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A new two-step iteration method for solving non-Hermitian positive definite linear systems

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Abstract

We introduce a new two-step iteration method for solving large and sparse non-Hermitian positive definite system of linear equations. We analyze the convergence properties of the method and show that under some conditions it is convergent. We extract the preconditioner of proposed method and apply it to expedite the convergence of the flexible version of GMRES method for solving the system. Numerical experiments support the theoretical results and shows that the new method and the corresponding preconditioner are more efficient than the well-known methods and preconditioners in the literature.

Keywords. Non-Hermitian, Positive definite, Hermitian, Skew-Hermitian, Convergence.
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1. Introduction

Many problems in scientific computing and engineering need to solve a system of linear equations. In this work, we consider the system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian positive definite matrix, $b \in \mathbb{C}^n$ is a known right-hand side vector, and $x \in \mathbb{C}^n$ is termed the unknown vector. Stationary iterative methods such as the Jacobi, Gauss-Seidel, SOR, AOR, HSS iteration methods use efficient splitting of the coefficient matrix A for solving the systems of form (1.1) [1, 25, 26, 28]. Among the stationary methods, the Hermitian and skew-Hermitian splitting (HSS) iteration method, proposed by Bai et al. in [1], has attracted considerable attention of the researchers because of its impressive performance and graceful mathematical attributes. In the sequel we review the HSS iteration method and some of its variants.

Using the Hermitian and skew-Hermitian splitting

$$A = H + S, \tag{1.2}$$

of the coefficient matrix A, where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*),$$

the HSS iteration method can be written as following.

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The HSS method [1]. Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ converges, we need to compute

$$\begin{cases}(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b,\\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b,\end{cases} \tag{1.3}$$

where α is a given positive constant.

In [1], it has been proved that for any positive α the HSS method converges unconditionally to the unique solution of the system of linear equations (1.1). Moreover, it was shown that

$$\alpha_{est}^{\text{HSS}} = \sqrt{\lambda_{\min}(H)\lambda_{\max}(H)}, \tag{1.4}$$

minimizes the specified upper bound on the spectral radius of the HSS iteration matrix, where $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ are the smallest and largest eigenvalues of H, respectively. In this method, the upper bound of the spectral radius of the iteration matrix is dependent on only the spectrum of the matrix H, but is independent of the spectrum of the matrix S, as well as the eigenvectors of all matrices involved [1].

Based on the HSS method, several efficient iterative methods for solving non-Hermitian (positive definite or positive semi-definite) systems of linear equations have been proposed by researchers, such as preconditioned HSS (PHSS) method [2], normal and skew-Hermitian splitting (NSS) method [3, 17], positive and skew-Hermitian splitting (PSS) method [4, 5], asymmetric and skew-Hermitian splitting method [20], inexact HSS method [6], generalized HSS (GHSS) method [11], relaxed HSS method [14], modified GHSS method [21], generalized PSS method [15], generalized PHSS method [27], accelerated HSS method [7], modified HSS (MHSS) method [8], preconditioned MHSS (PMHSS) method [9], local HSS method [19], dimensional splitting (DS) iteration method [13, 16] and so on.

The above iterative methods were first used as stationary iterative solvers, but they can also be used as preconditioners for the Krylov subspace methods [24], resulting in a far more efficient and robust class of solvers.

In [22], Li and Wu proposed the single-step Hermitian and skew-Hermitian splitting and shift-splitting [10] (SHSS-SS) method, for solving the system as follows.

The SHSS-SS method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2 \dots$, until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ by the following procedure

$$\begin{cases}(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b,\\ (\alpha I + A)x^{(k+1)} = (\alpha I - A)x^{(k+\frac{1}{2})} + 2b,\end{cases} \tag{1.5}$$

where α is a given positive constant.

It was shown that

$$\alpha_{est}^{\text{SHSS-SS}} = \arg\min_{\alpha}\left\{\frac{\sqrt{\alpha^2 + \sigma_{\max}^2}}{\alpha + \lambda_{\min}}\right\} = \frac{\sigma_{\max}^2}{\lambda_{\min}}, \tag{1.6}$$

minimizes an upper bound of the spectral radius of the SHSS-SS iteration matrix, where $\lambda_{\min}$ is the smallest eigenvalue of matrix H and $\sigma_{\max}$ is the largest singular value of matrix S [22].

In this paper, we develop a new two-step iteration method for solving positive definite system of linear equations. We prove that it converges under some conditions. One of the key advantages of this method is its insensitivity to the parameters employed. Numerical results demonstrate that it significantly outperforms several existing methods.

