral accuracy of the method and validate the theoretical findings.

Keywords. Continuous piecewise polynomials, Volterra integral equations of the third kind, Solvability, Convergence analysis, Collocation method.
2010 Mathematics Subject Classification. 65R20, 65Q20, 45L05, 45D05, 41A05.

1. Introduction

In the early 20th century, Volterra introduced a class of integral equations, which he initially formulated to address population dynamics. Since then, these equations have grown in importance across numerous fields, including mechanics, physics, astronomy, engineering, biology, and economics, where they serve as critical tools for modeling complex systems [4, 5, 10]. Over the last several decades, Volterra integral equations (VIEs) of the third kind, in particular, have garnered significant attention because of their applicability in accurately describing dynamic systems in science and engineering [18, 19, 21].

Integral equations of the third kind can be an ill-posed problem. Finding analytical solutions for integral equations of the third kind is usually very challenging. Therefore, reliable and efficient numerical methods are essential, and developing these methods can be difficult. Numerical methods for solving Volterra integral equations (VIEs) of the third kind have been studied, with collocation methods emerging as a central focus due to their flexibility and accuracy [6–9, 14]. The authors of [19], considered an extension of spline collocation techniques within piecewise polynomial functions and applied it to a certain class of VIEs of the third kind. Additionally, in [20] alternative researches have been proposed for approximating solutions to nonlinear VIEs of the third kind. The authors employed collocation methods in piecewise polynomial functions. The authors of [16] developed a spectral collocation method based on generalized Jacobi wavelets, combined with the Gauss-Jacobi quadrature formula, to solve a class of third-kind VIEs. Furthermore, [1] introduced a moving least squares approximation with shifted Chebyshev polynomials to solve linear and nonlinear third-kind VDIE. The authors of [15] developed a collocation method for solving third-kind Volterra integral equations. In order to achieve high order convergence for problems with nonsmooth solutions, they constructed a collocation scheme on a modified graded mesh using a basis of fractional polynomials, depending on a certain parameter. Despite these advancements, the development of numerical methods for third-kind VIEs remains relatively limited compared to first- and second-kind VIEs.

In this study, we propose a multi-interval continuous collocation method for solving Volterra integral equations of the third kind. The main idea is to approximate the solution by continuous piecewise polynomials on small subintervals,

Received: 24 June 2025; Accepted: 15 December 2025.
∗ Corresponding author. Email: a_davari@khc.ui.ac.ir.

which makes the method both flexible and efficient. Compared with traditional global polynomial or spline-based collocation schemes, the present approach offers several advantages. By decomposing the problem into smaller local systems, it can be extended naturally to problems with nonlinearities or nonsmooth kernels, and it achieves high-order convergence without relying on graded meshes or special basis functions. In this sense, the method addresses some of the limitations of earlier techniques while remaining simple to implement and supported by a rigorous convergence analysis.

Consider the following VIEs of the third kind,

$$x^{\alpha}y(x) = f(x) + \int_0^x k(x,t)y(t)dt, \quad x \in I = [0,T]. \tag{1.1}$$

Here, $\alpha > 0$ and $f(x) = x^{\alpha}g(x)$, where g is a continuous function. The function k is continuous on the domain $\Delta = \{(x,t) : 0 \leq t \leq x \leq T\}$.

The existence, uniqueness, and regularity of solutions to the model described in Equation (1.1) have been thoroughly analyzed in [18].

The remainder of the paper is structured as follows; Section 2 introduces the proposed collocation methods. Next, in section 3, the solvability of the collocation equations is discussed, and also the convergence of the collocation solution is proved. To approve the effectiveness of the proposed method, some numerical examples are given in section 4. Some conclusions are drawn in section 5.

2. Continuous collocation methods for third kind Volterra integral equations

2.1. **Collocation space and meshes.** Consider $I_h = \{x_j = jh : \quad j = 0, 1, \cdots, N\}$ as a mesh on the interval I. Without loss of generality, let $h = \frac{T}{N}$ be the mesh diameter and $I^{(n)} = [x_n, x_{n+1}]$. Moreover, let function y be the solution to the third-kind VIE (1.1). Then, y can be approximated by y_h, which is defind within

$$S_m^{(0)}(I_h) = \Big\{y \in C(I) : \quad y_{|_{I^{(n)}}} \in \mathcal{P}_m = \mathcal{P}_m(I^{(n)}), \quad n = 0, 1, \cdots, N-1\Big\},$$

where $\mathcal{P}_m$ represents the set of real polynomials of degree up to m.

Letting $\Omega_h = \Big\{x_{n,j} = x_n + \theta_j h : \quad 0 \leq \theta_1 < ... < \theta_m \leq 1, \quad n = 0, 1, 2, \cdots, N-1\Big\}$ denote the set of collocation points, the collocation solution $y_h \in S_m^{(0)}(I_h)$ is determined by the following collocation equation for (1.1)

$$x^{\alpha}y_h(x) = f(x) + \int_0^x k(x,t)y_h(t)dt, \quad x \in \Omega_h \tag{2.1}$$

It will be convenient to use the local Lagrange basis representations of y_h. Since y_h' belongs to the polynomial space $\mathcal{P}_{m-1}$, we expand the collocation polynomial as follows.

$$y_h'(x_n + sh) = \sum_{j=1}^{m} Y_{n,j} L_j(s), \quad t \in [0,1], \tag{2.2}$$

with $Y_{n,j} = y_h'(x_{n,j})$, and

$$L_j(s) = \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{s - \theta_k}{\theta_j - \theta_k}, \quad j = 1, 2, \cdots, m.$$

Remark 2.1. When interpolating y', we can achieve smoother results for y. If we interpolate y directly, any noise or irregularities in the data can propagate through the interpolated function. By interpolating the derivative, we can better control the smoothness of the resulting function.

We obtain from (2.2) the local representation of $y_h \in S_m^{(0)}(I_h)$ on $I^{(n)}$,

$$y_h(x_n + sh) = y_h(x_n) + h \sum_{j=1}^{m} Y_{n,j} (\int_0^s L_j(\tau)d\tau). \tag{2.3}$$

Setting $\beta_j(s) = \int_0^s L_j(\tau)d\tau$, we can rewrite (2.3) as

$$y_h(x_n + sh) = y_h(x_n) + h\sum_{j=1}^{m} \beta_j(s)Y_{n,j}. \tag{2.4}$$

By substituting a general point $x = x_{n,i} = x_n + \theta_i h$ into the collocation Equation (2.1) and using the local representation (2.4), we obtain

$$x_{n,i}^{\alpha} y_h(x_{n,i}) = f(x_{n,i}) + \int_0^{x_{n,i}} k(x_{n,i}, t)y_h(t)dt, \quad i = 1, 2, \cdots, m. \tag{2.5}$$

and

$$y_h(x_{n,i}) = y_h(x_n) + h\sum_{j=1}^{m} a_{ij}Y_{n,j}, \quad i = 1, 2, \cdots, m. \tag{2.6}$$

where $a_{ij} = \beta_j(\theta_i)$.

