Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-14
DOI:10.22034/cmde.2025.65862.3053

Positive solutions for a conformable fractional boundary value problem with a numerical approach

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Abstract

This work introduces a novel approach to analyzing a class of conformable fractional boundary value problems. By employing a combination of standard and newly developed fixed-point theorems, we establish sufficient conditions for the existence and uniqueness of positive solutions. The proposed method offers several advantages over traditional techniques, including its simplicity and computational efficiency. Moreover, we construct a sequence of successive approximations that converge to the unique positive solution, providing a practical tool for numerical simulations. Our findings have significant implications for various fields, including physics, engineering, and biology, where conformable fractional differential equations are used to model complex phenomena.

Keywords. Fractional boundary value problem,Fixed point theorems, Successive sequences.

2010 Mathematics Subject Classification. 34Axx,34Bxx,34A08.

1. Introduction

The field of fractional calculus is a captivating area of mathematical analysis that extends the notion of differentiation and integration to non-integer orders, it investigates the possibility of taking derivatives and integrals to fractional orders, such as half or three-quarters. This seemingly counterintuitive idea has applications in numerous fields, including physics, engineering, and signal processing ([2, 11, 13, 17, 20–23]).

Unlike integer-order derivatives, fractional derivatives can be defined in various ways, leading to a diverse range of properties. This diversity in definitions can sometimes introduce complexities into the analysis of fractional differential equations. Additionally, different fractional derivatives may exhibit varying degrees of smoothness or regularity, which can influence the behavior of solutions. While this diversity enriches the field of fractional calculus, it also highlights the need for careful consideration when selecting the appropriate fractional derivative for a given problem [3–5, 8, 13, 16–20].

In recent years, a novel definition known as the conformable fractional derivative has been introduced to address these limitations [14]. The conformable fractional derivative provides several advantages over traditional fractional derivatives. It preserves fundamental characteristics of classical calculus, such as the product and chain rules. Furthermore, it is easier to compute and analyze, which makes it a more accessible tool for researchers and practitioners.

Numerous studies have investigated the applications of the conformable fractional derivative in various fields. It has been used, for example, to model anomalous diffusion, a phenomenon where particles spread out in a non-standard way. Conformable fractional derivatives have also been utilized in the analysis of fractional differential equations, which arise in a variety of physical and engineering issues. The conformable fractional derivative represents a promising development in fractional calculus. Its simplicity, compatibility with classical calculus, and potential applications make it a useful tool for researchers and engineers alike [6, 9, 10, 12, 15].

Received: 11 February 2025 ; Accepted: 24 October 2025.
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Although conformable fractional derivatives have gained considerable attention in recent years due to their simplicity and compatibility with classical calculus, the literature on boundary value problems involving these derivatives remains relatively limited. While several studies have explored the existence and uniqueness of solutions for certain types of conformable fractional differential equations, there is still a substantial gap in our understanding of more complex boundary value problems [7, 26]. For instance, it is imperative to conduct additional research on nonlinear conformable fractional differential equations, systems of conformable fractional differential equations, and boundary value problems involving non-standard boundary conditions. Furthermore, the development of numerical methods for solving such problems is an active area of research.

In light of the preceding discussion, this article examines the conformable fractional boundary value problem defined as follows:

$$\begin{aligned} &\mathcal{T}_\upsilon \zeta(\tau) + \phi(\tau, \zeta(\tau)) + \psi(\tau, \zeta(\tau)) = 0, \quad \tau \in [0,1], \\ &\zeta(0) = \zeta(1) = 0. \end{aligned} \tag{1.1}$$

Here, $\mathcal{T}_\upsilon$ represents the conformable fractional derivative of order $\upsilon \in (1,2]$, while ϕ and ψ are continuous functions defined on the domain $[0,1] \times [0,\infty)$, possessing certain properties that will be elaborated upon in subsequent sections. The structure of the paper is organized as follows: Section 2 will present preliminary concepts and essential tools. In Section 3, the problem will be reformulated as an integral equation, followed by the computation of Green's function and an analysis of its properties, thereby laying the groundwork for the application of fixed-point theorems. Section 5 will demonstrate the existence and uniqueness of positive solutions through the application of various fixed-point theorems, alongside the construction of successive sequences aimed at obtaining numerical solutions. Finally, a series of numerical examples will be provided to showcase the effectiveness of our findings

2. Preliminaries

To enhance reader comprehension, we introduce essential notation and lemmas employed in our subsequent proofs.

Definition 2.1. *[14]: Let $\nu \in (0,1]$. The conformable derivative of $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ of order ν is formulated as*

$$D^\nu \varphi(\tau) = \lim_{\epsilon \to 0} \frac{\varphi(\tau + \epsilon \tau^{1-\nu}) - \varphi(\tau)}{\epsilon}. \tag{2.1}$$

If $D^\nu \varphi(\tau)$ exists on $(0,b)$, then $D^\nu \varphi(0) = \lim_{\tau \to 0^+} D^\nu \varphi(\tau)$.