We use the following notations throughout the paper. $\sigma(A)$ and $\rho(A)$ denote the spectrum and the spectral radius of the matrix A, respectively. $\|.\|_2$ denotes the Euclidean norm. For a vector $x \in \mathbb{C}^n$, x^* is used for the conjugate transpose of x. A parameter with the subscript "est", like α_{est}, stands for an estimation of the parameter.

The rest of the paper is organized as follows; In section 2, we introduce a new method for solving non-Hermitian positive-definite linear systems. In section 3, we analyze the convergence properties of the new method. In section 4, we establish the inexact new iteration method. In section 5, we extract the preconditioner of the proposed method. Numerical examples are presented in section 6, to illustrate the effectiveness of our methods. Finally, in section 7, we draw some conclusions.

2. The new two-step method

In this section, we introduce a new two-step iterative method for solving non-Hermitian positive definite system of linear Equations (1.1). We first split the matrix A as

$$A = \Big(\frac{1}{2}I + B_\alpha\Big) - \Big(\frac{1}{2}I - C_\alpha\Big), \tag{2.1}$$

where $B_\alpha = \frac{1}{2}(1+\alpha)A$ and $C_\alpha = \frac{1}{2}(1-\alpha)A$, in which $\alpha > 0$ and I is the identity matrix. Using the splitting (2.1), the first step of our method is stated as

$$\Big(\frac{1}{2}I + B_\alpha\Big)x^{(k+\frac{1}{2})} = \Big(\frac{1}{2}I - C_\alpha\Big)x^{(k)} + b,$$

and the second step is established as the first step of the HSS method with $\alpha = 0$. Since

$$\frac{1}{2}I + B_\alpha = \frac{1}{2}(I + \alpha A + A) = \frac{1}{2}(P + A),$$

and

$$\frac{1}{2}I + C_\alpha = \frac{1}{2}(I + \alpha A - A) = \frac{1}{2}(P - A).$$

with $P = I + \alpha A$, in fact, the proposed method employs a preconditioned version of the Shift-Splitting (SS) method in the first step [10]. So, from this point forward, we will refer to the proposed method as SSTHS, as it is derived using a Shift-Splitting-type (SST) approach combined with a Hermitian and skew-Hermitian (HS) splitting. Hence, the SSTHS method is stated as following.

The SSTHS method. Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} \Big(\frac{1}{2}I + B_\alpha\Big)x^{(k+\frac{1}{2})} = \Big(\frac{1}{2}I - C_\alpha\Big)x^{(k)} + b, \\ Hx^{(k+1)} = -Sx^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.2}$$

where α is a given positive constant and I is the identity matrix.

3. Convergence analysis of the SSTHS method

In this section, we study the convergence of the new iteration method. The following lemma describes a general convergence criterion for the two-step splitting iterations.

Lemma 3.1. *([1]) Let $A \in \mathbb{C}^{n\times n}$, $A = M_i - N_i$ (i = 1,2) be two splittings of the matrix A, and let $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $\{x^{(k)}\}$ is produced using a two-steps iteration method defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases} \tag{3.1}$$

$k = 0, 1, 2, \ldots,$ then

$$x^{(k+1)} = M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + M_2^{-1}(I + N_2M_1^{-1})b, \quad k = 0, 1, 2, \ldots.$$

Moreover, if the spectral radius of the iteration matrix $G = M_2^{-1}N_2M_1^{-1}N_1$ is less than 1, i.e., $\rho(G) < 1$, then the iteration method converges to the unique solution $x^* \in \mathbb{C}^n$ of the system of linear Equations (1.1) for all initial vectors $x^{(0)} \in \mathbb{C}^n$.

We apply the above results to obtain the following main theorem.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, and H and S be the Hermitian and the skew-Hermitian parts A. Let B_α and C_α be the ones introduced in Eq. (2.1) and $\alpha > 0$. Then the iteration matrix M_α of the SSTHS method is given by*

$$M_\alpha = -H^{-1}S\Big(\frac{1}{2}I + B_\alpha\Big)^{-1}\Big(\frac{1}{2}I - C_\alpha\Big), \tag{3.2}$$

and spectral radius of the iteration matrix M_α is bounded by

$$\rho(M_\alpha) \leq \frac{\sigma_{\max}(S)}{\lambda_{\min}(H)}\delta_2(\alpha), \tag{3.3}$$

where

$$\delta_2(\alpha) = \left(\max_{q,r>0} \frac{(\alpha-1)^2 q + 2(\alpha-1)r + 1}{(\alpha+1)^2 q + 2(\alpha+1)r + 1}\right)^{1/2}, \quad r = \frac{x^* H x}{x^* x}, \quad q = \frac{x^* A A^* x}{x^* x},$$

$\lambda_{\min}$ is the smallest eigenvalue of matrix H and $\sigma_{\max}$ is the largest singular value of matrix S. Moreover, if $\sigma_{\max}(S) < \lambda_{\min}(H)$, then $\rho(M_\alpha) < 1$, which means that the SSTHS method is convergent.