Hence, combining Equations (2.5) and (2.6), we derive:

$$\begin{aligned} x_{n,i}^{\alpha} y_h(x_{n,i}) &= f(x_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i}, x_l + sh)\Big(y_h(x_l) + h\sum_{j=1}^{m}\beta_j(s)Y_{l,j}\Big)ds \\ &\quad + h\int_0^{\theta_i} k(x_{n,i}, x_n + sh)\Big(y_h(x_n) + h\sum_{j=1}^{m}\beta_j(s)Y_{n,j}\Big)ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} x_{n,i}^{\alpha} y_h(x_{n,i}) &= x_{n,i}^{\alpha} y_h(x_n) + h x_{n,i}^{\alpha}\sum_{j=1}^{m} a_{i,j}Y_{n,j} \\ &= f(x_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i}, x_l + sh)y_h(x_l)ds + h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i}, x_l + sh)\Big(h\sum_{j=1}^{m}\beta_j(s)Y_{l,j}\Big)ds \\ &\quad + h\int_0^{\theta_i} k(x_{n,i}, x_n + sh)y_h(x_n)ds + h\int_0^{\theta_i} k(x_{n,i}, x_n + sh)\Big(h\sum_{j=1}^{m}\beta_j(s)Y_{n,j}\Big)ds, \quad i = 1, 2, \cdots, m. \end{aligned} \tag{2.7}$$

Setting $Y_n = (Y_{n,1}, \cdots, Y_{n,m})^T$, $e = (1, \cdots, 1)^T$ and $A = (a_{ij})_{m\times m}$,

$$F_n = (f(x_{n,1}), \cdots, f(x_{n,m}))^T, \; T_n = diag\Big(x_{n,1}^{\alpha}, \cdots, x_{n,m}^{\alpha}\Big),$$

$$B_n^{(l)} = \Big(\int_0^1 k(x_{n,i}, x_l + sh)\beta_j(s)ds\Big),$$

$$B_n = \Big(\int_0^{\theta_i} k(x_{n,i}, x_n + sh)\beta_j(s)ds\Big),$$

$$C_n^{(l)} = diag\Big(\int_0^1 k(x_{n,i}, x_l + sh)ds\Big),$$

$$C_n = diag\Big(\int_0^{\theta_i} k(x_{n,i}, x_n + sh)ds\Big).$$

By the above notation from (2.7), we have

$$\Big(hT_nA - h^2B_n\Big)Y_n = F_n + (hC_n - T_n)e\; y_h(x_n) + h\sum_{l=0}^{n-1}\Big(C_n^{(l)}ey_h(x_l) + hB_n^{(l)}Y_l\Big). \tag{2.8}$$

If f and K are continuous functions on appropriate domains, (2.8) determines a unique $y_h \in S_m^{(0)}(I_h)$ for all sufficiently small mesh diameters.

Lemma 2.2. *The matrix A is invertible.*

Proof. Consult [13]. □

Theorem 2.3. *Let $k \in C(\Delta)$ and $\alpha \in (0,1)$. Then, for collocation parameters $0 < \theta_1 < \cdots < \theta_m \leq 1$, there exists a constant $\bar{h} > 0$ such that the collocation equation (2.8) is solvable for all $h \in (0, \bar{h})$.*

Proof. Equation (2.8) can be written as

$$\Big(I - h(T_nA)^{-1}B_n\Big)Y_n = \frac{1}{h}(T_nA)^{-1}\Big(F_n + (hC_n - T_n)e\ y_h(x_n) + h\sum_{l=0}^{n-1}\Big(C_n^{(l)}ey_h(x_l) + hB_n^{(l)}Y_l\Big)\Big).$$

The matrix $I - h(T_nA)^{-1}B_n$ is invertible, since

$$||h(T_nA)^{-1}B_n||_\infty = ||hA^{-1}T_n^{-1}B_n||_\infty \leq h\ ||A^{-1}||_\infty\ ||T_n^{-1}B_n||_\infty.$$

We set $z = ||A^{-1}||_\infty$, Thus we have

$$(T_n^{-1}B_n)_{ij} = x_{n,i}^{-\alpha}\int_0^{\theta_i} k(x_{n,i}, x_n + sh)\beta_j(s)ds.$$

Then,

$$\begin{aligned}
||T_n^{-1}B_n||_\infty &= \max_{1\leq i\leq m}\sum_{j=1}^{m}|x_{n,i}^{-\alpha}\int_0^{\theta_i} k(x_{n,i}, x_n + sh)\beta_j(s)ds| \\
&\leq \max_{1\leq i\leq m} x_{n,i}^{-\alpha}\int_0^{\theta_i}|k(x_{n,i}, x_n + sh)|\ |\sum_{j=1}^{m}\beta_j(s)|ds \\
&\leq x_{n,1}^{-\alpha}||k||_\infty \max_{0\leq s\leq\theta_m}|\sum_{j=1}^{m}\beta_j(s)| \\
&\leq (\theta_1 h)^{-\alpha}||k||_\infty \max_{0\leq s\leq\theta_m}|\sum_{j=1}^{m}\beta_j(s)|.
\end{aligned}$$

Hence, if we set

$$\bar{h} = \Big(\frac{{\theta_1}^{\alpha}}{z\ ||k||_\infty \max\limits_{0\leq s\leq\theta_m}|\sum_{j=1}^{m}\beta_j(s)|}\Big)^{\frac{1}{1-\alpha}}, \tag{2.9}$$

for $h \in (0, \bar{h})$, then the matrix $I - h(T_nA)^{-1}B_n$ is invertible and the proof is complete. □

3. Convergence analysis

Theorem 3.1. *Assume that*

i. *The functions k and f in (1.1) are of class C^{m+2}.*
ii. *$y_h \in S_m^{(0)}(I_h)$ as the collocation solution for (1.1), associated to the collocation points Ω_h, is determined by (2.1).*
iii. *For any $h \in (0, \bar{h})$, with $\bar{h} > 0$ determined in (2.9), each of the collocation equation (2.8) has a unique solution.*

Then

$$\lim_{h\to 0} ||y - y_h||_\infty = 0,$$

if the collocation parameters $\{\theta_i\}$ *satisfy the condition* $-1 \le \prod_{i=1}^{m} \frac{\theta_i - 1}{\theta_i} \le 1$.

The attainable global order of convergence is described by

$$||y - y_h||_\infty = \max_{x\in I}|y(x) - y_h(x)| \le C \begin{cases} h^{m+1} & -1 \le \lambda_1 < 1 \\ h^m & \lambda_1 = 1 \end{cases} \tag{3.1}$$

and holds for any set Ω_h *of collocation parameters with* $0 \le \theta_1 < \cdots < \theta_m \le 1$. *The constant* C *depends on* $\{\theta_i\}$ *but not on* h.

