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Jump-diffusion optimization: An iterative solution to the HJB equation for investment value

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Abstract

This paper focuses on optimizing the investment value function by incorporating jump risk using the Merton Jump-Diffusion (MJD) model. Our main goal is to determine the optimal dynamic asset allocation strategy to maximize expected utility. We derive the governing nonlinear Hamilton-Jacobi-Bellman (HJB) equation and employ a linearized generalized Newton method, which generates an iterative sequence for the optimal control. The theoretical convergence of this sequence was rigorously established using the Contraction Mapping Theorem, confirming the method's strong stability and reliability. Applying the model to real Google stock data, which exhibited significant jump risks, we derived an optimal investment ratio (π^*) that suggests a notably aggressive allocation to the risky asset. This optimal strategy provides a direct, actionable benchmark for investors. Crucially, the derived dynamic control law functions as a powerful tool for investment management firms, enabling them to proactively adjust capital allocation strategies in response to potential future jump risk scenarios.

Keywords. Investment value function, Jump risk, Merton Jump-diffusion model, Hamilton-Jacobi-Bellman equation, Contraction mapping, Frechet derivative.

2010 Mathematics Subject Classification. 91G10, 93E20, 60J75, 65H10.

1. Introduction

Investing in financial markets is inherently complex and often fraught with significant risks. A core challenge for both individual investors and professional portfolio managers lies in developing strategies that not only maximize returns but also effectively manage these inherent market risks. Traditionally, financial models have often represented asset price movements as continuous processes, typically relying on concepts like Brownian motion. However, real-world markets frequently experience sudden, unpredictable events such as major economic announcements, geopolitical shifts, or financial crises that trigger abrupt and substantial changes in asset prices. These sudden movements, commonly known as jumps, are simply not adequately captured by continuous models alone.

The Merton jump-diffusion (MJD) model offers a powerful and more realistic framework for incorporating these discrete jumps into asset price dynamics. This model enhances the continuous component (Brownian motion) with a jump component, usually described by a Poisson process. By utilizing this advanced framework, researchers and practitioners can more accurately analyze complex price behaviors, which in turn leads to the design of more robust and truly optimal investment strategies. This pioneering model was first introduced by Merton (1976) [17] to address the limitations of continuous-time models in capturing discontinuous price movements.

The field of optimal investment control is deeply rooted in the foundational theories of optimization and stochastic processes, with significant contributions spanning several decades. A pivotal figure in this domain is Robert Merton, who, in the 1960s and 1970s, established much of the groundwork for optimal asset allocation in continuous-time financial models. His groundbreaking works, such as the Merton's Consumption-Investment Problem (Merton, 1969; Merton, 1971) [18, 19], were instrumental in applying Ito's lemma and the Hamilton-Jacobi-Bellman (HJB) equation

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to derive optimal strategies for dynamic portfolio management. Merton's early research laid the foundation for understanding how investors should allocate their wealth over time to maximize their utility.

However, Merton's early models primarily depicted asset price movements as continuous, typically relying on geometric Brownian motion. Over time, financial economists recognized that real markets often exhibit abrupt, discontinuous changes in prices phenomena that purely continuous processes could not fully explain. Responding to this critical limitation, Merton himself introduced the jump-diffusion model in 1976 (Merton, 1976) [17]. This pioneering model incorporated a jump component, driven by a Poisson process, alongside the traditional diffusion component. This innovation allowed for the robust modeling of sudden market shocks, providing a more comprehensive understanding of market dynamics and significantly enhancing the realism of financial models for optimal control.

Following Merton's introduction of the jump-diffusion model, a multitude of researchers extended its application to various problems of optimal investment control. Early studies explored how the presence of jumps influences asset allocation strategies, demonstrating that accounting for jump risk often necessitates different optimal policies compared to those derived from purely continuous models. A central objective in solving these optimal control problems is to derive the optimal investment ratio (or optimal portfolio weight), which provides the investor with the greatest expected utility.Economically, this optimal ratio can often be interpreted as a type of Reward-to-Risk ratio, balancing the expected excess return from taking on the investment against the total risk (both continuous diffusion risk and discrete jump risk) inherent in the asset. Given that the HJB equations arising from jump-diffusion models are frequently nonlinear and lack analytical solutions, the development of various numerical methods became crucial. Techniques like finite difference methods, Monte Carlo simulations, and iterative schemes have since been widely employed for approximating solutions to these complex problems (Pham, 2009; Kushner and Dupuis, 2001) [13, 22]. Specifically, Kushner and Dupuis (2001) [13] provide a comprehensive overview of numerical techniques for stochastic control problems.

More recently, the application of Newton's method, a powerful numerical tool for solving nonlinear equations, has gained notable traction in quantitative finance. Researchers have adapted generalized Newton methods to tackle intricate nonlinear partial differential equations that arise in areas such as option pricing, risk management, and dynamic portfolio optimization. This article builds upon this rich historical foundation by applying a linearized generalized Newton method to solve the HJB equation derived from a Merton jump-diffusion framework. Our aim is to provide a practical and computationally viable approach for optimal investment control, specifically designed to account for the critical presence of jump risk in financial markets (Andersen and Broadie, 2004; Forsyth and Labahn, 2007) [3, 10]. Crucially, the resulting optimal investment ratio derived through this method will be analyzed and framed as a Reward-to-Risk ratio, offering a clear, quantifiable measure of the economic trade-off an investor faces when managing a portfolio exposed to jump-diffusion risks. For instance, Andersen and Broadie (2004) [3] developed a primal-dual simulation algorithm for pricing American options, which highlights the importance of numerical methods in finance, while Forsyth and Labahn (2007) [10] discuss numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance, providing a relevant framework for our approach.

1.1. **Literature Review.** Recent studies extensively explore optimal investment strategies that account for sudden, discontinuous price movements, often termed jumps, in asset dynamics. These jumps are crucial for accurately modeling real-world financial markets, which frequently experience abrupt shifts due to unexpected events.

Sun, Li, and Zhang (2018) [24] investigated robust portfolio choice for defined contribution pension plans, specifically considering both stochastic income and stochastic interest rates. Their research provides valuable insights for pension managers seeking to make resilient investment decisions under multiple sources of market and personal income uncertainty. Cao (2017) [7] addressed the optimal investment-reinsurance problem for insurers within a jump-diffusion risk process framework. A key aspect of this work was modeling the correlated Brownian motions between financial market movements and insurance claim processes, enabling a more realistic depiction of these dependencies. While older, Vissing-Jorgensen's (2002) [25] foundational work on limited asset market participation and the elasticity of intertemporal substitution remains crucial for understanding investor behavior. This study highlights how barriers to full market participation influence individuals' ability to smooth consumption over time, a vital consideration for developing realistic optimal investment models.

Solving optimal control problems in finance, especially those involving jump-diffusion processes, often leads to complex, nonlinear Hamilton-Jacobi-Bellman (HJB) equations. Since analytical solutions are rare, numerical methods play a crucial role.