Proof. Let

$$M_1 = \frac{1}{2}\left(I + (1+\alpha)A\right), \quad N_1 = \frac{1}{2}\left(I - (1-\alpha)A\right), \quad M_2 = H, \quad N_2 = -S.$$

As both of the matrices $M_1 = \frac{1}{2}(I + (1+\alpha)A)$ and $M_2 = H$ are nonsingular for any $\alpha > 0$, using Lemma 3.1 we get the iteration matrix M_α in (3.2).

Now for the spectral radius of M_α, we have

$$\begin{aligned} \rho(M_\alpha) &= \rho\left(-H^{-1}S\Big(\frac{1}{2}(I + (1+\alpha)A)\Big)^{-1}\frac{1}{2}(I - (1-\alpha)A)\right) \\ &\leq \|H^{-1}S\left(I + (1+\alpha)A\right)^{-1}\left(I - (1-\alpha)A\right)\|_2 \\ &\leq \|H^{-1}S\|_2 \|\left((I + (1+\alpha)A\right)^{-1}\left(I - (1-\alpha)A\right)\|_2. \end{aligned} \tag{3.4}$$

Let

$$\delta_1 = \|H^{-1}\|_2 \|S\|_2, \quad \delta_2(\alpha) = \|((I + (1+\alpha)A)^{-1}(I - (1-\alpha)A)\|_2. \tag{3.5}$$

As H is Hermitian positive definite matrix, we get

$$\|H^{-1}\|_2 = \rho(H^{-1}) = \max_{i=1,2,\ldots,n} \frac{1}{\lambda_i(H)} = \frac{1}{\lambda_{\min}(H)}. \tag{3.6}$$

Also, we have

$$\|S\|_2 = \sqrt{\rho(S^H S)} = \sigma_{\max}(S). \tag{3.7}$$

Using (3.6) and (3.7), we have

$$\delta_1 = \frac{\sigma_{\max}(S)}{\lambda_{\min}(H)}. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned} \delta_2^2(\alpha) &= \|((I + (1+\alpha)A)^{-1}(I - (1-\alpha)A)\|_2^2 \\ &= \|(\alpha A + I + A)^{-1}(\alpha A + I - A)\|_2^2 \\ &= \max_{x \neq 0} \frac{x^*(\alpha A + I - A)(\alpha A + I - A)^* x}{x^*(\alpha A + I + A)(\alpha A + I + A)^* x} \end{aligned}$$

It is a simple task to show that

$$\delta_2^2(\alpha) = \max_{x \neq 0} \frac{(\alpha-1)^2 x^* A A^* x + (\alpha-1)x^*(A + A^*)x + x^* x}{(\alpha+1)^2 x^* A A^* x + (\alpha+1)x^*(A + A^*)x + x^* x}.$$

Considering the fact that $H = (A + A^*)/2$, and also dividing the numerator and denominator of the fraction by $x^* x$ leads to

$$\delta_2^2(\alpha) = \max_{q,r>0} \frac{(\alpha-1)^2 q + 2(\alpha-1)r + 1}{(\alpha+1)^2 q + 2(\alpha+1)r + 1} < 1, \tag{3.9}$$

where

$$r = \frac{x^*Hx}{x^*x} > 0, \quad \text{and} \quad q = \frac{x^*AA^*x}{x^*x} = \frac{\|Ax\|_2^2}{\|x\|_2^2} > 0.$$

Clearly, it follows from (3.9) that $\delta_2(\alpha) < 1$. Now, using (3.8) and (3.9), the upper bound of $\rho(M_\alpha)$ given in (3.3) is obtained. Therefore if $\sigma_{\max}(S) < \lambda_{\min}(H)$, then $\delta_1 < 1$, and

$$\rho(M_\alpha) < \delta_1\delta_2(\alpha) < 1,$$

which means that the SSTHS method is convergent. □

Remark 3.3. It is worth mentioning that the proposed method may converge even when $\lambda_{\min}(H) < \sigma_{\max}(S)$. In fact, while $\lambda_{\min}(H)$ can be less than $\sigma_{\max}(S)$, the value of $\delta_2(\alpha)$ may still be sufficiently small. This occurs in most of the text examples within the numerical experiments section.