Proof. Assumption (i) implies that $y \in C^{m+1}(I)$ hence $y' \in C^m(I)$. Thus, using Peano's theorem [3], for y' on $I^{(n)}$, we have

$$y'(x_n + sh) = \sum_{j=1}^{m} y'(x_{n,j})L_j(s) + h^m R^{(1)}_{m+1,n}(s), \quad s \in [0, 1]. \tag{3.2}$$

The Peano remainder term and Peano kernel are respectively given by

$$R^{(1)}_{m+1,n}(s) = \int_0^1 K_m(s,\nu)y^{(m+1)}(x_n + \nu h)d\nu$$

and

$$K_m(s,\nu) = \frac{1}{(m-1)!}\Big\{(s-\nu)_+^{m-1} - \sum_{k=1}^{m} L_k(s)(\theta_k - \nu)_+^{m-1}\Big\}, \quad s \in [0, 1].$$

Integrating (3.2) yields

$$y(x_n + sh) = y(x_n) + h\sum_{j=1}^{m} y'(x_{n,j})\beta_j(s) + h^{m+1}R_{m+1,n}(s), \quad 0 \le s \le 1, \tag{3.3}$$

where $R_{m+1,n}(s) = \int_0^s R^{(1)}_{m+1,n}(\nu)d\nu$.

First, we suppose $k(x,t) = 1$ and define the collocation error $e_h = y - y_h$ on $I^{(n)}$.

Employing (2.4) and (3.3), the collocation error can be written as

$$e_h(x_n + sh) = e_h(x_n) + h\sum_{j=1}^{m} \varepsilon_{n,j}\beta_j(s) + h^{m+1}R_{m+1,n}(s), \tag{3.4}$$

where $\varepsilon_{n,j} = y'(x_{n,j}) - Y_{n,j}$. By setting $s = \theta_i$ in (3.4), we have

$$e_h(x_{n,i}) = e_h(x_n) + h\sum_{j=1}^{m} a_{i,j}\varepsilon_{n,j} + h^{m+1}R_{m+1,n}(\theta_i). \tag{3.5}$$

Then, based on (1.1)-(2.1) and using (3.4), we can write

$$\begin{aligned} x^\alpha_{n,i} e_h(x_{n,i}) &= \int_0^{x_{n,i}} e_h(s)ds = h\sum_{l=0}^{n-1}\int_0^1 e_h(x_l + sh)ds + h\int_0^{\theta_i} e_h(x_n + sh)ds \\ &= h\sum_{l=0}^{n-1} e_h(x_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j} + h\theta_i e_h(x_n) + h^2\sum_{j=1}^{m} b_{i,j}\varepsilon_{n,j} + h^{m+1}\tilde{R}_{m+1,n}(\theta_i), \end{aligned} \tag{3.6}$$

where $\gamma_j(s) = \int_0^s \beta_j(\nu)d\nu, \quad b_{i,j} = \gamma_j(\theta_i)$ and

$$\tilde{R}_{m+1,n}(\theta_i) = h\sum_{l=0}^{n-1}\int_0^1 R_{m+1,l}(\nu)d\nu + h\int_0^{\theta_i} R_{m+1,n}(\nu)d\nu.$$

Thus, from (3.5) and (3.6), we will have

$$x_{n,i}^{\alpha}\Big(e_h(x_n) + h\sum_{j=1}^{m} a_{i,j}\varepsilon_{n,j} + h^{m+1}R_{m+1,n}(\theta_i)\Big) = h\sum_{l=0}^{n-1} e_h(x_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j} + h\theta_i e_h(x_n) + h^2\sum_{j=1}^{m} b_{i,j}\varepsilon_{n,j} + h^{m+1}\tilde{R}_{m+1,n}(\theta_i). \tag{3.7}$$

We can modify Equation (3.7) by substituting n with $n-1$ and setting i equal to m. By subtracting this modified equation from the original equation given in (3.7), we get

$$\begin{aligned} & x_{n,i}^{\alpha}e_h(x_n) - x_{n-1,m}^{\alpha}e_h(x_{n-1}) + hx_{n,i}^{\alpha}\sum_{j=1}^{m} a_{i,j}\varepsilon_{n,j} - hx_{n-1,m}^{\alpha}\sum_{j=1}^{m} a_{m,j}\varepsilon_{n-1,j} \\ & + h^{m+1}x_{n,i}^{\alpha}R_{m+1,n}(\theta_i) - h^{m+1}x_{n-1,m}^{\alpha}R_{m+1,n-1}(\theta_m) \\ & = he_h(x_{n-1}) + h^2\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{n-1,j} + h\theta_i e_h(x_n) - h\theta_m e_h(x_{n-1}) \\ & + h^2\sum_{j=1}^{m} b_{i,j}\varepsilon_{n,j} - h^2\sum_{j=1}^{m} b_{m,j}\varepsilon_{n-1,j} + h^{m+1}\tilde{R}_{m+1,n}(\theta_i) - h^{m+1}\tilde{R}_{m+1,n-1}(\theta_m). \end{aligned} \tag{3.8}$$

We set $\varepsilon_n = (\varepsilon_{n,1}, \cdots, \varepsilon_{n,m})^T$, $C = diag(\theta_1, \cdots, \theta_m)$, $B = (b_{ij})_{m\times m}$ and define $\gamma = (\gamma_1(1), \cdots, \gamma_m(1))^T$.

Then, (3.8) can be rewritten as

$$\begin{aligned} & \Big(e_h(x_n)T_n - e_h(x_{n-1})\ x_{n-1,m}^{\alpha}I_m\Big)e + hT_nA\varepsilon_n - hx_{n-1,m}^{\alpha}ee_m^TA\varepsilon_{n-1} \\ & = he_h(x_{n-1})e + h^2e\gamma^T\varepsilon_{n-1} + he_h(x_n)Ce - hc_me_h(x_{n-1})e + h^2B\varepsilon_n - h^2ee_m^TB\varepsilon_{n-1} + h^{m+1}R_{m+1,n}, \end{aligned} \tag{3.9}$$

Since e_h is continuous in the interval I, and of course at the grid points, from (3.4) and $e_h(0) = 0$, we obtain

$$\begin{aligned} e_h(x_n) & = e_h(x_{n-1} + h) = e_h(x_{n-1}) + h\sum_{j=1}^{m} b_j\varepsilon_{n-1,j} + h^{m+1}R_{m+1,n-1}(1) \\ & = h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j\varepsilon_{l,j} + h^{m+1}\sum_{l=0}^{n-1}R_{m+1,l}(1) = h\sum_{l=0}^{n-1} b^T\varepsilon_l + h^{m+1}\sum_{l=0}^{n-1}R_{m+1,l}(1), \end{aligned} \tag{3.10}$$

with $b_j := \int_0^1 L_j(s)ds$ and $b^T := (b_1, \cdots, b_m)$.