Ma and Li (2018) [15] provided a detailed analysis of finite difference methods for solving HJB equations arising from stochastic optimal control problems. Their work significantly contributes to the foundational understanding and practical application of these numerical techniques, which are widely used to approximate solutions to complex partial differential equations. For a deeper theoretical foundation, Fleming and Soner's (2006) [9] book, Controlled Markov Processes and Viscosity Solutions, remains a cornerstone. This seminal work establishes the rigorous mathematical framework for viscosity solutions to HJB equations, offering essential tools for proving the existence and uniqueness of solutions in stochastic control theory, including problems with discontinuous dynamics.

More recently, cutting-edge numerical approaches have emerged. Andersson, Andersson, and Oosterlee (2023) [4] contributed with their research on the convergence of a robust deep FBSDE method for stochastic control. This highlights the growing trend of leveraging advanced computational techniques, such as those inspired by deep learning and forward-backward stochastic differential equations (FBSDEs), to tackle challenging optimal control problems, including those involving jumps, and to ensure method reliability. Additionally, Witte and Reisinger (2011) [26] introduced a penalty method for the numerical solution of HJB equations in financial contexts. Their work offers an effective approach to handle constraints often present in optimal control problems, providing a robust numerical technique for approximating solutions to these complex partial differential equations that arise in quantitative finance.

To understand the models used in the aforementioned studies, it's essential to reference foundational work on jump processes in finance. Cont and Tankov (2003) [8], in their book Financial Modelling with Jump Processes, provide a comprehensive and widely cited exposition of the mathematical theory, statistical properties, and applications of jump processes in financial markets. This book serves as a critical reference for anyone developing or working with jump-diffusion models for asset prices.

The modern financial literature increasingly employs complex Mixed Fractional Brownian Motion (MFBM) models to accurately capture market anomalies like long-range dependence. Pricing complex financial derivatives under such models presents significant computational hurdles. Specifically, the work by Shokrollahi et al. (2024) [23] addresses the valuation of Asian options in an MFBM environment complicated by the inclusion of jumps. The incorporation of both fractional dynamics and discontinuities transforms the standard pricing problem into a partial integro-differential equation (PIDE) that defies analytical solution. Similarly, Ahmadian et al. (2022) [2] tackled the highly demanding task of pricing multi-asset Asian rainbow options under MFBM, relying entirely on sophisticated adaptive Monte Carlo techniques due to the immense complexity.

Collectively, these studies highlight an urgent and broad demand within computational finance for developing highly accurate, flexible, and robust numerical methodologies capable of managing these sophisticated, non-linear stochastic models, particularly those involving jumps and time/path dependencies that necessitate solutions to complex boundary and time-dependent valuation problems.

This article focuses on optimal investment control within the Merton jump-diffusion model. Our primary objective is to identify the optimal investment value function and determine the optimal allocation between risky and risk-free assets. To achieve this, we employ the powerful framework of dynamic programming and the associated Hamilton-Jacobi-Bellman (HJB) equation. Given the inherently nonlinear nature of the HJB equation, an analytical solution is typically unobtainable. Therefore, we utilize a linearized generalized Newton method to construct an iterative sequence for approximating the optimal value function. Finally, the convergence conditions of this iterative sequence are rigorously examined using the Contraction Mapping Theorem, providing a solid theoretical foundation for our approach. Our numerical analysis, drawing on real stock data from Google, offers empirical insights into the practical applicability and inherent limitations of our methodology.

2. Mathematical model

This section details the mathematical models used to formulate the optimal investment control problem.

2.1. **Merton Jump Diffusion model.** First, the Merton jump-diffusion model is employed to describe the price dynamics of a risky asset, S_t, as follows [16]:

$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dB_t + (y_{t-1})S_t dN_t. \tag{2.1}$$

in which

- α: Expected return (drift) of the risky asset.
- λ: Jump occurrence rate (Poisson process intensity).
- k: Average size of relative jumps, where $k = E[y_t - 1]$.
- σ: Volatility of the continuous component.
- dB_t: Continuous Brownian motion component.
- dN_t: Poisson process component with rate λ.
- y_t: Jump size (as a multiple of S_t).

2.2. **Value Function of Investment.** The general form of the investment value function is defined as follows [20]:

$$v(t,x) = \max_{\pi, c \geq 0} E\left[\int_t^T F(u, c_u, x_u)du + U(x_T)|x_t = x\right], \tag{2.2}$$

in which:

- $v(t,x)$: Value function at time t and with capital x.
- π_t: Investment ratio in the risky asset at time t.
- c_t: Consumption rate at time t.
- $F(u, c_u, x_u)$: Instantaneous utility function of consumption and capital.
- $U(x_T)$: Terminal utility function of capital at time T.

2.3. **Self-Financing Portfolio.** The dynamics of a self-financing portfolio are as follows [16]:

$$dx_t = x_t\left(rdt + \pi_t\left(\frac{dS_t}{S_t} - rdt\right) - c_t dt\right). \tag{2.3}$$

Such that:

- r: Risk-free interest rate.
- x_t: Portfolio value at time t.
- π_t: Investment ratio in the risky asset.
- c_t: Consumption rate.

To simplify the problem, we assume a zero consumption rate ($c_t = 0$) and an instantaneous utility function ($F = 0$). Additionally, the final utility function is set to be logarithmic, $U(x) = \log(x)$. Consequently, the value function simplifies as follows:

$$v(t,x) = \max_{\pi, c \geq 0} E\left[\log(x_T)|x_t = x\right]. \tag{2.4}$$

2.4. **Ito's Lemma.** Applying Ito's Lemma to Equation (2.4), we obtain the following [6, 12]:

$$\begin{aligned} dv &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dx_t + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}dx_t^2 + \frac{1}{2}\frac{\partial^2 v}{\partial t^2}dt^2 + \frac{\partial^2 v}{\partial x \partial t}dtdx \\ &= \frac{1}{x}dx_t - \frac{1}{2x_t^2}(dx_t^2). \end{aligned} \tag{2.5}$$

By substituting Equation (2.1) into the self-financing portfolio dynamics Equation (2.3) (with $c_t = 0$), we obtain:

$$dx_t = x_t\left(rdt + \pi_t(\alpha - \lambda k - r)dt + \pi_t \sigma dB_t + \pi_t(y_t - 1)dN_t\right). \tag{2.6}$$

Subsequently, we calculate $(dx_t)^2$:

$$dx_t^2 = x_t^2\left(rdt + \pi_t(\alpha - \lambda k - r)dt + \pi_t \sigma dB_t + \pi_t(y_t - 1)dN_t\right)^2. \tag{2.7}$$

Next, we determine the square of the change in wealth, $(dx_t)^2$, by applying the standard rules of Ito's calculus:

$$\begin{aligned} dx_t^2 = x_t^2 \big[& (rdt + \pi_t(\alpha - \lambda k - r)dt + \pi_t \sigma dB_t)^2 + \pi_t^2 (y_t - 1)^2 dN_t^2 \\ & + 2\,(rdt + \pi_t(\alpha - \lambda k - r)dt + \pi_t \sigma dB_t) \times (\pi_t (y_t - 1) dN_t) \big]. \end{aligned} \tag{2.8}$$

We now consider the properties of the following random process:

(1) $(dN_t)^2 = dN_t$.
(2) $dNt \times dB_t = dN_t \times dt = dB_t \times dt = 0$.
(3) $(dB_t)^2 = dt$.
(4) $(dt)^2 = 0$.