4. Inexact version of SSTHS

In each iteration of the SSTHS method, we have to solve two subsystems with the coefficient matrices $\frac{1}{2}(I+A+\alpha A)$ and H. Solving these systems using direct methods are impractical for large systems. So we solve them using iteration methods. We solve the system with the coefficient matrix $\frac{1}{2}(I + A + \alpha A)$ by a Krylov subspace method like GMRES and the system with H using the conjugate gradient (CG) method [26]. To do so, similar to the HSS iteration method we obtain the inexact version of the SSTHS iteration method as follows.

Let us denote

$$x^{(k+\frac{1}{2})} = x^{(k)} + \delta.$$

Then, by substituting this into the first relation of Eq. (2.2), we deduce

$$(\frac{1}{2}I + B_\alpha)\delta = b - Ax^{(k)} = r^{(k)}.$$

Also, denoting

$$x^{(k+1)} = x^{(k+\frac{1}{2})} + \zeta,$$

and substituting it into the second relation of Eq. (2.2), we get

$$H\zeta = -Hx^{(k+\frac{1}{2})} - Sx^{(k+\frac{1}{2})} + b = b - Ax^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}.$$

Then, the following algorithm describes the inexact version of SSTHS.

Algorithm 1. Inexact version of SSTHS (ISSTHS).

for $k = 0, 1, \ldots,$ until convergence, **do**
  $r^{(k)} = b - Ax^{(k)}$,
  Solve $\left(\frac{1}{2}I + B_\alpha\right) z^{(k)} = r^{(k)}$ by GMRES,
  such that the residual $p^{(k)} = r^{(k)} - \left(\frac{1}{2}I + B_\alpha\right) z^{(k)}$ of the iteration satisfies $\|p^{(k)}\| \le \epsilon_k \|r^{(k)}\|$,
  $x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}$,
  $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$,
  Solve $Hz^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}$ by the conjugate gradient,
  such that the residual $q^{(k)} = r^{(k+\frac{1}{2})} - Hz^{(k+\frac{1}{2})}$ of the iteration satisfies $\|q^{(k)}\| \le \eta_k \|r^{(k+\frac{1}{2})}\|$,
  $x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}$,
end for

Remark 4.1. In Algorithm 1, the only system whose coefficient matrix depends on the parameter α is the system of Step 3. This system is solved using an iterative method such as GMRES. As we have seen, the coefficient matrix of this system can be represented as

$$\frac{1}{2}I + B_\alpha = \frac{1}{2}(I + A + \alpha A).$$

Hence, the parameter

$$\alpha_{est} = \frac{\|I + A\|_F}{\|A\|_F}, \tag{4.1}$$

creates a balance between the matrices A and $A + I$ (see [12]).

5. The SSTHS preconditioner

In this section, we aim to derive the preconditioner for the SSTHS iterative method. We know that the single-step iterative method corresponding to (2.2) is the following:

$$x^{(k+1)} = -H^{-1}S(\frac{1}{2}I + B_\alpha)^{-1}(\frac{1}{2}I - C_\alpha)x^{(k)} + H^{-1}(I - S(\frac{1}{2}I + B_\alpha))b, \quad k = 0, 1, 2, \dots. \tag{5.1}$$

Eq. (5.1) can be rewritten as follows:

$$x^{(k+1)} = M_\alpha^{-1}N_\alpha x^{(k)} + M_\alpha^{-1}b.$$

It is easy to obtain a split of matrix A from Equation (5.1) as follows:

$$A = M_\alpha - N_\alpha, \tag{5.2}$$

where

$$M_\alpha = (\frac{1}{2}I + B_\alpha)(\frac{1}{2}I + B_\alpha - S)^{-1}H,$$
$$N_\alpha = -(\frac{1}{2}I + B_\alpha)(\frac{1}{2}I + B_\alpha - S)^{-1}S(\frac{1}{2}I + B_\alpha)^{-1}(\frac{1}{2}I - C_\alpha).$$

As a matter of fact, any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for the Krylov subspace methods. The splitting preconditioner corresponding to the SSTHS iteration (5.1) is given by

$$\mathcal{P}_{\text{SSTHS}} = M_\alpha = (\frac{1}{2}I + B_\alpha)(\frac{1}{2}I + B_\alpha - S)^{-1}H. \tag{5.3}$$

At each iteration of a Krylov subspace method in conjunction with the SSTHS preconditioner,we need solving a system of the form

$$\mathcal{P}_{\text{SSTHS}}z = r, \tag{5.4}$$

where $r, z \in \mathbb{C}^n$. This can be done using the following algorithm.

Algorithm 2. Solution of $\mathcal{P}_{\text{SSTHS}}z = r$.