From Definition 2.2 to Lemma 2.6, we set $\nu \in (k, k+1]$, $k \in \mathbb{N}$.

Definition 2.2. *[14]: The conformable derivative of $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ is formulated as*

$$D^\nu \varphi(\tau) = D^\varsigma \varphi^{(k)}(t),$$

where $\varsigma = \nu - k$.

Definition 2.3. *[14]: The conformable integral of $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ of order ν is given by*

$$I^\nu \varphi(\tau) = \frac{1}{k!} \int_0^\tau (\tau - \jmath)^k \jmath^{\nu - k - 1} \varphi(\jmath) d\jmath.$$

Lemma 2.4. *[14]: For each $\tau > 0, D^\nu I^\nu \varphi(\tau) = \varphi(\tau)$ whenever φ is continuous on $\mathbb{R}^{\geq 0}$.*

Lemma 2.5. *[14] For all $\tau \in [0,1]$, $D^\nu \tau^\ell = 0$ if $\ell = 1, \ldots, k$.*

Lemma 2.6. *(See [14]) If $D^\nu \varphi(\tau)$ is continuous on $\mathbb{R}^{\geq 0}$, then*

$$I^\nu D^\nu \varphi(\tau) = \varphi(\tau) + C_1 + C_2 \tau^2 + \cdots + C_n \tau^n,$$

for some real numbers C.

Now let's recall some facts in Analysis that we will use throughout this paper.

A non-empty, closed, convex subset $\mathcal{P}$ of a real Banach space $(\mathcal{E}, \|\cdot\|)$ qualifies as a cone if it adheres to the properties of positive homogeneity and positivity:

(1) $\forall \zeta \in \mathcal{P}, \lambda > 0 : \lambda\zeta \in \mathcal{P}$.
(2) $\zeta, -\zeta \in \mathcal{P} \implies \zeta = \theta$.

The cone $\mathcal{P}$ establishes a partial order on the set $\mathcal{E}$, defined by the relation $\zeta \preceq \xi$ if and only if the difference $\xi - \zeta$ belongs to the cone $\mathcal{P}$.

A cone $\mathcal{P}$ is termed **Normal** if there exists a constant $N > 0$ such that for all elements $\zeta, \xi \in \mathcal{E}$, the condition $0 \preceq \zeta \preceq \xi$ implies that $\|\zeta\| \le N\|\xi\|$.

An operator $X : \mathcal{E} \to \mathcal{E}$ is classified as **increasing** (or **decreasing**) if the relation $\zeta \preceq \xi$ leads to $X\zeta \preceq X\xi$ (or $X\zeta \succeq X\xi$, respectively).

Two elements $\zeta, \xi \in \mathcal{E}$ are said to be **equivalent** ($\zeta \sim \xi$) if there exist positive constants λ and μ such that $\lambda\zeta \preceq \xi \preceq \mu\zeta$. This relationship establishes an equivalence relation on the set $\mathcal{E}$.

For a given element $h \succ \theta$ (where $h \in \mathcal{P}$ and $h \neq \theta$), the collection $\mathcal{P}_h = \{\zeta \in \mathcal{E} : \zeta \sim h\}$ constitutes an equivalence class within the cone $\mathcal{P}$.

Definition 2.7. *Let α be a real number such that $0 < \alpha < 1$. An operator $X : \mathcal{P} \to \mathcal{P}$ is defined as α-**concave** if it fulfills the condition $X(\tau\zeta) \geq \tau^{\alpha} X(\zeta)$ for every $\tau \in (0, 1)$ and $\zeta \in \mathcal{P}$.*

*An operator $X : \mathcal{E} \to \mathcal{E}$ is termed **homogeneous** if it meets the criterion $X(\tau\zeta) = \tau X(\zeta)$ for all $\tau > 0$ and $\zeta \in \mathcal{E}$.*

*Furthermore, an operator $X : \mathcal{P} \to \mathcal{P}$ is classified as **sub-homogeneous** if it satisfies the equation $X(\tau\zeta) = \tau X(\zeta)$ for all $\tau > 0$ and $\zeta \in \mathcal{P}$.*

Theorem 2.8. *Let $\mathcal{P}$ denote a normal cone within a real Banach space $\mathcal{E}$ and consider the operator equation*

$$\mathcal{A}\zeta + \mathcal{B}\zeta = \zeta, \tag{2.2}$$

where the operator $A : \mathcal{P} \to \mathcal{P}$, is an increasing γ-concave operator, and the operator $B : \mathcal{P} \to \mathcal{P}$, is an increasing sub-homogeneous operator. We assume the following conditions hold:

(1) There exists an element $h \succ \theta$ such that $Ah \in P_h$ and $Bh \in \mathcal{P}_h$.
(2) There exists a positive constant $\delta_0 > 0$ such that for all $x \in P$, the inequality $Ax \geq \delta_0 Bx$ is satisfied.