Based on the above assumptions and simplifications, we conclude:

$$dx_t^2 = x_t^2 \left(\pi_t^2 \sigma^2 dt + \pi_t^2 (y_t - 1)^2 dN_t \right). \tag{2.9}$$

By substituting the above relationship into Equation (2.5), we conclude:

$$\begin{aligned} dv(t, x) = dlog(x_T) = & \left(r + \pi_t(\alpha - \lambda - k - r)\right) dt + \pi_t \sigma dB_t + \pi_t (y_t - 1) dN_t \\ & - \frac{1}{2} \left(\pi_t^2 \sigma^2 dt + \pi_t^2 (y_t - 1)^2 dN_t \right). \end{aligned} \tag{2.10}$$

Next, we have:

$$\begin{aligned} dlog(x_T) = & \left(r + \pi_t(\alpha - \lambda k - r) - \frac{1}{2} \pi_t^2 \sigma^2 \right) dt + \pi_t \sigma dB_t \\ & - \frac{1}{2} \pi_t^2 (y_t - 1)^2 dN_t + \pi_t (y_t - 1) dN_t. \end{aligned} \tag{2.11}$$

$$\begin{aligned} \int_t^T dlog(x_T) = & \int_t^T \left(r + \pi_t(\alpha - \lambda k - r) - \frac{1}{2} \pi_t^2 \sigma^2 \right) dt + \pi_t \sigma dB_t \\ & - \frac{1}{2} \pi_t^2 (y_t - 1)^2 dN_t + \pi_t (y_t - 1) dN_t. \end{aligned} \tag{2.12}$$

Consequently:

$$\begin{aligned} log(x_T) = log(x_t) + & \int_t^T \left(r + \pi_t(\alpha - \lambda k - r) - \frac{1}{2} \pi_t^2 \sigma^2 \right) dt + \pi_t \sigma dB_t \\ & - \frac{1}{2} \pi_t^2 (y_t - 1)^2 dN_t + \pi_t (y_t - 1) dN_t. \end{aligned} \tag{2.13}$$

Considering Equation (2.4), we have:

$$\begin{aligned} \underset{\max \pi_t \geq 0}{v(t, x)} = & \, log(x_t) + E \int_t^T \left(r + \pi_t(\alpha - \lambda k - r) - \frac{1}{2} \pi_t^2 \sigma^2 \right) dt \\ & + E \int_t^T \pi_t \sigma dB_t + E \int_t^T \left(-\frac{1}{2} \pi_t^2 (y_t - 1)^2 + \pi_t (y_t - 1) \right) dN_t. \end{aligned} \tag{2.14}$$

We consider the following properties:

(1) $E[dN_t] = \lambda dt$.
(2) $E[dB_t] = 0$.

we have:

$$\underset{\max \pi_t \geq 0}{v(t, x)} = log(x_t) + E \int_t^T \big((r + \pi_t(\alpha - \lambda k - r)) - \frac{1}{2} \pi_t^2 \sigma^2 - \frac{1}{2} \pi_t^2 \lambda (y_t - 1)^2 + \pi_t (y_t - 1) \lambda \big) dt. \tag{2.15}$$

Therefore:

$$\underset{\max \pi_t \geq 0}{v(t,x)} = log(x_t) + E\Bigg[\int_t^T \Big((r+\pi_u(\alpha-\lambda k - r)) - \frac{1}{2}\pi_u^2\left(\sigma^2+\lambda(y_t-1)^2\right) + \pi_u(y_t-1)\lambda\Big)du|x_t = x\Bigg]. \quad (2.16)$$

Taking the derivative with respect to π of the right-hand side of Equation (2.16) yields:

$$\alpha - \lambda k - r - \pi_t\Big(\sigma^2 + \lambda(y_t-1)^2\Big) + (y_t-1)\lambda = 0. \quad (2.17)$$

Consequently, the optimal investment ratio (π^*) is given by:

$$\pi^* = \frac{\alpha - \lambda k - r + (y_t-1)\lambda}{\sigma^2 + \lambda(y_t-1)^2}. \quad (2.18)$$

2.5. **Hamilton-Jacobi-Bellman equation.** The general dynamic programming equation for the optimal investment value function is given by [21]:

$$v(t,x) = \max_{\pi, c\geq 0} E\Big[\int_t^{t+\Delta t} F(u,c_u,x_u)du + v(t+\Delta t, x_t+\Delta t)|x_t = x\Big]. \quad (2.19)$$

Considering the interval $[t, t+\Delta t]$, Ito's Lemma applied to $v(t,x_t)$ gives:

$$dv(t,x_t) = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dx_t + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}dx_t^2. \quad (2.20)$$

Substituting Equations (2.6) and (2.9) into Equation (2.20), we obtain:

$$\begin{aligned} dv(t,x_t) = {} & \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}x\Big((r+\pi_t(\alpha-\lambda k - r) - c_t)dt + \pi_t\sigma dB_t + \pi_t(y_t-1)dN_t\Big) \\ & + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}x^2\Big(\pi_t^2\sigma^2 dt + \pi_t^2(y_t-1)^2 dN_t\Big). \end{aligned} \quad (2.21)$$

In conclusion:

$$\begin{aligned} v(t+\Delta t, x_t+\Delta t) = v(t,x_t) + & \int_t^{t+\Delta t}\Big(\frac{\partial}{\partial t} + \frac{\sigma^2\pi_u^2x_u^2}{2}\frac{\partial^2}{\partial x^2} \\ & + (r+\pi_u(\alpha-\lambda k - r) - c_u)x_u\frac{\partial}{\partial x}\Big)v(u,x_u)du \\ & + \sigma\int_t^{t+\Delta t}\pi_u x_u\frac{\partial}{\partial x}v(u,x_u)dB_u \int_t^{t+\Delta t}\Big(\pi_u(y_t-1)\frac{\partial}{\partial x}x_u + \frac{1}{2}\frac{\partial^2}{\partial x^2}x_u^2\pi_u^2(y_t-1)^2\Big)v(u,x_u)dN_u. \end{aligned} \quad (2.22)$$

We also consider the following properties:

(1) $E[dB_t] = 0$.
(2) $E[dN_t] = \lambda dt$.