1. Solve $\left(\frac{1}{2}I + B_\alpha\right)v = r$ for v;
2. Set $w = \left(\frac{1}{2}I + B_\alpha - S\right)v$;
3. Solve $Hz = w$.

In Algorithm 2, we need solving two systems of linear equations with the coefficient matrices $\frac{1}{2}I + B_\alpha$ and H. The system in Step 1 can be solved exactly using the LU factorization and the system in Step 3 may be solved using the Cholesky factorization. However, solving these two systems exactly is not practical. So, in our numerical results we solve these systems using iteration methods. In fact, the system in Step 1 is solved using the GMRES method and the system in Step 3 is solved by the CG method. It is worth noting that, in this case, the preconditioned system should be solved using the flexible GMRES (FGMRES) method, see [23] or [24, Section 9.4] for more details.

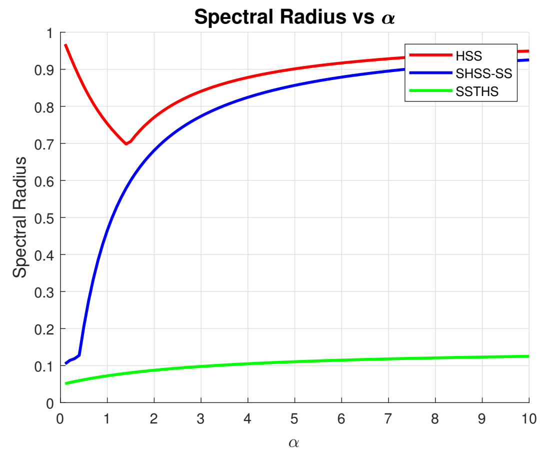

FIGURE 1. The spectral radius of the iteration matrices of HSS, SHSS-SS and SSTHS methods for Example 6.1 with $n = 64^2$.

6. Numerical Experiments

In this section, we compare some numerical results to show the effectiveness of SSTHS iterations and the induced preconditioner. The numerical results are compared with two known methods. The numerical experiments were performed in MATLAB (R2020a), on an Intel(R) Core(TM) i5-3320M CPU @ 2.60GHz 2.60 GHz and 8GB RAM. We present the results in two subsections. The first subsection in devoted to iterations and the next one to their preconditioners.

6.1. **Numerical results of the SSTHS iteration method.** We use Algorithm 1 for solving the system (1.1). The outer iteration is stopped as the Euclidean norm of the residual is reduced by a factor of 10^6 or the number of iterations exceeds 1000. The inner systems in Steps 3 and 6 of Algorithm 1 are respectively solved by the restarted GMRES(20) and the CG method. For the subsystems, the iterations are terminated as soon as the residual of their linear systems is reduced by a factor of 10^3 and the maximum number of iterations is set to be 100. We always use a zero vector as an initial guess. The numerical results are compared with those of the HSS and SHSS-SS iteration method. It is noted that, similar to the ISSTHS, we use inexact version of the HSS (IHSS) and SHSS-SS (ISHSS-SS) iteration methods. Numerical results are presented in the tables. In these tables, "IT" and "CPU" stand for the number of iterations and elapsed CPU time (in second).

Example 6.1. We consider the two-dimensional convectiondiffusion problem

$$-(u_{xx} + u_{yy}) + u_x + u_y = g, \tag{6.1}$$

where g is a given function and u satisfies Dirichlet boundary conditions, see [18]. Using the five-point centered finite difference discretization on the unit square $[0, 1] \times [0, 1]$ with mesh-size $h = 1/(m + 1)$ to discretize the two-dimensional convectiondiffusion problem (6.1), we obtain the non-Hermitian positive definite linear systems (1.1) with the coefficient matrix

$$A = T \otimes I + T \otimes I,$$

where $\otimes$ denotes the Kronecker product, T is a tridiagonal matrix given by

$$T = \text{tridiag}\big(- 1 - R_e, 2, -1 + R_e\big),$$

and $R_e = h/2$ is the mesh Reynolds number. The right-hand side vector b (here and the subsequent examples) is taken such a way that the exact solution of the system is a vector of all ones.

The spectral radius of the iteration matrix is a useful criterion for comparing iterative methods. Figure 1 shows the spectral radius $\rho(M_\alpha)$ of the iteration matrices of HSS, SHSS-SS and SSTHS methods for Example 6.1 with $n = 64^2$. We observe, the SSTHS method is less sensitive than the other two methods with respect to the parameter α.