Under these assumptions, the operator Equation (2.2) possesses a unique solution $x^* \in \mathcal{P}_h$. Furthermore, the sequence $\{y_n\}$, defined recursively by $y_{n+1} = Ay_n + By_n$ with any initial value $y_0 \in \mathcal{P}_h$, converges to the solution x^* as n approaches infinity.

Proof. See [25]. □

Theorem 2.9. *Let P denote a normal cone within a real Banach space E. Consider the operator $A : \mathcal{P} \to \mathcal{P}$ as an increasing operator, and $B : \mathcal{P} \to \mathcal{P}$ as a decreasing operator. We assume the following conditions hold:*
1. For every $x \in \mathcal{P}$ and $t \in (0, 1)$, there exist functions $\phi_1(t)$ and $\phi_2(t)$ within the interval $(t, 1)$ such that $A(tx) \geq \phi_1(t)Ax$ and $B(tx) \leq \frac{1}{\phi_2(t)}Bx$.
2. There exists an element $h_0 \in \mathcal{P}_h$ such that $Ah_0 + Bh_0 \in \mathcal{P}_h$.

Under these circumstances, the operator equation (7) possesses a unique solution $x^* \in \mathcal{P}_h$. Furthermore, the sequences $\{x_n\}$ and $\{y_n\}$, defined by the relations $x_{n+1} = Ax_n + By_n$ and $y_{n+1} = Ay_n + Bx_n$ with arbitrary initial values $x_0, y_0 \in P_h$, will converge to x^* as n approaches infinity.

Proof. See [24]. □

Remark 2.10. *If B is a null operator, then Theorems 2.8 and 2.9 are held too.*

3. Green Function and Bounds

The essential significance of the Green function in the utilization of fixed point theorems requires its determination for the specified operator. In the following sections, we will not only focus on the computation of this Green function but also examine its fundamental properties, which will act as crucial components for the rest of this paper. As an initial step, we present the subsequent lemma:

Lemma 3.1. *Let $\varrho : [0,1] \to [0,\infty)$ be a continuous function. The fractional boundary value problem given by*

$$\begin{aligned} &\mathcal{T}_\upsilon \zeta(\tau) + \varrho(\tau) = 0, \\ &\zeta(0) = \zeta(1) = 0, \end{aligned} \tag{3.1}$$

has a unique solution represented as

$$\zeta(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\varrho(\varsigma)d\varsigma, \tag{3.2}$$

where the Green's function is defined by

$$\mathcal{G}(\tau,\varsigma) = \begin{cases} (1-\tau)\varsigma^{\upsilon-1}, & \varsigma \le \tau, \\ \tau(1-\varsigma)\varsigma^{\upsilon-2} & \varsigma \ge \tau. \end{cases} \tag{3.3}$$

Proof. The integration of the order of υ from both sides of the conformable fractional differential Equation (3.1) yields the following expression:

$$\zeta(\tau) = -\int_0^\tau (\tau-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma + c_1 + c_2\tau.$$

Applying the first boundary condition results in $c_1 = 0$. The second boundary condition leads to:

$$\zeta(1) = -\int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma + c_2 = 0,$$

which implies that

$$c_2 = \int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma.$$

Consequently, we can express $\zeta(\tau)$ as follows:

$$\begin{aligned} \zeta(\tau) &= -\int_0^\tau (\tau-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma + \int_0^1 \tau(1-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma \\ &= \int_0^\tau (1-\tau)\varsigma^{\upsilon-1}\varrho(\varsigma)d\varsigma + \int_\tau^1 \tau(1-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma \\ &= \int_0^1 \mathcal{G}(\tau,\varsigma)\varrho(\varsigma)d\varsigma. \end{aligned}$$

□

Lemma 3.2. *The function $\mathcal{G}(\tau,\varsigma)$, as defined by Equation (3.3), adheres to the following criteria:*

(i) *$\mathcal{G} \ge 0$ for all $\tau,\varsigma \in [0,1]$,*
(ii) *$\mathcal{G}$ is continuous,*
(iii) *for every $\tau,\varsigma \in [0,1]$, it holds that*

$$(1-\tau)\tau(1-\varsigma)\varsigma^{\upsilon-1} \le \mathcal{G}(\tau,\varsigma) \le \tau(1-\tau)\varsigma^{\upsilon-2}. \tag{3.4}$$

Proof. Statements (i) and (ii) are evidently valid. We will now verify the accuracy of Statement (iii). Consider the case where $\varsigma \le \tau$, in this scenario, we have

$$g_1(\tau,\varsigma) := (1-\tau)\varsigma^{\upsilon-1} \le \tau(1-\tau)\varsigma^{\upsilon-2},$$