By substituting e_h given by (3.10) into (3.9), we obtain

$$\begin{aligned} & \Big(T_nA - hB\Big)\varepsilon_n = \Big(x_{n-1,m}^{\alpha}ee_m^TA - hee_m^TB + he\gamma^T\Big)\varepsilon_{n-1} + Ce\Big(h\sum_{l=0}^{n-1} b^T\varepsilon_l + h^{m+1}\sum_{l=0}^{n-1}R_{m+1,l}(1)\Big) \\ & + \Big(h\sum_{l=0}^{n-2} b^T\varepsilon_l + h^{m+1}\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big)e - \theta_m e\Big(h\sum_{l=0}^{n-2} b^T\varepsilon_l + h^{m+1}\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big) + h^m\tilde{R}_{m+1,n} \\ & - \Bigg(\Big(\sum_{l=0}^{n-1} b^T\varepsilon_l + h^m\sum_{l=0}^{n-1}R_{m+1,l}(1)\Big)T_n - \Big(\sum_{l=0}^{n-2} b^T\varepsilon_l + h^m\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big)\ x_{n-1,m}^{\alpha}I_m\Bigg)e. \end{aligned} \tag{3.11}$$

We can rewrite (3.11) as

$$\Big(T_nA - hB\Big)\varepsilon_n = \Big(x^{\alpha}_{n-1,m}ee^T_mA - hee^T_mB + he\gamma^T - eb^TT_n + hCeb^T\Big)\varepsilon_{n-1}$$
$$+(he\Big(C + (1-\theta_m)I\Big) - T_n + x^{\alpha}_{n-1,m}I_m)\sum_{l=0}^{n-2} b^T\varepsilon_l + h^m\tilde{R}_{m+1,n}.$$

We assume $\underline{\Lambda} = x^{\alpha}_{n-1,m}ee^T_mA - eb^TT_n$, then

$$\varepsilon_n = \Big((T_nA)^{-1}\ \underline{\Lambda} + O(h)\Big)\varepsilon_{n-1} + hD\sum_{l=0}^{n-2}\varepsilon_l + h^m\bar{R}_{m+1,n}. \tag{3.12}$$

Let $(T_nA)^{-1} = (\xi_{ij})^m_{i,j=1}$. Since

$$\underline{\Lambda} = \begin{bmatrix} x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1 & x^{\alpha}_{n-1,m}a_{m2} - x^{\alpha}_{n-1,2}b_2 & \dots & x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m \\ x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1 & x^{\alpha}_{n-1,m}a_{m2} - x^{\alpha}_{n-1,2}b_2 & \dots & x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m \\ \vdots & \vdots & \ddots & \vdots \\ x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1 & x^{\alpha}_{n-1,m}a_{m2} - x^{\alpha}_{n-1,2}b_2 & \dots & x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m \end{bmatrix},$$

then

$$(T_nA)^{-1}\underline{\Lambda} = \begin{bmatrix} \Big(x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1\Big)\sum_{j=1}^{m}\xi_{1j} & \dots & \Big(x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m\Big)\sum_{j=1}^{m}\xi_{1j} \\ \Big(x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1\Big)\sum_{j=1}^{m}\xi_{2j} & \dots & \Big(x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m\Big)\sum_{j=1}^{m}\xi_{2j} \\ \vdots & \ddots & \vdots \\ \Big(x^{\alpha}_{n-1,m}a_{m1} - x^{\alpha}_{n-1,1}b_1\Big)\sum_{j=1}^{m}\xi_{mj} & \dots & \Big(x^{\alpha}_{n-1,m}a_{mm} - x^{\alpha}_{n-1,m}b_m\Big)\sum_{j=1}^{m}\xi_{mj} \end{bmatrix}.$$

The rank of $(T_nA)^{-1}\underline{\Lambda}$ is one because the rank of $\underline{\Lambda}$ is one. Therefore, the only non-zero eigenvalue of $(T_nA)^{-1}\underline{\Lambda}$ is equal to its trace. Then,

$$\lambda_{\mathbf{1}} = tr\Big((T_nA)^{-1}\underline{\Lambda}\Big) = \sum_{i=1}^{m}(x^{\alpha}_{n-1,m}a_{mi} - x^{\alpha}_{n-1,i}b_i)\sum_{j=1}^{m}\xi_{ij} = x^{\alpha}_{n-1,m}x^{-\alpha}_{n,m} - b^TT_{n-1}A^{-1}T_n^{-1}e.$$

Let $N(x) = \dfrac{1}{m!}\displaystyle\prod_{i=1}^{m}(x-\theta_i)$ be the collocation polynomial based on the points $\theta_1, \theta_2, \cdots, \theta_m$. The rational stability function, defined as $R(x) = \frac{P(x)}{Q(x)}$, has the value $R(\infty) = x^{\alpha}_{n-1,m}x^{-\alpha}_{n,m} - b^TT_{n-1}A^{-1}T_n^{-1}e$. $P(x)$ and $Q(x)$ are polynomials of degree at most m, as

$$P(x) = N(1)x^m + \cdots + N^{(m-1)}(1)x + N^{(m)}(1), \quad Q(x) = N(0)x^m + \cdots + N^{(m-1)}(0)x + N^{(m)}(0).$$

The following relation also holds for $R(\infty)$ [3, 17]:

$$R(\infty) = \frac{N(1)}{N(0)} = \prod_{i=1}^{m}\frac{\theta_i - 1}{\theta_i},$$

that is, $x_{n-1,m}^{\alpha} x_{n,m}^{-\alpha} - b^T T_{n-1} A^{-1} T_n^{-1} e = \prod_{i=1}^{m} \frac{\theta_i - 1}{\theta_i} = \lambda_{\mathbf{1}}$.

Since $(T_n A)^{-1} \underline{\Lambda}$ is diagonalizable, there exists a nonsingular matrix $\bar{\mathbf{P}}$ such that $\bar{\mathbf{P}}^{-1}(T_n A)^{-1} \underline{\Lambda} \bar{\mathbf{P}} = \bar{\mathbf{D}}$, with $\bar{\mathbf{D}} = diag(\lambda_{\mathbf{1}}, 0, \cdots, 0)$. We next multiply the recursion (3.12) by $\bar{\mathbf{P}}^{-1}$ and define $\mathrm{E}_n = \bar{\mathbf{P}}^{-1} \varepsilon_n$. We find that

$$\bar{\mathbf{P}}^{-1} \varepsilon_n = \Big(\bar{\mathbf{P}}^{-1} A^{-1} T_n^{-1} \ \underline{\Lambda} + O(h) \Big) \varepsilon_{n-1} + h \bar{\mathbf{P}}^{-1} D \sum_{l=0}^{n-2} \varepsilon_l + h^m \bar{R}_{m+1,n}$$

or

$$\mathrm{E}_n = (\bar{\mathbf{D}} + O(h)) \mathrm{E}_{n-1} + h \bar{\mathbf{P}}^{-1} D \bar{\mathbf{P}} \sum_{l=0}^{n-2} \mathrm{E}_l + h^m \bar{\mathbf{P}}^{-1} \bar{R}_{m+1,n}. \tag{3.13}$$

The following three cases are considered:

Case(i): $-1 < \lambda_{\mathbf{1}} < 1$.