Tables 1–3 present the numerical results for the three methods for different values of the parameter α and the size of the system to highlight the advantage of our proposed method. The numerical results clearly show that the ISSTHS method is superior to the IHSS and ISHSS-SS iteration methods from the point of the number of iterations and the CPU time. We also see that IHSS and ISHSS-SS iteration methods fail to converge for some values of α, whereas the ISSTHS method is convergent. Here, a dagger (†) indicates that the method fails to converge.

We compute α_{est} by means of the formulas (1.4), (1.6), and (4.1) for IHSS method, ISHSS-SS method, and our method, respectively, and summarize the results of implementing the methods with this parameter in Table 4. As we see the ISSTHS method is superior to the other methods.

Table 1. Numerical results for Example 6.1 with $n = 64^2$.

Method	α	0.1	0.2	0.3	0.5	0.7	0.9
IHSS	IT	332	254	296	493	690	887
	CPU	3.89	1.23	1.41	1.74	2.00	2.36
ISHSS-SS	IT	67	132	198	329	460	592
	CPU	0.82	0.80	1.04	1.24	1.38	1.62
ISSTHS	IT	5	5	5	5	5	5
	CPU	0.15	0.15	0.15	0.16	0.16	0.15

Table 2. Numerical results for Example 6.2 with $n = 128^2$.

Method	α	0.1	0.2	0.3	0.5	0.7	0.9
IHSS	IT	498	692	†	†	†	†
	CPU	7.16	8.06	†	†	†	†
ISHSS-SS	IT	231	462	692	†	†	†
	CPU	4.21	5.11	5.13	†	†	†
ISSTHS	IT	5	5	4	4	4	4
	CPU	0.31	0.32	0.26	0.26	0.27	0.28

Table 3. Numerical results for Example 6.1 with $n = 200^2$.

Method	α	0.1	0.2	0.3	0.5	0.7	0.9
IHSS	IT	777	†	†	†	†	†
	CPU	24.44	†	†	†	†	†
ISHSS-SS	IT	516	†	†	†	†	†
	CPU	15.51	†	†	†	†	†
ISSTHS	IT	5	5	4	4	4	4
	CPU	0.55	0.57	0.47	0.49	0.49	0.49

Example 6.2. We consider the three-dimensional convection-diffusion problem

$$-(u_{xx} + u_{yy} + u_{zz}) + (u_x + u_y + u_z) = f(x, y, z), \tag{6.2}$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with Dirichlet boundary conditions, see [1]. The problem is discretized using the finite differences. To discretize the problem (6.2), in all three directions, we assume that the numbers (n) of

TABLE 4. Numerical results for Example 6.1 for α_{est}.

Method		$n=64^2$	$n=128^2$	$n=200^2$
IHSS	α_{est}	0.19	0.09	0.06
	IT	255	497	879
	CPU	1.26	7.33	31.86
ISHSS-SS	α_{est}	0.20	0.20	0.20
	IT	134	467	†
	CPU	0.78	5.12	†
ISSTHS	α_{est}	1.17	1.17	1.17
	IT	5	4	4
	CPU	0.16	0.27	0.50

grid points are the same. Using the centered differences for discretizing the diffusive terms and the first order upwind for discretizing the convective terms, respectively, we obtain the non-Hermitian positive definite linear systems of the form (1.1) with the coefficient matrix

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z,$$

where $\otimes$ denotes the Kronecker product and

$$T_x = \text{tridiag}(t_2, t_1, t_3), \quad T_y = T_z = \text{tridiag}(t_2, 0, t_3).$$

When the centered difference scheme is used to approximate the first order derivatives, we get

$$t_1 = 6,\ t_2 = -1 - r,\ t_3 = -1 + r.$$

When the first order derivatives are approximated by the upwind difference scheme, we obtain

$$t_1 = 6 + 6r,\ t_2 = -1 - 2r,\ t_3 = -1.$$

Here, $r = h/2$ is the mesh Reynolds number where $h = 1/(n+1)$ is the step size.

Figures 2 and 3 display the spectral radius of the iteration matrices corresponding to the HSS, SHSS-SS, and the SSTHS methods for Example 6.2, using upwind and centered difference schemes, respectively, for $n = 10^3$. As seen, the sensitivity of the HSS and the SHSS-SS iterations method is greater than that of SSTHS.

The results of implementing the three methods with different values of α are summarized in Tables 5–8. We also use α_{est} by the means of the value in (1.4), (1.6), and (4.1) for the IHSS method, ISHSS-SS method, and the ISSTHS method, respectively. The results of implementing the methods for their associated parameter α_{est} is summarized in Table 9. The numerical results reported in this example consistently demonstrate the effectiveness and superiority of the ISSTHS method.

TABLE 5. The results of Example 6.2 for the upwind difference scheme using different α with $n = 20^3$.