The outcome for every pair (π, c) over the interval $[t, t+\Delta t]$ is:

$$\begin{aligned} E\Bigg[\int_t^{t+\Delta t} & \Big(F(u,c_u,x_u) + \frac{\partial}{\partial t} + \frac{\sigma^2\pi_u^2x_u^2}{2}\frac{\partial^2}{\partial x^2} + (r+\pi_u(\alpha-\lambda k-r)-c_u)x_u\frac{\partial}{\partial x} \\ & + \pi_u\lambda(y_t-1)\frac{\partial}{\partial x}x_u + \frac{1}{2}\frac{\partial^2}{\partial x^2}x_u^2\pi_u^2\lambda(y_t-1)^2\Big)v(u,x_u)du|x_t = x\Bigg] \leq 0. \end{aligned} \quad (2.23)$$

For simplicity, we set $F = 0$ (instantaneous utility) and $c = 0$ (consumption rate). Optimizing the dynamic programming equation with respect to π and c, the preceding equation holds. By dividing this equation by Δt and taking the limit as $\Delta t \longrightarrow 0$, we rigorously derive the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}+rx\frac{\partial}{\partial x}\Big)v(t,x)+\underset{\pi}{max}\Big(\frac{\sigma^2\pi^2x^2+x^2\pi^2\lambda^2(y_t-1)^2}{2}\Big)\frac{\partial^2}{\partial x^2}v(t,x)\\
&+\Big(\pi(\alpha-\lambda k-r)+\lambda\pi(y_t-1)\Big)x\frac{\partial}{\partial x}v(t,x)=0.\\
&V(T,x)=U(x).
\end{aligned} \tag{2.24}$$

We then obtain the optimal investment ratio (π^*) by taking the derivative of Equation (2.24) with respect to π:

$$\frac{1}{2}\Big(2\pi\sigma^2x^2+2\pi x^2\lambda^2(y_t-1)^2\Big)\frac{\partial^2}{\partial x^2}v(t,x)+\Big((\alpha-\lambda k-r)+\lambda(y_t-1)\Big)x\frac{\partial}{\partial x}v(t,x)=0. \tag{2.25}$$

Therefore:

$$\pi\Big(\sigma^2x^2+x^2\lambda^2(y_t-1)^2\Big)\frac{\partial^2}{\partial x^2}v(t,x)=-\Big(\alpha-\lambda k-r+\lambda(y_t-1)\Big)x\frac{\partial}{\partial x}v(t,x). \tag{2.26}$$

Thus, the optimal investment ratio (π^*) is given by:

$$\pi^*=-\frac{\big(\alpha-\lambda k-r+\lambda(y_t-1)\big)\frac{\partial}{\partial x}v(t,x)}{x\big(\sigma^2+\lambda^2(y_t-1)^2\big)\frac{\partial^2}{\partial x^2}v(t,x)}. \tag{2.27}$$

Ultimately, by substituting the optimal investment ratio π^* into Equation (2.24), a nonlinear equation for the optimal investment value function $v(t,x)$ is obtained as follows:

$$\Big(\frac{\partial}{\partial t}+rx\frac{\partial}{\partial x}\Big)v(t,x)-\frac{\big(\alpha-\lambda k-r+\lambda(y_t-1)\big)^2\Big(\frac{\partial}{\partial x}v(t,x)\Big)^2}{2\big(\sigma^2+\lambda^2(y_t-1)^2\big)\frac{\partial^2}{\partial x^2}v(t,x)}. \tag{2.28}$$

For notational simplicity, we define the following parameters:

$$\begin{aligned}
&\Delta=\big(\alpha-\lambda k-r+\lambda(y_t-1)\big),\\
&\Omega=2\big(\sigma^2+\lambda^2(y_t-1)^2\big).
\end{aligned} \tag{2.29}$$

As a result, the nonlinear Equation (2.28) takes the following simplified form:

$$F(v):=v_t+rxv_x-\frac{\Delta^2(v_x)^2}{\Omega v_{xx}}. \tag{2.30}$$

Consequently, the simplified form of the nonlinear Equation (2.28) (using the defined notational parameters) is as follows:

$$F(v):=\Omega v_tv_{xx}+rx\Omega v_xv_{xx}-\Delta^2(v_x)^2. \tag{2.31}$$

3. Methods

In this section, we will explain the methods used, including the Newton's iterative method, the Fréchet derivative, and the convergence analysis of the iterative method using the Contraction Mapping Theorem.

3.1. **Generalized Newton method.** We employ a generalized Newton's method to solve the nonlinear partial differential Equation (2.31). This method provides an iterative sequence to approximate its solution.

Definition 3.1. The operator F (mapping from a Banach space V to another Banach space W) at $u_0\in V$ is Fréchet differentiable if and only if there exists a bounded linear operator $A\in L(V,W)$ such that:

$$f(u_0+h)=f(u_0)+Ah+o(\|h\|),\quad \text{as } h\longrightarrow 0.$$

In this case, the operator F at u_0 is Fréchet differentiable, and we denote the Fréchet derivative of F at u_0 by $A=F'(u_0)$ [5]. The term A represents the bounded linear operator defined in the previous statement.

Definition 3.2. Suppose U and V are two Banach spaces, and $F : U \longrightarrow V$ is a differentiable function. We are interested in solving the following nonlinear equation:

$$F(u) = 0.$$

The generalized Newton's method approximates the solution by choosing an initial guess $u_0 \in U$, and then, for $n = 0, 1, 2, \dots,$ iterating as follows [5]:

$$u_{n+1} = u_n - [F'(u_n)]^{-1}F(u_n).$$

Here, $F'(u_n)$ represents the Fréchet derivative of the operator F evaluated at u_n.

3.1.1. *Formulating the Newton's method.* According to Definition 3.2, the extended Newton's method for a nonlinear equation $F(v) = 0$ is defined as follows:

$$\begin{aligned} &v_{n+1} = v_n - [F'(v_n)]^{-1}F(v_n) \\ &\text{or equivalently, by solving the linear equation:} \quad F'(v_n)(v_{n+1} - v_n) = -F(v_n). \end{aligned} \tag{3.1}$$

Here, $F'(v_n)$ represents the Fréchet derivative of the operator F evaluated at the current iteration v_n. The term $\delta_n(x,t) = v_{n+1}(x,t) - v_n(x,t)$ is defined as the correction term (or Newton step) for each iteration step n.

3.2. **Fréchet derivative.** The Fréchet derivative $F'(v_n)(\delta)$ in Equation (2.31), as defined in Definition 3.1, is calculated using the definition of the limit as follows [1]:

$$F'(v_n)(\delta) = \lim_{h\to 0} \frac{1}{h}\Big[F(v + \delta h)(t) - F(v)(t)\Big]. \tag{3.2}$$

In result:

$$\begin{aligned} F'(v_n)(\delta) = \lim_{h\to 0} \frac{1}{h}\Big[&\Omega(v_t + h\delta_t)(v_{xx} + h\delta_{xx}) + rx\Omega(v_x + h\delta_x)(v_{xx} + h\delta_{xx}) \\ &- \Delta^2(v_x^2 + 2h\delta_x v_x + h^2\delta_x^2) - \big(\Omega v_t v_{xx} + rx\Omega v_x v_{xx} - \Delta^2 v_x^2\big)\Big]. \end{aligned} \tag{3.3}$$