On the other hand, if $\varsigma \ge \tau$, it follows that $(1-\varsigma) < (1-\tau)$, leading to

$$g_2(\tau,\varsigma) := \tau(1-\varsigma)\varsigma^{\upsilon-2} \le \tau(1-\tau)\varsigma^{\upsilon-2}.$$

Thus, for all values of τ and ς within the interval [0, 1], we conclude that

$$\mathcal{G}(\tau,\varsigma) \le \tau(1-\tau)\varsigma^{\upsilon-2}.$$

□

4. Existence Results

In this section, we utilize Theorems 2.8 and 2.9 to investigate the conformable fractional boundary value problem (1.1), leading to new findings regarding the existence and uniqueness of positive solutions. To achieve this, we examine the Banach space $C[0,1]$ equipped with the standard norm defined by $\|\zeta\| = \sup\{|\zeta(\tau)| : \tau \in [0,1]\}$. We define the set $\mathcal{P} = \{\zeta \in C[0,1] : \zeta(\tau) > 0, \tau \in [0,1]\}$, which denotes the positive cone within $C[0,1]$. It is noteworthy that $\mathcal{P}$ is recognized as a normal cone, characterized by a normality constant of $N = 1$. Additionally, we establish a partial order on $C[0,1]$ such that $\zeta \preceq \xi$ if and only if $\zeta(\tau) \leq \xi(\tau)$ for all τ in the interval $[0,1]$.

Before we commence with the results pertaining to our existence for convenience, it is essential to examine the following assumptions.

- (H1) The functions $\phi, \psi : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous and exhibit monotonicity in their second argument, with the stipulations that $\tau^{\upsilon-2}\phi(\tau,\zeta(\tau)) \in L^1[0,1]$ and $\psi(\tau,0) \not\equiv 0$.
- (H2) For every $\eta \in (0,1)$, $\tau \in [0,1]$, and $\zeta \geq 0$, it is established that $\psi(\tau,\eta\zeta) > \eta\psi(\tau,\zeta)$. Furthermore, there exists an $\alpha \in (0,1)$ such that for all $\tau \in [0,1]$, $\eta \in (0,1)$, and $\zeta \geq 0$, the inequality $\phi(\tau,\eta\zeta) > \eta^\alpha\phi(\tau,\zeta)$ holds.
- (H3) There exists a constant $\delta_0 > 0$ such that for all $\tau \in [0,1]$ and $\zeta > 0$, the relation $\phi(\tau,\zeta) > \delta_0\psi(\tau,\zeta)$ is satisfied.
- (H4) Let $\phi : [0,1] \times [0,+\infty) \to [0,+\infty)$ be a function that is continuous and increasing in its second argument, fulfilling the conditions $f(t,0) \neq 0$ for all $t \in [0,1]$ and $\tau^{\upsilon-2}\phi(\tau,\zeta(\tau)) \in L^1[0,1]$.
- (H5) Let $\psi : [0,1] \times [0,+\infty) \to [0,+\infty)$ be a function that is continuous and monotonically decreasing with respect to its second variable, under the stipulation that $g(t,1) \neq 0$ for every $\tau \in [0,1]$.
- (H6) For every $\delta \in (0,1)$, there exist parameters $\rho_i(\delta) \in (\delta,1)$ for $i = 1,2$ such that the following inequalities hold for all $\tau \in [0,1]$ and $\zeta \in [0,+\infty)$:

$$\phi(\tau,\delta\zeta) > \rho_1(\delta)\phi(\tau,\zeta),$$

$$\psi(\tau,\delta\zeta) \leq \frac{1}{\rho_2(\delta)}\psi(\tau,\zeta).$$

The initial findings presented here are derived from Theorem 2.8.

Theorem 4.1. *Let conditions (H1)-(H3) be satisfied. In this case, the conformable fractional boundary value problem described by Equation (1.1) possesses a unique positive solution denoted as $\zeta^* \in \mathcal{P}_\eta$, where $\mu(\tau) = \tau(1-\tau)$ for $\tau \in [0,1]$. Furthermore, for any chosen ζ_0 within the set $\mathcal{P}_\mu$, the iterative sequence defined by*

$$\zeta_{n+1}(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\zeta_n(\varsigma)) + \psi(\varsigma,\zeta_n(\varsigma))]d\varsigma, \quad n = 0,1,2,\ldots \tag{4.1}$$

Converges to $\zeta^*(\tau)$ as n approaches infinity, where $\mathcal{G}(\tau,\varsigma)$ is specified by Equation (3.3).

Proof. In view of Lemma 3.1 it is concluded that the conformable fractional boundary value problem (1.1) is equivalent with the integral equation

$$\zeta(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\zeta(\tau)) + \psi(\varsigma,\zeta(\tau))]d\varsigma. \tag{4.2}$$

where $\mathcal{G}(\tau,\varsigma)$ defined by (3.3).