Method	α	0.7	0.9	1.2	1.5	1.7	1.9
IHSS	IT	92	74	98	123	139	155
	CPU	0.73	0.68	0.70	0.74	0.75	0.79
ISHSS-SS	IT	39	50	66	82	93	104
	CPU	0.43	0.41	0.44	0.46	0.50	0.54
ISSTHS	IT	6	6	6	6	5	5
	CPU	0.21	0.22	0.23	0.23	0.21	0.21

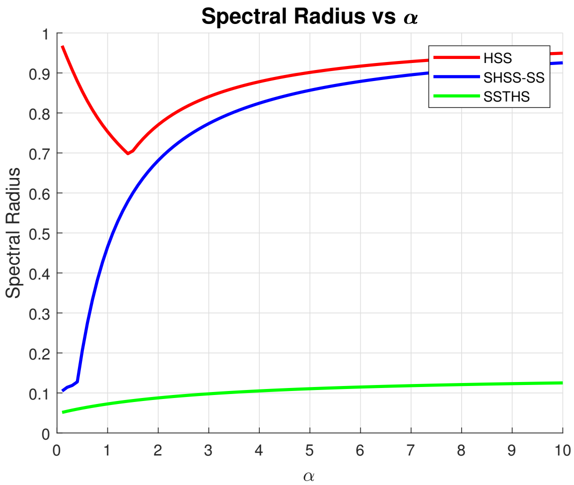

FIGURE 2. The spectral radius $\rho(M_\alpha)$ of the iteration matrices of HSS, SHSS-SS and SSTHS for Example 6.2 with the upwind difference scheme for $n = 10^3$.

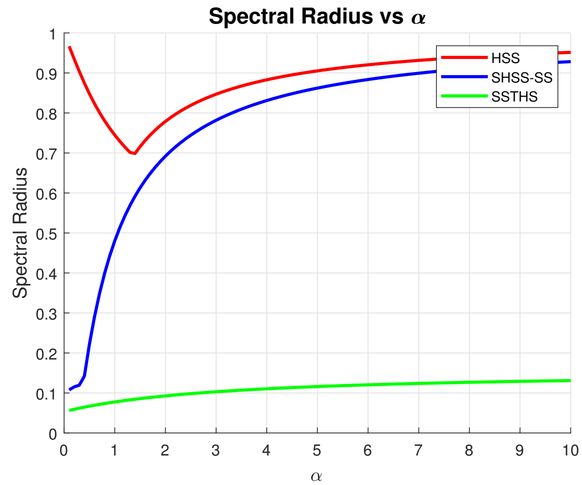

FIGURE 3. The spectral radius $\rho(M_\alpha)$ of the iteration matrices of HSS, SHSS-SS and SSTHS for Example 6.2 with the centered difference scheme for $n = 10^3$.

TABLE 6. The results of Example 6.2 for the upwind difference scheme using different α with $n = 30^3$.

Method	α	0.7	0.9	1.2	1.5	1.7	1.9
IHSS	IT	119	153	204	255	289	223
	CPU	2.53	2.64	2.93	3.32	3.60	3.79
ISHSS-SS	IT	80	103	137	171	193	216
	CPU	1.58	1.69	1.95	2.23	2.38	2.56
ISSTHS	IT	5	5	5	5	5	5
	CPU	0.46	0.47	0.51	0.52	0.54	0.53

TABLE 7. The results of Example 6.2 for the centered difference scheme using different α with $n = 20^3$.

Method	α	0.7	0.9	1.2	1.5	1.7	1.9
IHSS	IT	89	75	100	125	142	159
	CPU	0.72	0.67	0.70	0.74	0.77	0.83
ISHSS-SS	IT	40	51	68	84	95	107
	CPU	0.41	0.41	0.46	0.47	0.51	0.55
ISSTHS	IT	6	6	6	6	6	6
	CPU	0.21	0.21	0.22	0.22	0.23	0.24

6.2. **Numerical results of the SSTHS preconditioner.** In this subsection, the SSTHS preconditioner is compared with the preconditioners induced by the HSS and SHSS-SS iteration methods. The HSS and the SHSS-SS preconditioners are given as follows (See [2, 22])

$$\mathcal{P}_{\text{HSS}} = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S), \quad \mathcal{P}_{\text{SHSS-SS}} = (\alpha I + H)(3\alpha I + A^*)^{-1}(\alpha I + A).$$

We consider the systems in Examples 6.1 and 6.2. We use the flexible version of the GMRES (FGMRES) method for solving the preconditioned system and the inner systems are solved by the CG and GMRES methods. We always use

Table 8. The results of Example 6.2 for the centered difference scheme using different α with $n = 30^3$.