Next:

$$\begin{aligned} F'(v_n)(\delta) = \lim_{h\to 0} \frac{1}{h}\Big[&\Omega v_t v_{xx} + \Omega v_t h\delta_{xx} + \Omega h\delta_t v_{xx} \\ &+ \Omega h^2\delta_t\delta_{xx} + rx\Omega v_x v_{xx} + rx\Omega v_x\delta_{xx}h + rx\Omega h\delta_x v_{xx} + rx\Omega h^2\delta_x\delta_{xx} - \Delta^2 v_x^2 \\ &- 2\Delta^2 h\delta_x v_x - \Delta^2 h^2\delta_x^2 - \Omega v_t v_{xx} - rx\Omega v_x v_{xx} + \Delta^2 v_x^2\Big]. \end{aligned} \tag{3.4}$$

Applying this definition to the nonlinear operator $F(v)$ from Equation (2.31) and simplifying the terms, the Fréchet derivative $F'(v_n)(\delta)$ is obtained as follows:

$$F'(v_n)(\delta) = \Omega v_t\delta_{xx} + \Omega\delta_t v_{xx} + rx\Omega v_x\delta_{xx} + rx\Omega\delta_x v_{xx} - 2\Delta^2\delta_x v_x. \tag{3.5}$$

According to Definition 3.2, we substitute the linearized Fréchet derivative $F'(v_n)(\delta)$ into Equation (3.1). Therefore, the linear system to be solved for the correction term δ_n is:

$$\begin{aligned} \Omega v_t(n)\big(v_{xx}(n+1) - v_{xx}(n)\big) + \Omega v_{xx}(n)\big(v_t(n+1) - v_t(n)\big) + rx\Omega v_x(n)\big(v_{xx}(n+1) - v_{xx}(n)\big) \\ + rx\Omega\big(v_x(n+1) - v_x(n)\big)v_{xx}(n) - 2\Delta^2\big(v_x(n+1) - v_x(n)\big)v_x(n) \\ = -\Omega v_t(n)v_{xx}(n) - rx\Omega v_x(n)v_{xx}(n) + \Delta^2(v_x(n))^2. \end{aligned} \tag{3.6}$$

As a result:

$$\begin{aligned} \Omega v_t(n)v_{xx}(n+1) + \Omega v_{xx}(n)v_t(n+1) - \Omega v_{xx}(n)v_t(n) + rx\Omega v_x(n)v_{xx}(n+1) + rx\Omega v_x(n+1)v_{xx}(n) \\ - rx\Omega v_x(n)v_{xx}(n) - 2\Delta^2 v_x(n+1)v_x(n) + \Delta^2(v_x(n))^2 = 0. \end{aligned} \tag{3.7}$$

In this case, the partial derivatives of $v(x,t)$ exhibit a specific functional form, meaning:

$$v_x = v_{xx} = v_t = v(x,t) = e^{t+x}. \tag{3.8}$$

From Equation (3.7), we obtain the following expression:

$$\begin{aligned}&\Omega v(n)v(n+1) + \Omega v(n)v(n+1) - \Omega(v(n))^2 + rx\Omega v(n)v(n+1)\\&+ rx\Omega v(n+1)v(n) - rx\Omega(v(n))^2 - 2\Delta^2 v(n+1)v(n) + \Delta^2(v(n))^2 = 0.\end{aligned} \tag{3.9}$$

Therefore:

$$v(n)v(n+1)\Big(2\Omega + 2rx\Omega - 2\Delta^2\Big) = (-\Delta^2 + rx\Omega + \Omega)v(n)^2. \tag{3.10}$$

Consequently, we have:

$$v(n+1) = \frac{\big(-\Delta^2 + rx\Omega + \Omega\big)v(n)}{\big(2\Omega + 2rx\Omega - 2\Delta^2\big)}. \qquad \forall n = 0,1,2,\dots \tag{3.11}$$

3.3. **Convergence Conditions.** The Contraction Mapping Theorem, also known as the Banach Fixed-Point Theorem, is a fundamental tool in mathematical analysis. It guarantees the existence and uniqueness of a fixed point for a specific type of mapping. This theorem has broad applications across various fields, including solving differential equations, optimal control theory, and numerical methods. In this article, it is particularly crucial for demonstrating the convergence of our iterative sequence.

Theorem 3.3. *The Contraction Mapping Theorem states that if T is a contraction mapping on a complete metric space (X,d), then [14]:*

(1) *Existence of a Unique Fixed Point: The mapping T has exactly one unique fixed point x^*. That is, there exists a unique $x^* \in X$ such that $T(x^*) = x^*$.*
(2) *Convergence of Iterative Sequence: For any arbitrary initial point $x_0 \in X$, the iterative sequence $x_{n+1} = T(x_n)$ converges to this unique fixed point. This means:*

$$\lim_{n\to\infty} x_n = x^*.$$

3.4. **Convergence Analysis of the Iterative Sequence $G(v_n)$ Using the Contraction Mapping Theorem.** Here, the mapping (or operator) T from the theorem's general definition corresponds to our iteration operator $G(v)$. The metric space X is considered to be the set of real numbers, $\mathbb{R}$, or a suitable subset where $G(v)$ is well-defined. The metric $d(v_1,v_2)$ is the standard absolute difference: $d(v_1,v_2) = |v_1 - v_2|$. Since $\mathbb{R}$ with the absolute difference metric forms a complete metric space, the Contraction Mapping Theorem can be applied if $G(v)$ is a contraction mapping on this space. The iteration function (or operator) $G(v)$ under consideration is defined as:

$$G(v) = v(n+1) = \frac{(-\Delta^2 + rx\Omega + \Omega)v}{\Big(2\Omega + 2rx\Omega - 2\Delta^2\Big)}. \tag{3.12}$$

For simplicity, we can define a constant K:

$$K = \frac{-\Delta^2 + rx\Omega + \Omega}{2\Omega + 2rx\Omega - 2\Delta^2}. \tag{3.13}$$

Thus, utilizing the constant K defined previously, the iteration function can be simply written as $G(v) = Kv$.

1. Finding the Fixed Point A fixed point v^* is a value such that $v^* = G(v^*)$. Substituting v^* into the iteration function $G(v) = Kv$, we obtain the following condition for the fixed point:

$$\begin{aligned}v^* &= Kv^*,\\ v^* - Kv^* &= 0,\\ v^*(1-K) &= 0.\end{aligned} \tag{3.14}$$

Table 1. Daily stock data of Google Inc.

Row	Date	Final Price (USD)	Return Rate
1	2024/05/13	169.14	
2	2024/05/14	170.34	0.0071
3	2024/05/15	172.51	0.0127
4	2024/05/16	174.18	0.0097
5	2024/05/17	176.06	0.0108
...	...	...	...
...	...	...	...
227	2025/04/08	144.7	- 0.014
228	2025/04/09	158.71	0.0968
229	2025/04/10	144.7	-0.0371
...	...	...	...
...	...	...	...
248	2025/05/08	154.28	0.0192
249	2025/05/09	152.75	-0.0099

From this equation, $v^* = Kv^*$, we conclude that either $v^* = 0$ or $K = 1$. As our numerical results indicated that the contraction factor satisfies $|K| < 1$ (meaning $K \neq 1$), the unique fixed point for this iteration must be:

$$v^* = 0.$$

2. Calculating the Derivative of the Iterative Function $G'(v)$

To verify the contraction condition, we need to calculate the derivative of $G(v)$ with respect to v. Given that $G(v) = Kv$, where K is a constant, the derivative is simply:

$$G'(v) = K = \frac{-\Delta^2 + rx\Omega + \Omega}{2\Omega + 2rx\Omega - 2\Delta^2}. \tag{3.15}$$

Note that since $G'(v)$ is constant and does not depend on v, the derivative evaluated at the fixed point is $G'(v^*) = K$.