Consider two mappings, $X : \mathcal{P} \to \mathcal{E}$ and $Y : \mathcal{P} \to \mathcal{E}$, defined as follows:

$$X(\zeta)(\tau) = \int_0^1 \mathcal{G}(\varsigma,\zeta(\varsigma))\phi(\varsigma,\zeta(\tau))d\varsigma, \tag{4.3}$$

$$Y(\zeta)(\tau) = \int_0^1 \mathcal{G}(\varsigma,\zeta(\varsigma))\psi(\varsigma,\zeta(\tau))d\varsigma. \tag{4.4}$$

We can assert that ζ is a solution to Equation (1.1) if and only if $\zeta = X(\zeta) + Y(\zeta)$.

Given conditions (H1), (3.3), and (3.2), it's evident that X and Y map $\mathcal{P}$ to itself. Subsequently, we verify that X and Y adhere to all the prerequisites of Theorem 2.8.

First, we demonstrate that X and Y are two mappings that increase. For any $\zeta, \xi \in \mathcal{P}$ where $\zeta \geq \xi$, we find that $\zeta(\tau) \geq \xi(\tau)$ for $\tau \in [0,1]$, and by (H1), (3.3), and (3.2),

$$X(\zeta)(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\zeta(\varsigma))d\varsigma \geq \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\xi(\varsigma))d\varsigma = X(\xi)(\tau). \tag{4.5}$$

This means that $X(\zeta) \geq X(\xi)$. Similarly, $Y(\zeta) \geq Y(\xi)$. Next, we establish that X is a α-concave mapping and Y is a sub-homogeneous mapping. For any $\eta \in (0,1)$ and $\zeta \in \mathcal{P}$, using (H2), we get

$$X(\gamma\zeta)(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\gamma\zeta(\varsigma))d\varsigma \geq \gamma^\alpha \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\zeta(\varsigma))d\varsigma = \gamma^\alpha X(\zeta)(\tau).$$

This implies that $X(\gamma\zeta) \geq \gamma^\alpha X(\zeta)$ for $\gamma \in (0,1), \zeta \in \mathcal{P}$. Therefore, the mapping X is a α-concave mapping. Additionally, for any $\gamma \in (0,1)$ and $\zeta \in \mathcal{P}$, using (H2), we obtain

$$Y(\gamma\zeta)(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\psi(\varsigma,\gamma\zeta(\varsigma))d\varsigma \leq \gamma \int_0^1 \mathcal{G}(\tau,\varsigma)\psi(\varsigma,\zeta(\varsigma))d\varsigma = \gamma Y(\zeta)(\tau).$$

This means that $Y(\gamma\zeta) \leq \gamma Y(\zeta)$ for $\gamma \in (0,1), \zeta \in \mathcal{P}$. Hence Y is a sub-homogeneous operator.
Now we must show that both $X\mu$ and $Y\mu$ are in $\mathcal{P}_\mu$. Assumption (H1) and (3.4) implies

$$X\mu(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\varsigma(1-\varsigma)) \leq (1-\tau)\tau \int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\phi(\varsigma,1)d\varsigma := (1-\tau)\tau\lambda_2,$$

$$X\mu(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\varsigma(1-\varsigma))d\varsigma \geq (1-\tau)\tau \int_0^1 (1-\varsigma)\varsigma^{\upsilon-1}\phi(\varsigma,0)d\varsigma := (1-\tau)\tau\lambda_1.$$

Using (H1) and (H3) we get

$$\phi(\varsigma,1) \geq \phi(\varsigma,0) \geq \delta_0\psi(\varsigma,0) \geq 0.$$

On the other hand, since $\upsilon - 1 > 0$ and $\psi(\tau,0) \not\equiv 0$ we have

$$\begin{aligned}\lambda_2 &= \int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\phi(\varsigma,1)d\varsigma \geq \int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\phi(\varsigma,0) = \lambda_1 \\ &\geq \delta_0 \int_0^1 (1-\varsigma)\varsigma^{\upsilon-2}\psi(\varsigma,0)d\varsigma > 0.\end{aligned}$$

Consequently we have $\lambda_1\mu(\tau) \leq X\zeta(\tau) \leq \lambda_2\mu(\tau)$. Thus $X(\zeta) \in \mathcal{P}_\mu$. Similarly, one can prove that $Y(\mu) \in \mathcal{P}_\mu$ too. Thus, the first condition of Theorem 2.8 is established. Now we check the condition 2. In view of (H3) for any $\zeta \in \mathcal{P}$ we have

$$X\zeta(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\zeta(\varsigma))d\varsigma \geq \delta_0 \int_0^1 \mathcal{G}(\tau,\varsigma)\psi(\varsigma,\zeta(\varsigma))d\varsigma = \delta_0 Y\zeta(\tau).$$