Method	α	0.7	0.9	1.2	1.5	1.7	1.9
IHSS	IT	121	158	207	259	293	328
	CPU	2.54	2.66	2.97	3.32	3.62	3.75
ISHSS-SS	IT	81	104	139	178	196	219
	CPU	1.54	1.72	1.99	2.25	2.43	2.62
ISSTHS	IT	5	5	5	5	5	5
	CPU	0.45	0.46	0.49	0.51	0.52	0.54

Table 9. The results of Example 6.2 for the centered and upwind difference scheme using α_{est} with $n = 30^3$.

Method		upwind difference scheme	centered difference scheme
IHSS	α_{est}	0.61	0.60
	IT	106	106
	CPU	2.54	2.65
ISHSS-SS	α_{est}	0.29	0.30
	IT	34	36
	CPU	1.28	1.32
ISSTHS	α_{est}	1.14	1.14
	IT	5	5
	CPU	0.48	0.47

a zero vector as an initial guess and choose the right-hand vector in (1.1) such that the solution of the system is a vector of all ones. The iteration of the FGMRES method is terminated if the current iteration satisfies

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq 10^{-6},$$

where $r_k = b - Ax_k$ is the residual at the kth iteration. Also the CG and GMRES methods is stopped as soon as the residual of the system is reduced by a factor of 10^2. The maximum number of iterations for FGMRES (resp., CG and GMRES) is set to be 1000 (resp., 600).

The numerical results have been reported for different values of α in Tables 10–12. As we see the SSTHS preconditioner is more effective that the other methods. Another observations which can be posed here that the FGMRES method in conjunction with the SSTHS preconditioner is not sensitive with respect to the parameter α and the number of iterations remains constant when the parameter α varies in the interval $[0, 0.6]$.

Table 10. Numerical results of FGMRES for Example 6.1 with $n = 300^2$.

Method	α	0.1	0.2	0.3	0.4	0.5	0.6
$\mathcal{P}_{\text{HSS}}$	IT	42	56	67	76	84	90
	CPU	3.96	4.58	5.30	5.89	6.51	7.11
$\mathcal{P}_{\text{SHSS-SS}}$	IT	26	35	42	48	53	58
	CPU	6.38	5.97	5.98	6.35	6.51	6.65
$\mathcal{P}_{\text{SSTHS}}$	IT	5	5	5	5	5	5
	CPU	1.94	1.99	2.01	2.03	2.06	2.09

Table 11. Numerical results of FGMRES for Example 6.2 for the upwind difference scheme using different α with $n = 60^3$.

Method	α	0.1	0.2	0.3	0.4	0.5	0.6
$\mathcal{P}_{\text{HSS}}$	IT	21	24	29	32	35	38
	CPU	7.72	7.46	8.43	8.82	9.36	10.04
$\mathcal{P}_{\text{SHSS-SS}}$	IT	10	14	17	19	21	24
	CPU	10.89	10.59	10.64	10.89	10.85	11.72
$\mathcal{P}_{\text{SSTHS}}$	IT	5	5	5	5	5	5
	CPU	3.81	3.84	3.85	3.88	3.89	3.90

Table 12. Numerical results of FGMRES for Example 6.2 for the centered difference scheme using different α with $n = 60^3$.

Method	α	0.1	0.2	0.3	0.4	0.5	0.6
$\mathcal{P}_{\text{HSS}}$	IT	21	25	29	32	35	38
	CPU	7.67	7.79	8.46	8.77	9.32	10.05
$\mathcal{P}_{\text{SHSS-SS}}$	IT	10	14	17	19	22	24
	CPU	10.86	10.61	10.63	10.70	11.33	11.59
$\mathcal{P}_{\text{SSTHS}}$	IT	5	5	5	5	5	5
	CPU	3.77	3.81	3.85	3.89	3.91	3.94

7. Conclusion

A new iterative method, say SSTHS, has been introduced for solving non-Hermitian positive-definite systems of linear equations. Theoretical analysis shows that, under a mild condition the SSTHS method converges to the unique solution of system (1.1) for any positive value of α. Furthermore, we derived a preconditioner induced by the proposed method and applied it to accelerate the convergence of the flexible GMRES method for solving the system. Numerical comparisons demonstrate that the proposed method outperforms the SHSS-SS and HSS methods for two known text examples. Additionally, the numerical results indicate that the SSTHS iterative method is relatively insensitive with respect to the parameter α, as the number of iterations does not change significantly by changing the value of α. In contrast, this is not the case for other methods.

Conflict of Interest

We declare that there is no conflict of interest.

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