Using error analysis for collocation solutions of VIEs [3, 12, 13], there exists a constant C_1, such that

$$||\varepsilon_n||_\infty \leq C_1 h^m.$$

We have $x_{n,i}^{\alpha} = (x_n + \theta_i h)^{\alpha} \geq (\theta_i h)^{\alpha}$ then $\frac{1}{x_{n,i}^{\alpha}} \leq \frac{1}{(\theta_i h)^{\alpha}} \leq \frac{1}{(\theta_1 h)^{\alpha}}$ or $x_{n,i}^{-\alpha} \leq (\theta_1 h)^{-\alpha}$.

From (3.7), we have

$$e_h(x_n) + h \sum_{j=1}^{m} a_{i,j} \varepsilon_{n,j} + h^{m+1} R_{m+1,n}(\theta_i) = x_{n,i}^{-\alpha} \Big(h \sum_{l=0}^{n-1} e_h(x_l) + h^2 \sum_{l=0}^{n-1} \sum_{j=1}^{m} \gamma_j(1) \varepsilon_{l,j} + h \theta_i e_h(x_n)$$
$$+ h^2 \sum_{j=1}^{m} b_{i,j} \varepsilon_{n,j} + h^{m+1} \tilde{R}_{m+1,n}(\theta_i) \Big),$$

or

$$\Big(1 - h\theta_i \ x_{n,i}^{-\alpha} \Big) e_h(x_n) = -h \sum_{j=1}^{m} a_{i,j} \varepsilon_{n,j} - h^{m+1} R_{m+1,n}(\theta_i) + x_{n,i}^{-\alpha} \Big(h \sum_{l=0}^{n-1} e_h(x_l)$$
$$+ h^2 \sum_{l=0}^{n-1} \sum_{j=1}^{m} \gamma_j(1) \varepsilon_{l,j} + h^2 \sum_{j=1}^{m} b_{i,j} \varepsilon_{n,j} + h^{m+1} \tilde{R}_{m+1,n}(\theta_i) \Big).$$

Hence, we have

$$|e_h(x_n)| \leq h C_2 \sum_{l=0}^{n-1} |e_h(x_l)| + C_3 h^{m+1}.$$

The above inequality is a generalized discrete Gronwall inequality [2, 3, 11]. Therefore, there exists a constant C_4 such that

$$|e_h(x_n)| \leq C_4 h^{m+1} \ \ n = 1, 2, \cdots, N.$$

Case(ii): $\lambda_{\mathbf{1}} = -1$.

We replace n in (3.13) by $n-1$ and subtract it from (3.13) to get

$$\mathrm{E}_n - \mathrm{E}_{n-1} = (\bar{\mathbf{D}} + O(h))(\mathrm{E}_{n-1} - \mathrm{E}_{n-2}) + h \bar{\mathbf{P}}^{-1} D \bar{\mathbf{P}} \mathrm{E}_{n-1} + h^m \bar{\mathbf{P}}^{-1} (\bar{R}_{m+1,n} - \bar{R}_{m+1,n-1}).$$

Hence,

$$\begin{bmatrix} \mathrm{E}_n \\ \mathrm{E}_{n-1} \end{bmatrix} = \begin{bmatrix} I_m + \bar{\mathbf{D}} + O(h) & -\bar{\mathbf{D}} + O(h) \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \mathrm{E}_{n-1} \\ \mathrm{E}_{n-2} \end{bmatrix} + \begin{bmatrix} h^m \bar{\mathbf{P}}^{-1} (\bar{R}_{m+1,n} - \bar{R}_{m+1,n-1}) \\ 0 \end{bmatrix}.$$

Since $\bar{R}_{m+1,n} - \bar{R}_{m+1,n-1} = O(h)$ for $y \in C^{m+2}$, we define $r_{m+1,n} = \frac{\bar{R}_{m+1,n} - \bar{R}_{m+1,n-1}}{h}$ and set

$$X_n = \begin{bmatrix} \mathrm{E}_n \\ \mathrm{E}_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} I_m + \bar{\mathbf{D}} & -\bar{\mathbf{D}} \\ I_m & 0 \end{bmatrix}, \quad \bar{r}_{m+1,n} = \begin{bmatrix} \bar{\mathbf{P}}^{-1} r_{m+1,n} \\ 0 \end{bmatrix}.$$

Then,

$$X_n = GX_{n-1} + O(h)X_{n-1} + h^{m+1}\bar{r}_{m+1,n}. \tag{3.14}$$

The eigenvalues of the matrix G are $1, -1, 0$ of multiplicities $m, 1$ and $m-1$. There are m linearly independent eigenvectors corresponding to 1, $m-1$ linearly independent eigenvectors associated with 0. Accordingly, G is diagonalizable, and there is an invertible matrix $\tilde{\mathbf{P}}$ such that $\tilde{\mathbf{P}}^{-1}G\tilde{\mathbf{P}}$ is a diagonal matrix:

$$\tilde{P}^{-1}G\tilde{P} = \underline{\Lambda} = diag(\underbrace{1, 1, \cdots, 1}_{m}, -1, \underbrace{0, 0, \cdots, 0}_{m-1}).$$

Defining $U_n = \tilde{\mathbf{P}}^{-1}X_n$, we find

$$U_n = (\Lambda + O(h))U_{n-1} + h^{m+1}\tilde{\mathbf{P}}^{-1}\bar{r}_{m+1,n}. \tag{3.15}$$

Similar to [13] we can assert that there exist constants C_5 and C_6 so that

$$||U_n||_\infty \leq (1 + C_5h)||U_{n-1}||_\infty + C_6h^{m+1}.$$

Using an inductive argument, we will have

$$||U_n||_\infty \leq (1 + C_5h)^n||U_0||_\infty + \frac{(1 + C_5h)^n - 1}{C_5h}C_6h^{m+1}$$

and also, there exists a constant C_7 such that

$$||\varepsilon_n||_\infty \leq C_7h^m. \tag{3.16}$$

Based on (3.7), and similar to Case(i), there are a constant C_8 such that

$$|e_h(x_n)| \leq C_8h^{m+1}. \tag{3.17}$$

Case(iii): $\lambda_1 = 1$.