3. Applying the Contraction Condition: $|G'(v^*)| < 1$

For the iterative method $v_{n+1} = G(v_n)$ to converge to the unique fixed point, the Contraction Mapping Theorem requires that the absolute value of the derivative at the fixed point be strictly less than 1, i.e., $|G'(v^*)| < 1$. Substituting the previously derived result $G'(v^*) = K$, the condition becomes:

$$\left| \frac{-\Delta^2 + rx\Omega + \Omega}{2\Omega + 2rx\Omega - 2\Delta^2} \right| < 1. \tag{3.16}$$

Based on the numerical results obtained from the market parameters, we have found that the value of K satisfies $|K| < 1$. This empirically confirms that the contraction condition is met for this iterative function $G(v)$, thereby guaranteeing the existence and uniqueness of the solution and the convergence of the Newton iterative method (as per the Contraction Mapping Theorem).

4. Numerical results and analysis

In this section, we empirically examine the convergence conditions derived from the Contraction Mapping Theorem using real market data. Specifically, we utilize the daily prices of Google (GOOGL) shares, spanning the period from May 13, 2024, to May 9, 2025. This historical data was obtained from the Investing.com website.

4.1. **Calculations of Merton Jump-Diffusion Model Parameters.** Table 1 presents the daily prices and the corresponding expected return rates utilized in the subsequent analysis.

According to Table 1, the average rate of return (α) and the standard deviation of the rate of return (σ) for the Google stock price data are obtained as follows:

Table 2. 95% confidence intervals for MJD parameters via Monte Carlo simulation.

Parameter	Point Estimate	Lower Bound (95% CI)	Upper Bound (95% CI)
Mean Return Rate (α)	-0.0002	-0.1344	0.1864
Standard Deviation (σ)	0.0195	0.0179	0.2015
Jump Intensity (λ)	0.3700	0.0000	2.0241
Average Jump Size (k)	0.0221	0.0000	0.1881

Based on the Google (GOOGL) stock price data from Table 1, the calculated statistical parameters for the model are:

- The Mean Rate of Return (α):

$$\alpha = -0.0002.$$

- The Standard Deviation of the Return Rate (σ):

$$\sigma = 0.01947.$$

These values will be utilized in the subsequent sections to solve the Hamilton-Jacobi-Bellman (HJB) equation and determine the optimal investment ratio π^*.

Based on the daily price changes presented in Table 1, we define price jumps as days where the daily price change exceeds one percent. Furthermore, the average size of these price jumps is calculated as the average magnitude of price changes greater than one percent. The resulting jump parameters, computed via Python programming, are as follows:

Based on the analysis of the daily price changes, the calculated jump parameters for the Merton model are:

- The Average Number of Price Jumps (λ):

$$\lambda = 0.37.$$

- The Average Size of Price Jumps (k):

$$k = 0.02212.$$

These jump parameters, along with the previously calculated α and σ, complete the empirical calibration of the stock price process, allowing for the full numerical solution of the HJB equation.

According to Table 1, the maximum price jump occurred on April 9, 2025 (the 228^{th} day). Hence, the size of the price jump on this day is observed as:

$$y_{228} = 1.0968.$$

4.2. **Monte Carlo Validation of Merton Jump-Diffusion Parameters.** Following the initial point estimation, the statistical validity of the model parameters $(\alpha, \sigma, \lambda, k)$ is assessed using a Monte Carlo bootstrap simulation. We perform $N = 10,000$ iterations of the simulation, resampling the stock price paths to derive the 95% confidence interval for each Merton Jump Diffusion (MJD) parameter. The fact that the initial point estimates fall well within these confidence intervals confirms the reliability of the parameters used for the optimal control problem. The results of this reliability assessment are summarized in Table 2.

The graphical analysis presented in Figure 1 serves two critical purposes: data visualization and model validation. The blue curve tracks the daily price movements of Google stock, clearly demonstrating the high-frequency trading activity.

Most notably, the red vertical marker highlights the largest absolute daily jump recorded, corresponding to an absolute return of 9.68%. This substantial shock underscores the fat tails and leptokurtic nature of the observed return distribution. The inclusion of this 9.68% jump event is essential, as it significantly contributes to the overall risk component (λ and k) of the MJD model, which, in turn, critically influences the determination of the optimal investment strategy (π^*) derived from the Hamilton Jacobi Bellman (HJB) equation.

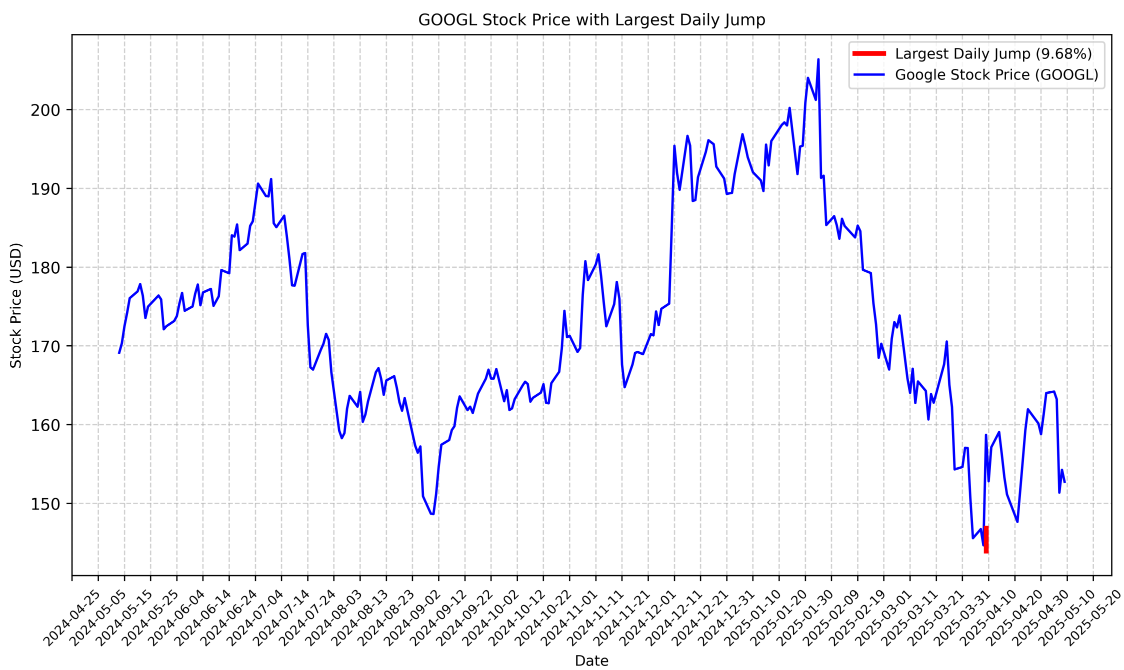

FIGURE 1. Time series plot of the Google (GOOGL) stock price.