Thus for any $\zeta \in \mathcal{P}$, we have $X(\zeta) \geq Y(\zeta)$. Consequently all conditions of Theorem 2.8 are satisfied and the equation $\zeta = X(\zeta) + Y(\zeta)$ has a unique fixed point ζ^* in $\mathcal{P}_\mu$ which it is the unique solution of the conformable fractional boundary value problem (1.1). Also from Theorem 2.8 we conclude that for any $\zeta_0 \in \mathcal{P}_\mu$, the successive sequence $\zeta_n = X(\zeta_{n-1}) + Y(\zeta_{n-1})$, $n = 1,2,\ldots$ tends to ζ^* as $n \to \infty$. Thus for any initial value ζ_0, the sequence

$$\zeta_{n+1}(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\zeta_n(\varsigma)) + \psi(\varsigma,\zeta_n(\varsigma))]d\varsigma \to \zeta^*, \quad n = 1,2,\ldots, \tag{4.6}$$

as $n \to \infty$. □

Theorem 4.2. *Let $(H4)-(H6)$ are hold, then conformable fractional boundary value problem (1.1) has a unique postive solution $\zeta^* \in \mathcal{P}_\mu$ where $\mu(\tau)=1-\tau$, $\tau\in[0,1]$. Also for arbitrary ζ_0 and ξ_n in $\mathcal{P}_\mu$, the successive sequences*

$$\zeta_{n+1}(\tau)=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\zeta_n(\varsigma))+\psi(\varsigma,\xi_n(\varsigma))]d\varsigma, \quad n=0,1,2... \tag{4.7}$$

$$\xi_{n+1}(\tau)=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\xi_n(\varsigma))+\psi(\varsigma,\zeta_n(\varsigma))]d\varsigma, \quad n=0,1,2... \tag{4.8}$$

tends to $\zeta^*(\tau)$ as $n\to\infty$, where $\mathcal{G}(\tau,\varsigma)$ defined by (3.3).

Proof. Again similar to the proof of the previous theorem, consider the operators (4.3) and (4.4). Because of (H4) and (H5), easily one can conclude that the operator X is an increasing and the operator Y is a decreasing operator from $\mathcal{P}$ to $\mathcal{P}$. Now we show that the first condition of Theorem 2.9 is hold. For this by (H6) we have

$$\begin{aligned}X(\mu)(\tau)+Y(\mu)(\tau)&=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\varsigma(1-\varsigma))+\psi(\varsigma,\varsigma(1-\varsigma))]d\varsigma\\&\le \tau(1-\tau)\int_0^1(1-\varsigma)\varsigma^{\upsilon-2}[\phi(\varsigma,1)+\psi(\varsigma,0)]d\varsigma := \tau(1-\tau)\ell_1,\\X(\mu)(\tau)+Y(\mu)(\tau)&=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\varsigma(1-\varsigma))+\psi(\varsigma,\varsigma(1-\varsigma))]d\varsigma\\&\ge \tau(1-\tau)\int_0^1(1-\varsigma)\varsigma^{\upsilon-2}[\phi(\varsigma,0)+\psi(\varsigma,1)]d\varsigma := \tau(1-\tau)\ell_2.\end{aligned}$$

Considering assuption (H4) and (H5) we get

$$\psi(\varsigma,1)+\psi(\varsigma,0)\ge\phi(\varsigma,0)+\psi(\varsigma,1)\ge 0.$$

On the other hand since $\varsigma^{\upsilon-2}[\phi+\psi]\in L^1[0,1]$ and $\phi(\tau,0)+\psi(\tau,1)\not\equiv 0$, we have

$$\ell_2=\int_0^1 \varsigma^{\upsilon-2}(1-\varsigma)[\phi(\varsigma,1)+\psi(\varsigma,0)]d\varsigma \ge \int_0^1 \varsigma^{\upsilon-1}(1-\varsigma)[\phi(\varsigma,0)+\psi(\varsigma,1)]d\varsigma=\ell_1>0.$$

Thus for all $\tau\in[0,1]$ we have $\ell_1\mu(\tau)\le X(\mu)(\tau)+Y(\mu)(\tau)\le \ell_2\mu(\tau)$. Consequently

To conclude, according to Theorem 2.9, the equation $X(\zeta)+Y(\zeta)=\zeta$ possesses a unique solution $\zeta^*\in\mathcal{P}_\mu$. For any starting points $\zeta_0,\xi_0\in\mathcal{P}_\mu$, we can generate the sequences

$$\zeta_n=X\zeta_{n-1}+Y\xi_{n-1},\quad \xi_n=X\xi_{n-1}+Y\zeta_{n-1},\quad n=1,2,...$$

such that $\zeta_n\to\zeta^*$ and $\xi_n\to\zeta^*$ as $n\to\infty$. This means that the conformable fractional boundary value problem (1.1) has a unique positive solution $\zeta^*\in\mathcal{P}_\mu$, where $\mu(\tau)=\tau(1-\tau)$, with $\tau\in[0,1]$. Furthermore, for any starting points $\zeta_0,\xi_0\in\mathcal{P}_\mu$, we can construct the sequences