The matrix G, in this case, has eigenvalues 1 of multiplicity $m+1$ and 0 of multiplicity $m-1$. The eigenspace corresponding to 1 is of dimension m. Therefore, G is not diagonalizable. However, there is a nonsingular matrix $\hat{\mathbf{P}}$ such that

$$\hat{\mathbf{P}}^{-1}G\hat{\mathbf{P}} = \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

Setting $\bar{\Lambda} = \hat{\mathbf{P}}^{-1}G\hat{\mathbf{P}}, \bar{U}_n = \hat{\mathbf{P}}^{-1}X_n$ and using (3.14), we conclude

$$\bar{U}_n = (\bar{\Lambda} + O(h))\bar{U}_{n-1} + h^{m+1}\hat{\mathbf{P}}^{-1}\bar{r}_{m+1,n}. \tag{3.18}$$

Based on an inductive argument, we obtain

$$\bar{U}_n = (\bar{\Lambda} + O(h))^n\bar{U}_0 + h^{m+1}\sum_{l=0}^{n-1}(\bar{\Lambda} + O(h))^l\hat{\mathbf{P}}^{-1}\bar{r}_{m+1,n-1},$$

and hence there are constants C_9 and C_{10} such that

$$||\bar{U}_n||_\infty \leq C_9||\bar{\Lambda}^n||||\bar{U}_0||_\infty + h^{m+1}C_{10}\sum_{l=0}^{n-1}||\bar{\Lambda}^l||_\infty.$$

For matrix $\overline{\Lambda}$, we have

$$\overline{\Lambda}^n = \begin{bmatrix} 1 & n & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

Thus, there are a constant C_{11} so that

$$||\varepsilon_n||_\infty \leq C_{11} h^{m-1}. \tag{3.19}$$

By similar argument to Case(i), and using discrete Gronwall inequality, there exists a constant C_{12} such that

$$|e_h(x_n)| \leq C_{12} h^m. \tag{3.20}$$

If $|\lambda_{\mathbf{1}}| > 1$, then the collocation solution y_h is divergent. Therefore, in the case of constant kernel, the proof is completed.

In the following, we show the convergence analysis in the case of a general, non-constant kernel. From (3.5), (1.1)-(2.1) and using (2.4), we have

$$\begin{aligned} e_h(x_{n,i}) &= e_h(x_n) + h\sum_{j=1}^{m} a_{i,j}\varepsilon_{n,j} + h^{m+1}R_{m+1,n}(\theta_i) \\ &= x_{n,i}^{-\alpha} \int_0^{x_{n,i}} k(x_{n,i},s)e_h(s)ds \\ &= x_{n,i}^{-\alpha}\Big(h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i},x_l+sh)e_h(x_l+sh)ds + h\int_0^{\theta_i} k(x_{n,i},x_n+sh)e_h(x_n+sh)ds\Big) \\ &= x_{n,i}^{-\alpha}\Big(h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i},x_l+sh)(e_h(x_l) + h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{l,j})ds \\ &+ h^{m+1}\sum_{l=0}^{n-1} h\int_0^1 k(x_{n,i},x_l+sh)R_{m+1,l}(s)ds + h^{m+2}\int_0^{\theta_i} k(x_{n,i},x_n+sh)R_{m+1,n}(s)ds\Big) \\ &= x_{n,i}^{-\alpha}\Big(h\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i},x_l+sh)ds\ e_h(x_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^1 k(x_{n,i},x_l+sh)\beta_j(s)ds\ \varepsilon_{l,j} \\ &+ h\int_0^{\theta_i} k(x_{n,i},x_n+sh)ds\ e_h(x_n) + h^2\sum_{j=1}^{m}\int_0^{\theta_i} k(x_{n,i},x_n+sh)\beta_j(s)ds\ \varepsilon_{n,j} \\ &+ h^{m+1}\sum_{l=0}^{n-1}\int_0^1 k(x_{n,i},x_l+sh)R_{m+1,l}(s)ds + h^{m+2}\int_0^{\theta_i} k(x_{n,i},x_n+sh)R_{m+1,n}(s)ds\Big). \end{aligned} \tag{3.21}$$

We first rewrite (3.21) with n replaced by $n-1$ and with $i=m$ and subtract it from (3.21) to obtain

$$
\begin{aligned}
&x_{n,i}^{\alpha}\Big(e_h(x_n)+h\sum_{j=1}^{m}a_{i,j}\varepsilon_{n,j}\Big)-x_{n-1,m}^{\alpha}\Big(e_h(x_{n-1})+h\sum_{j=1}^{m}a_{m,j}\varepsilon_{n-1,j}\Big)\\
&\quad=h\int_0^1 k(x_{n,i},x_{n-1}+sh)ds\ e_h(x_{n-1})+h^2(\theta_i+1-\theta_m)\sum_{l=0}^{n-2}\int_0^1\frac{\partial k}{\partial t}(\eta_{n,i},x_l+sh)ds\ e_h(x_l)\\
&\quad+h^2\sum_{j=1}^{m}\int_0^1 k(x_{n,i},x_{n-1}+sh)\beta_j(s)ds\ \varepsilon_{n-1,j}+h^3(\theta_i+1-\theta_m)\sum_{l=0}^{n-2}\sum_{j=1}^{m}\int_0^1\frac{\partial k}{\partial t}(\eta_{n,i},x_l+sh)\beta_j(s)ds\ \varepsilon_{l,j}\\
&\quad+h\int_0^{\theta_i}k(x_{n,i},x_n+sh)ds\ e_h(x_n)-h\int_0^{\theta_m}k(x_{n-1,m},x_{n-1}+sh)ds\ e_h(x_{n-1})\\
&\quad+h^2\sum_{j=1}^{m}\int_0^{\theta_i}k(x_{n,i},x_n+sh)\beta_j(s)ds\ \varepsilon_{n,j}-h^2\sum_{j=1}^{m}\int_0^{\theta_m}k(x_{n-1,m},x_{n-1}+sh)\beta_j(s)ds\ \varepsilon_{n-1,j}\\
&\quad+h^{m+1}\Re_{m+1,n}(\theta_i)
\end{aligned}
$$

or

$$
\begin{aligned}
&\Big(e_h(x_n)T_n-e_h(x_{n-1})\ x_{n-1,m}^{\alpha}I_m\Big)e+hT_nA\varepsilon_n-hx_{n-1,m}^{\alpha}e_m^TA\varepsilon_{n-1}e\\
&\quad=he_h(x_{n-1})C_n^{(n-1)}e+h^2(C+(1-\theta_m)I_m)\sum_{l=0}^{n-2}e_h(x_l)\partial C_n^{(l)}e+h^2B_n^{(n-1)}\varepsilon_{n-1}\\
&\quad+h^3(C+(1-\theta_m)I_m)\sum_{l=0}^{n-2}\partial B_n^{(l)}\varepsilon_l+he_h(x_n)C_ne-he_h(x_{n-1})ee_m^TC_{n-1}e\\
&\quad+h^2B_n\varepsilon_n-h^2ee_m^TB_{n-1}\varepsilon_{n-1}+h^{m+1}\Re_{m+1,n}(\theta_i),
\end{aligned}
\tag{3.22}
$$

with obvious meaning of $\Re_{m+1,n}(\theta_i)$, and with

$$
\begin{aligned}
\partial C_n^{(l)}&=diag\Big(\int_0^1\frac{\partial k}{\partial t}(\eta_{n,i},x_l+sh)ds\Big),\quad l=0,1,\cdots,N-1,\\
\partial B_n^{(l)}&=diag\Big(\int_0^1\frac{\partial k}{\partial t}(\eta_{n,i},x_l+sh)\beta_j(s)ds\Big)\quad l=0,1,\cdots,N-1.
\end{aligned}
$$