4.3. **Valuation of the Investment Portfolio.** Next, we construct a hypothetical investment portfolio consisting of a risk-free asset (represented by a bank account earning interest) and a risky asset (Google stock). We will calculate the portfolio's value on the day the maximum price jump occurred ($t = 228$). Assuming an initial capital of \$1,000, we allocate it equally (50% each) between the risky asset and the risk-free asset, which has an annual interest rate of $r = 0.24$. Using the data from Table 1, the portfolio's value will be calculated for the day of the maximum price jump, specifically the 228^{th} day. Also, we have the following parameters:

The initial capital and its allocation for the hypothetical portfolio (x_t) are defined as follows:

- The Initial Capital (x_0):
$$x_0 = 1000 \text{ dollars.}$$
- The Amount Invested in the Risky Asset (Google Stock):
$$1000 \times 0.5 = 500 \text{ dollars.}$$
- The Amount Invested in the Risk-Free Asset (Bank Account):
$$1000 \times 0.5 = 500 \text{ dollars.}$$

This equal allocation implies an initial investment ratio of $\pi_0 = 0.5$. These initial conditions will be used to calculate the portfolio value at the time of the maximum price jump ($t = 228$). According to Table 1, the initial value of Google stock on May 13, 2024, is as follows:

$$S_0 = 169.14 \text{ dollars.}$$

Additionally, the number of shares that can be purchased based on the available assets is given by:

$$n = \frac{500}{169.14} = 2.95.$$

The initial value in the risk-free asset is as follows:

$$B_0 = 500 \text{ dollars.}$$

As a result, the value of the investment portfolio on the t^{th} day (x_t) is represented as follows [6, 11]:

$$x_t = nS_t + B_0(1+r)^t. \tag{4.1}$$

Let r denote the daily interest rate, B_0 be the initial value of the risk-free asset, S_t be the stock price at time t, and n be the number of shares purchased. The value of the portfolio x_t at time t is then given by:

$$x_t = nS_t + B_0(1+r)^t.$$

According to Table 1, the Google (GOOGL) stock price on April 9, 2025 (the 228^{th} day, corresponding to the largest price jump), is observed as follows, denoted as S_{228}:

$$S_{228} = 158.71 \text{ dollars}.$$

The daily return rate (r) is given by:

$$r = \frac{0.24}{365} = 0.0006.$$

Consequently, the value of the portfolio on April 9, 2025, according to Equation (4.1), is calculated as follows:

$$x_{228} = nS_{228} + 500(1+r)^{228} = 2.95(158.71) + 500(1+0.0006)^{228}.$$

As a result:

$$x_{228} = 1041.47 \text{ dollars}.$$

The calculated value of the investment portfolio on the 228th day is known. Our primary objective, however, is to determine the optimal value of this portfolio using the iterative sequence derived from Newton's method.To achieve this, it is necessary to numerically establish the convergence conditions using the Contraction Mapping Theorem.We have thus far obtained all the necessary parameters to numerically investigate the conditions of the Contraction Mapping Theorem, which are as follows:

(1) $\alpha = -0.0002$.
(2) $\sigma = 0.01947$.
(3) $y_t = 1.0968$.
(4) $\lambda = 0.37$.
(5) $k = 0.02221$.
(6) $x_t = 1041.47$.
(7) $r = 0.0006$.

4.4. **Examining the Conditions for Convergence with Specific Parameters.** We will now examine the conditions for convergence of our iterative method $v_{n+1} = G(v_n)$, where $G(v) = Kv$. The constant K is defined as:

$$K = \frac{-\Delta^2 + rx\Omega + \Omega}{2\Omega + 2rx\Omega - 2\Delta^2}. \tag{4.2}$$

Based on the analysis of the Contraction Mapping Theorem, the key condition for convergence of the iterative method is related to the value of the Contraction Factor K: specifically, $|K| < 1$.

1. Condition for the Existence of a Real Fixed Point

For the linear iteration function $G(v) = Kv$, the fixed point is given by $v^* = Kv^*$, which simplifies to $v^*(1-K) = 0$. Since K is a real number (derived from real-valued parameters Δ, Ω, r, and x), a real fixed point always exists, which is the trivial solution $v^* = 0$ (provided $K \neq 1$). Thus, there is no separate condition required for the existence of a real fixed point other than K being a real constant.

According to Equation (2.29), the necessary parameters for calculating the contraction factor K are Δ and Ω, which are defined as follows:

$$\begin{aligned} \Delta &= 0.0268 \\ \Omega &= 0.00332. \end{aligned} \tag{4.3}$$

2. Contraction Condition

The contraction condition for the iterative method $v_{n+1} = G(v_n)$ requires that the absolute value of the derivative of $G(v)$ at the fixed point v^* must be less than 1, i.e., $|G'(v^*)| < 1$.

For our simplified linear function $G(v) = Kv$, the derivative $G'(v)$ is simply the constant K. Therefore, the convergence condition simplifies to:

$$|K| < 1.$$

By substituting the calculated parameter values from Equation (4.3) into the expression for the contraction factor K in Equation (4.2), we obtain the numerical value for K:

$$|K| = 0.5.$$

Since the calculated value of the Contraction Factor is $K = 0.5$ and this satisfies the condition $0.5 < 1$, the contraction condition $|K| < 1$ holds true.

This confirms that the mapping $G(v)$ is indeed a contraction, since the condition $|K| < 1$ is numerically satisfied ($0.5 < 1$). Consequently, based on the Contraction Mapping Theorem (Banach Fixed-Point Theorem), the iterative sequence $v_{n+1} = G(v_n)$ is guaranteed to converge rapidly to the unique fixed point $v^* = 0$.

5. Real-World Application: Optimal Portfolio Allocation

The optimal allocation ratio, π^*, represents the percentage of total wealth that should be continuously held in the risky asset (GOOGL) to maximize the investor's expected utility. This value is fundamentally driven by the real-time market dynamics and sudden jump events, as defined by the optimal control law:

$$\pi^* = \frac{\alpha - \lambda k - r + (y_t - 1)\lambda}{\sigma^2 + \lambda(y_t - 1)^2}. \tag{5.1}$$

Based on the estimated market parameters $(\alpha, \sigma, \lambda, \mathbf{k})$ and the numerical solution to the Hamilton-Jacobi-Bellman (HJB) equation, the calculated optimal investment ratio is:

$$\pi^* = 70\%.$$

This result suggests that, to maximize expected utility, the investor should continuously hold 70% of their total wealth in the risky asset (Google stock) under the defined market conditions and jump dynamics.