$$\zeta_{n+1}(\tau)=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\zeta_n(\varsigma))+\psi(\varsigma,\xi_n(\varsigma))]d\varsigma, \tag{4.9}$$

$$\xi_{n+1}(\tau)=\int_0^1 \mathcal{G}(\tau,\varsigma)[\phi(\varsigma,\xi_n(\varsigma))+\psi(\varsigma,\zeta_n(\varsigma))]d\varsigma, \tag{4.10}$$

for $n=0,1,2,...$, we observe that $\zeta_n(\tau)\to\zeta^*(\tau)$ and $\xi_n(\tau)\to\zeta^*(\tau)$ as $n\to\infty$.

□

Corollary 4.3. *Let $(H4)$ hold and For any $\delta\in(0,1)$, there exist values $\rho_i(\delta)\in(\delta,1)$ for $i=1,2$ such that the following inequalities are satisfied for all $\tau\in[0,1]$ and $\zeta\in[0,+\infty)$:*

$$\phi(\tau,\delta\zeta)>\rho_1(\delta)\phi(\tau,\zeta),$$

then the conformable fractional boundary value problem (1.1) has a unique postive solution $\zeta^*\in\mathcal{P}_\mu$ where $\mu(\tau)=1-\tau$, $\tau\in[0,1]$. Also for arbitrary ζ_0 and ξ_n in $\mathcal{P}_\mu$, the successive sequences

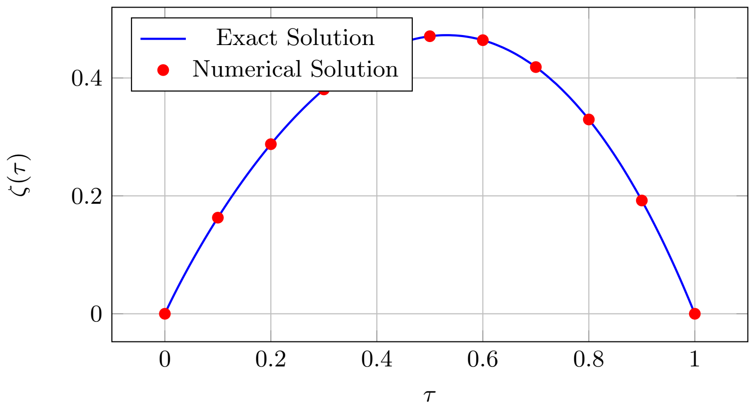


FIGURE 1. Comparison of Exact and Numerical Solutions of Example 5.1.

$$\zeta_{n+1}(\tau) = \int_0^1 \mathcal{G}(\tau,\varsigma)\phi(\varsigma,\zeta_n(\varsigma))d\varsigma, \quad n = 0, 1, 2... \tag{4.11}$$

tends to $\zeta^*(\tau)$ *as* $n \to \infty$*, where* $\mathcal{G}(\tau,\varsigma)$ *defined by* (3.3).

Proof. The proof will be obtained directly by considering Remark 2.10 and Theorem 4.2. □

5. Numerical Examples

Example 5.1. *We consider the equation*

$$\begin{aligned} & -T_{3/2}\zeta(\tau) = \phi(\tau,\zeta(\tau)) + \psi(\tau,\psi(\tau)), \\ & \zeta(0) = 0, \zeta(1) = 0, \end{aligned} \tag{5.1}$$

where $\phi(\tau,\zeta(\tau)) = 2\tau^{\frac{3}{2}}(e^{\tau}-1) + \tau^2 + \frac{1}{2} + \frac{2}{3}\tau^{\frac{1}{2}}\zeta(\tau)$ *and* $\psi(\tau,\zeta(\tau)) = \tau^{\frac{1}{2}} + \frac{1}{4} + \frac{1}{3}\tau^{\frac{1}{2}}\zeta(\tau)$*.The exact solution is* $\zeta(\tau) = 2\tau - 1 - e^{\tau}(-1+\tau) - \tau^{\frac{3}{2}}$.
Let the initial point be chosen $\zeta_0(\tau) = \tau - \tau^2$*. The higher iterates are given by the following iterative procedure:*

$$\begin{aligned} \zeta_{n+1}(\tau) = & \int_0^{\tau} (1-\tau)\,\varsigma^{\nu-1}[2\varsigma^{3/2}(e^{\varsigma}-1) + \varsigma^2 + 1/2 + (2/3)\,\varsigma^{1/2}\zeta_n(\varsigma) + \varsigma^{\frac{1}{2}} + 1/4 + (1/3)\,\varsigma^{\frac{1}{2}}\zeta_n(\varsigma)]d\varsigma \\ & + \int_{\tau}^{1} \tau\,(1-\varsigma)\,\varsigma^{\nu-2}[2\varsigma^{3/2}(e^{\varsigma}-1) + \varsigma^2 + 1/2 + (2/3)\,\varsigma^{1/2}\zeta_n(\varsigma) + \varsigma^{\frac{1}{2}} + 1/4 + (1/3)\,\varsigma^{\frac{1}{2}}\zeta_n(\varsigma)]d\varsigma. \end{aligned} \tag{5.2}$$