Substituting (3.10) into (3.22), we have

$$\Big(\Big(h\sum_{l=0}^{n-1}b^T\varepsilon_l+h^{m+1}\sum_{l=0}^{n-1}R_{m+1,l}(1)\Big)T_n-\Big(h\sum_{l=0}^{n-2}b^T\varepsilon_l+h^{m+1}\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big)\ x^{\alpha}_{n-1,m}I_m\Big)e$$
$$+hT_nA\varepsilon_n-hx^{\alpha}_{n-1,m}e_m^TA\varepsilon_{n-1}e=h\Big(h\sum_{l=0}^{n-2}b^T\varepsilon_l+h^{m+1}\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big)C_n^{(n-1)}e$$
$$+h^2(C+(1-\theta_m)I_m)\sum_{l=0}^{n-2}\Big(h\sum_{\theta=0}^{l-1}b^T\varepsilon_\theta+h^{m+1}\sum_{\theta=0}^{l-1}R_{m+1,\theta}(1)\Big)\partial C_n^{(l)}e+h^2B_n^{(n-1)}\varepsilon_{n-1}$$
$$+h^3(C+(1-\theta_m)I_m)\sum_{l=0}^{n-2}\partial B_n^{(l)}\varepsilon_l+h\Big(h\sum_{l=0}^{n-1}b^T\varepsilon_l+h^{m+1}\sum_{l=0}^{n-1}R_{m+1,l}(1)\Big)C_ne$$
$$-h\Big(h\sum_{l=0}^{n-2}b^T\varepsilon_l+h^{m+1}\sum_{l=0}^{n-2}R_{m+1,l}(1)\Big)ee_m^TC_{n-1}e+h^2B_n\varepsilon_n-h^2ee_m^TB_{n-1}\varepsilon_{n-1}+h^{m+1}\Re_{m+1,n}(\theta_i),$$

or

$$\varepsilon_n=\Big((T_nA-hB_n)^{-1}(x^{\alpha}_{n-1,m}ee_m^TA+T_neb^T)+O(h)\Big)\varepsilon_{n-1}+h\hat{\mathbf{D}}\sum_{l=0}^{n-2}b^T\varepsilon_l+h^m\hat{\mathbf{R}}_{m+1,n} \tag{3.23}$$

with obvious meaning of $\hat{\mathbf{R}}_{m+1,n}$ and $\hat{\mathbf{D}}$.

By comparing (3.12) for the case of constant kernel, i.e. $K(t,s)=1$, with (3.23), and following a similar approach used to proof the case of constant kernel, we can find

$$\mathrm{E}_n=(\bar{\mathbf{D}}+O(h))\mathrm{E}_{n-1}+h\bar{\mathbf{P}}^{-1}\hat{\mathbf{D}}\bar{\mathbf{P}}\sum_{l=0}^{n-2}\mathrm{E}_l+h^m\bar{\mathbf{P}}^{-1}\bar{R}_{m+1,n}. \tag{3.24}$$

Here, we consider three cases: $-1<\lambda_{\mathbf{1}}<1$, $\lambda_{\mathbf{1}}=-1$ and $\lambda_{\mathbf{1}}=1$, with the similar argument as in the case of $K(t,s)=1$, we see that the inequality (3.1) holds. □

4. Numerical results

In this section some examples are given to demonstrate the applicability, the efficiency and accuracy of the proposed adaptive continuous collocation method for VIEs of the third kind. We illustrate the theoretical results discussed in section 3. In order to complement the tables of numerical errors, we also include the error function $e(x)=|y(x)-y_h(x)|$, for the test examples.

Example 4.1. Consider the following VIE

$$x^{\alpha}y(x)=\frac{6}{7}x^3\sqrt{x}+\int_0^x\frac{1}{2}y(t)dt,\quad x\in I=[0,1], \tag{4.1}$$

whose exact solution is $y(x)=x^{\frac{5}{2}}$ for the case $\alpha=1$. The computational results comparing the maximum absolute errors and the order of convergence for Example 4.1 with Chebyshev points, $m=2$, and various values of N are reported in Table 1. The results demonstrate that the proposed method acheives spectral accuracy and performs better than earlier collocation techniques. Figure 1 illustrates the pointwise distribution of the error and confirms the convergence behavior predicted by the theory.

Example 4.2. Consider the following VIE

$$xy(x)=x^2(1-\frac{x}{3})+\int_0^x ty(t)dt,\quad x\in I=[0,1], \tag{4.2}$$

in which $\alpha=1$, the equation has a unique solution in $C^{m+1}[0,1]$, which is given by $y(x)=x$. The computational results of the maximum absolute error for $N=4$ and $m=2$, obtained with both Chebyshev and Gauss points, are

TABLE 1. Comparison of maximum absolute errors and order of convergence for Example 4.1 with Chebyshev points, $m = 2$.

	Present method		Method[19]	
N	$\|\|e\|\|_\infty$	p	$\|\|e\|\|_\infty$	p
4	2.519×10^{-3}	$--$	3.26×10^{-2}	$--$
8	6.954×10^{-4}	1.857	1.08×10^{-2}	1.59
16	1.819×10^{-4}	1.934	3.03×10^{-3}	1.83
32	4.651×10^{-5}	1.968	7.80×10^{-4}	1.92
64	1.176×10^{-5}	1.984	2.05×10^{-4}	1.96
128	2.959×10^{-6}	1.991	5.18×10^{-5}	1.98
256	7.420×10^{-7}	1.995	1.30×10^{-5}	1.99

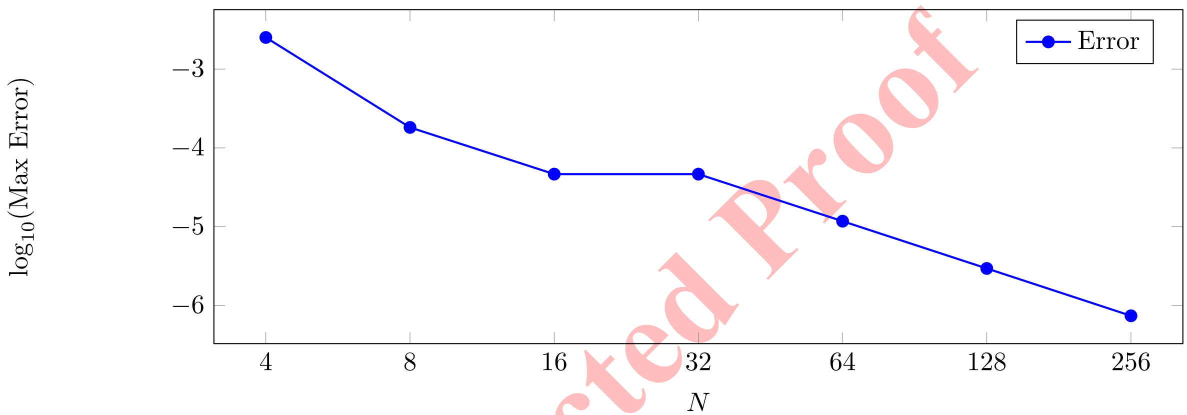

FIGURE 1. Plot of $\log_{10}$(Max Error) versus N (horizontal axis equally spaced).