Specific Case: The Largest Jump Event (Day 228). To illustrate the dynamic nature of this ratio, we specifically analyze the optimal allocation π^* at the moment of the most significant market shock recorded. The optimal ratio $\pi^* = \mathbf{70}\%$ was determined using the jump size, y_{228}, corresponding to the day with the largest recorded jump in the Google stock data (Day 228).

The results of the optimal control problem yield the following key interpretations regarding the investor's strategy and the role of jump dynamics:

Optimal Strategy: Aggressive Allocation. The resulting $\pi^* = \mathbf{70}\%$ suggests a significantly aggressive portfolio. The optimal investor should consistently hold 70% of their total wealth in the risky asset (GOOGL stock), with the remaining 30% allocated to the risk-free asset. This high allocation is justified by the estimated market parameters and the investor's utility preferences.

Dynamic Risk Compensation and Jump Factor. The numerator of the optimal control law (Equation (5.1)) incorporates the term $(\alpha - \lambda k - r)$, which represents the expected excess return adjusted for jump risk. Critically, the presence of the jump factor, $(y_t - 1)\lambda$, in both the numerator and the denominator ensures that the allocation decision is immediately and dynamically adjusted to reflect the massive change in the stock price movement (y_t) that occurred, for instance, on Day 228. This mechanism highlights the model's ability to provide dynamic risk compensation during periods of market shock.

The high optimal ratio ($\pi^* = \mathbf{70}\%$), even when incorporating the elevated volatility and explicit jump risk recorded on Day 228, signifies that the perceived potential for reward outweighs the explicitly modeled risk in the GOOGL market. This finding crucially highlights the investor's utility function (specifically, the assumed coefficient of relative risk aversion) favoring substantial exposure to high-growth assets, even in the presence of extreme price shocks.

6. Interpretation of Results

Our analysis of the iterative sequence, based on the linear iteration function $G(v) = Kv$ with a calculated contraction factor of $K = 0.5$, yields clear insights into its convergence. For this linear form, a unique real fixed point unequivocally exists at $v^* = 0$ because $K \neq 1$. Furthermore, the contraction condition $|K| < 1$ is definitively satisfied, as $|0.5| < 1$, guaranteeing the iterative sequence's rapid convergence. This confirms the model's stability and predictability under the linear $G(v)$ framework, providing strong mathematical validity for the underlying optimal control problem.

From Theoretical Robustness to Real-World Application. The mathematical proof of convergence and stability is critical, but the ultimate contribution lies in translating this theoretical framework into actionable financial strategy. Having established the computational feasibility and stability of the solution method, we now proceed to its practical application in the market. Based on the estimated parameters for GOOGL stock derived from empirical data, we derived an optimal allocation ratio of $\pi^* = 70\%$.

The derived results offer critical insights into the practical implementation of the optimal strategy:

Practical Application: Concrete Capital Allocation. This result demonstrates that our robust mathematical methodology directly translates into a concrete recommendation for capital allocation. The $\pi^* = 70\%$ mandate advises a utility-maximizing investor to allocate 70% of their total wealth to the risky asset (GOOGL stock) and 30% to the risk-free asset. This high proportion is the precise, mathematically justified ratio for maximizing expected utility under the model's defined risk structure.

Strategic Significance: Robustness Against Jump Risk. This high allocation is particularly meaningful because the Merton Jump Diffusion (MJD) model explicitly accounts for the extreme, non-normal risks observed in the market (e.g., the 9.68% daily jump). The stability of our solution, confirmed by the contraction condition $|K| < 1$, assures the investor that this aggressive 70% position is theoretically justified and dynamically optimal even in the presence of sudden jump risk. This underscores the value of using jump-diffusion models over classical continuous models.

In conclusion, our study not only validates the theoretical effectiveness of the iterative method for solving the HJB equation but also firmly grounds the derived optimal control law in the real-world context of high-jump equity markets, providing a reliable and dynamic allocation benchmark.

7. Conclusion

This paper successfully applied a linearized generalized Newton method to solve the nonlinear Hamilton Jacobi Bellman (HJB) equation within the Merton Jump-Diffusion (MJD) framework, yielding an iterative sequence for optimal investment control. Our analysis provides both strong theoretical assurance (via the Contraction Mapping Theorem and the condition $|K| < 1$) and a concrete practical application (establishing the optimal allocation benchmark $\pi^* = 70\%$ based on empirical GOOGL data). In conclusion, our study not only validates the theoretical effectiveness of the iterative method for solving the Hamilton Jacobi Bellman (HJB) equation (confirmed by $|K| < 1$) but also firmly grounds the derived optimal control law in the real-world context of high-jump equity markets (using GOOGL data), thereby providing a reliable and dynamically optimal allocation benchmark ($\pi^* = 70\%$).

1. Mathematical Stability and Convergence. The convergence analysis for the derived linear iteration function, $G(v) = Kv$, successfully confirmed the mathematical stability of the solution method. Utilizing the Contraction Mapping Theorem, the numerical results yielded a contraction coefficient of $K = 0.5$. This value definitively satisfies the condition $|K| < 1$, thereby demonstrating the method's robust stability and reliability where the linearization assumption is applicable and guaranteeing the rapid convergence to the unique fixed point $v^* = 0$.

2. Optimal Allocation in the Face of Jump Risk. The model was successfully applied to empirical data from Google stock (GOOGL), which clearly exhibited non-normal jump risks. The solution to the optimal control law, derived from the Hamilton-Jacobi-Bellman (HJB) equation, yielded an optimal investment ratio of $\pi^* = 70\%$. This high allocation confirms that the expected excess reward of GOOGL stock is sufficient to warrant an aggressive allocation, even when meticulously accounting for the severe discontinuous risks explicitly captured by the Merton Jump-Diffusion (MJD) parameters.

3. Strategic Utility for Portfolio Management. The derived optimal ratio $\pi^* = 70\%$ holds significant strategic utility. Given the model's dynamic nature, which explicitly incorporates jump intensity (λ) and jump size (k), the resulting optimal strategy serves as a powerful tool for investment management firms. By easily adjusting the jump risk parameters (λ and k) to reflect potential future scenarios or shifts in market volatility, this methodology allows portfolio managers to immediately calculate and implement an optimal, forward-looking capital allocation strategy, thereby managing jump risk proactively rather than reactively.

8. Suggestions for Future Research

The successful application and validation of the linearized iterative method open several promising avenues for future investigation. Firstly, to enhance model accuracy, future work should apply the Newton method directly to the full, non-linear HJB equation without simplifying the value function. The solution should then be approximated using robust numerical techniques, such as finite difference methods. Secondly, to gain deeper insight into investor behavior, research should explore alternative utility functions (e.g., Power Utility) to determine their impact on the optimal value function and the associated convergence conditions. Thirdly, for enhanced realism, the study should incorporate stochastic dynamics for the market parameters $(\alpha, \sigma, \lambda, k)$; this could involve modeling volatility or jump rates via Stochastic Volatility (SV) models. Finally, a comprehensive sensitivity analysis is warranted to systematically examine the impact of marginal changes in key parameters $(\alpha, r, \lambda, k, \sigma)$ on both the convergence behavior (contraction factor K) and the resulting optimal allocation ratio (π^*).

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