TABLE 2. The L^∞ errors for Example 4.2 with various collocation points, $N = 4$, $m = 2$

	Chebyshev points	Gauss points
Digits	$\|\|e\|\|_\infty$	$\|\|e\|\|_\infty$
10	7×10^{-10}	5×10^{-10}
20	3×10^{-20}	2×10^{-20}
30	6×10^{-30}	4×10^{-30}
40	4×10^{-40}	3×10^{-40}

reported in Table 2. The performance of the method is further examined under different precision levels, denoted by d. As d increases, the error $||e||_\infty$ decreases rapidly, illustrating the exponential convergence of the scheme.

In a second experiment, we study the pointwise error distribution of the method. The corresponding values are presented in Table 2, and Figure 2 depicts Error as a function of x. The error remains consistently small, between 10^{-10} and 10^{-9}, across the entire interval, highlighting the high level of accuracy achieved. As expected, the error is essentially zero at the initial point and shows a mild growth toward the right endpoint. This increase is natural and can be attributed to the accumulation of local approximation and quadrature errors. Despite this, the overall error stays well within the predicted bounds, confirming the stability of the method and its agreement with the theoretical convergence results established earlier.

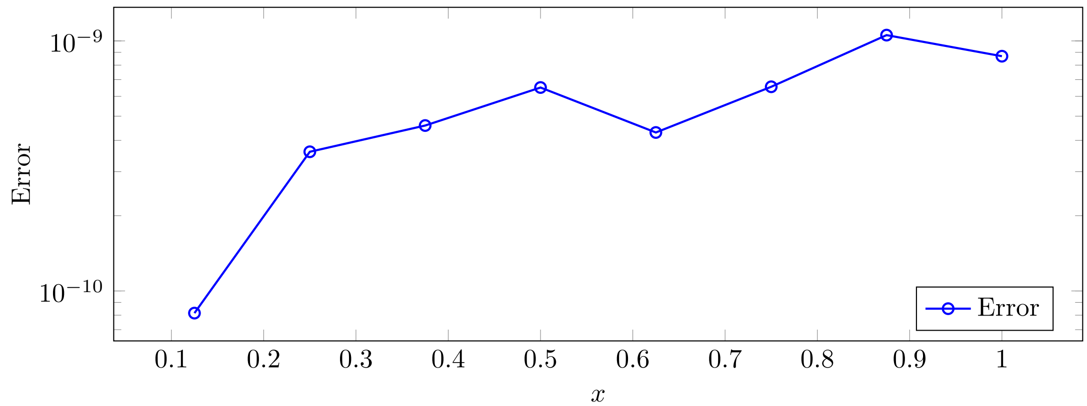

FIGURE 2. Pointwise error distribution for Example 4.2 with $N = 8$.

5. Concluding remarks

In this paper, we have developed and analyzed a multi-interval continuous collocation method for a class of VIEs of the third kind. By employing localized polynomial approximations on subintervals, the method combines flexibility with high accuracy. A detailed solvability analysis and convergence proof were provided, ensuring the theoretical foundation of the approach. Numerical experiments further confirmed the efficiency of the scheme, showing that it not only achieves spectral accuracy but also performs reliably in situations where existing methods face difficulties. These results demonstrate that the proposed method offers a practical and powerful tool for tackling third-kind Volterra problems and can be extended to more general settings in future research.

References

[1] E. Aourir, N. Izem, and H. Laeli Dastjerdi, *Numerical solution of third-kind Volterra integral equations with proportional delays based on moving least squares collocation method*, International Journal of Computer Mathematics, *101*(4) (2024), 447-464.

[2] R. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J., *10*(4) (1943), 643-647.

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004.

[4] H. Brunner, *Volterra Integral Equations: An Introduction to Theory and Applications*, Cambridge University Press: Cambridge, UK, 2017.

[5] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[6] T. Diogo, *Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations*, J. Comput. Appl. Math., *229*(2) (2009), 363-372.

[7] T. Diogo, N. J. Ford, and P. Lima, *Solution of a singular integral equation by a split-interval method*, Int. J. Numer. Anal. Model., *4*(1) (2007), 63-73.

[8] T. Diogo, S. McKee, and T. Tang, *A Hermite-type collocation method for the solution of an integral equation with a certain weakly singular kernel*, IMA J. Numer. Anal. 11, *11*(4) (1991), 595-605.

[9] T. Diogo and G. Vainikko, *Applicability of spline collocation to cordial Volterra equations*, Math. Model. Anal., *18*(1) (2013), 1-21.

[10] G. Gripenberg, S. O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, 1990.

[11] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math., *20* (1919), 292-296.

[12] E. Hairer, C. Lubich, and S. P. Norsett, *Order of convergence of cne-step methods for Volterra integral equations of the second kind*, SIAM J. Numer. Anal., *20*(3) (1983), 569-579.

[13] J. P. Kauthen and H. Brunner, *Continuous collocation approximations to solutions of first kind Volterra equations*, Math. Comput., *66*(220) (1997), 1441-1459.

[14] H. Liang and H. Brunner, *On the convergence of collocation solutions in continuous piecewise polynomial spaces for Volterra integral equations*, BIT Numer. Math., *56*(4) (2016), 1339 - 1367.

[15] Z. Ma and C. Huang, *Fractional Collocation Method for Third-Kind Volterra Integral Equations with Nonsmooth Solutions*, J. Sci. Comput., *95*(26) (2023) https://doi.org/10.1007/s10915-023-02155-z.

[16] S. Nemati, P. M. Lima, and D. F. M. Torres, *Numerical solution of a class of third-kind Volterra integral equations using Jacobi wavelets*, Numer. Algor., *86*(2) (2021), 675-691.

[17] S. P. Norsett, *Splines and collocation for ordinary initial value problems*, Approximation Theory and Spline Functions, Dordrecht, Springer Netherlands, (1984), 397-417.

[18] S. Seyed Allaei, Z. W. Yang, and H. Brunner, *Existence, uniqueness and regularity of solutions to a class of third-kind Volterra integral equations*, J. Integral Equ. Appl., *27*(3) (2015), 325-342.

[19] S. Seyed Allaei, Z. Yang, and H. Brunner, *Collocation methods for third-kind VIEs*, IMA J. Numer. Anal., *37*(3) (2017), 1104-1124.

[20] H. Song, Z. Yang, and H. Brunner, *Analysis of collocation methods for nonlinear Volterra integral equations of the third kind*, Calcolo., *56*(7) (2019).

[21] Li. Yuping, Y. Zhanwen, and Zhang Chiping, *Theoretical and numerical analysis of third-kind auto-convolution Volterra integral equations*, Comp. Appl. Math., *38*(170) (2019).