The iterative approach outlined in Equation (5.2) *is utilized to generate numerical solutions for Example 5.3. The maximum absolute errors, represented as* $|\zeta_n(\tau) - \zeta(\tau)|$*, for different counts of iterations are presented in Table 1. Furthermore, Table 2 lists the values of the approximate solution* ζ_{12}*, which corresponds to the twelfth iteration, at various specific points. Figure 1 shows the numerical and exact solutions for this problem.*

TABLE 1. Maximum absolute error of Example 5.1 using (5.2) for various iterations.

Number of iterations	**1**	**3**	**5**	**7**
Max.abs.error	1.6×10^{-2}	1.6×10^{-4}	1.6×10^{-6}	1.7×10^{-8}

TABLE 2. Numerical results of Example 5.1 using (5.2).

τ	Numerical solution ζ_{12}	Max.Abs.error
0.0	0	0
0.1	0.163031	2.4×10^{-13}
0.2	0.287679	6.3×10^{-13}
0.3	0.380584	1.1×10^{-12}
0.4	0.442113	1.5×10^{-12}
0.5	0.470807	1.7×10^{-12}
0.6	0.46409	1.8×10^{-12}
0.7	0.418464	1.6×10^{-12}
0.8	0.329566	1.3×10^{-12}
0.9	0.192145	$1.7. \times 10^{-13}$
1.0	0	1.8×10^{-15}

Example 5.2. *Consider the conformable fractional boundary value problem*

$$\begin{aligned}&\mathcal{T}_{\frac{3}{2}}\zeta(\tau)+\frac{\pi^2}{4}\tau^{\frac{1}{2}}\zeta(\tau)+\frac{\pi^2\tau^2}{4}+\frac{3}{4},\\&\zeta(0)=\zeta(1)=0.\end{aligned} \tag{5.3}$$

Here $\phi(\tau,\zeta(\tau))=\frac{\pi^2}{4}\tau^{\frac{1}{2}}\zeta(\tau)+\frac{\pi^2\tau^2}{4}+\frac{3}{4}$*, It is clear that* $\phi(\tau,\zeta)$ *is an increasing function and* $\phi(\tau,0)\not\equiv 0$ *for any* $\tau\in[0,1]$*. Also for any* $\delta\in(0,1)$ *we have*

$$\phi(\tau,\delta\zeta(\tau))=\frac{\pi^2}{4}\tau^{\frac{1}{2}}\delta\zeta(\tau)+\frac{\pi^2\tau^2}{4}+\frac{3}{4}\geq\delta\left(\frac{\pi^2}{4}\tau^{\frac{1}{2}}\zeta(\tau)+\frac{\pi^2\tau^2}{4}+\frac{3}{4}\right):=\rho_1(\delta)\phi(\tau,\zeta(\tau)).$$

So all conditions of Corollary 4.3 are held and consequently the conformable fractional boundary value problem (5.3) *has unique solution in* $\mathcal{P}_\mu$ *with* $\mu(\tau)=1-\tau$*, which can be obtaibed by succsessive sequence*

$$\zeta_{n+1}(\tau)=\int_0^1\mathcal{G}(\tau,\varsigma)\phi(\varsigma,\zeta_n(\varsigma))d\varsigma\to\zeta^*,\quad n=1,2,3,\ldots.$$

On the other hand one can verify that the function $\zeta(\tau)=\sin\frac{\pi}{2}\tau-\tau^{\frac{3}{2}}$ *is the exact solution of the problem. Now we try to obtain the numerical solution of the conformable fractional boundary value problem by the sequence* (4.11).

Let the starting point is chosen as $\zeta_0(\tau)=\tau-\tau^2$*. The higher iterates are given by the following iterative procedure:*

$$\begin{aligned}\zeta_{n+1}(\tau)=&\int_0^\tau(1-\tau)\varsigma^{\nu-1}\left[\frac{\pi^2}{4}\varsigma^{\frac{1}{2}}\zeta_n(\varsigma)+\frac{\pi^2}{4}\varsigma^2+\frac{3}{4}\right]d\varsigma\\&+\int_\tau^1\tau(1-\varsigma)\varsigma^{\nu-2}\left[\frac{\pi^2}{4}\varsigma^{\frac{1}{2}}\zeta_n(\varsigma)+\frac{\pi^2}{4}\varsigma^2+\frac{3}{4}\right]d\varsigma.\end{aligned} \tag{5.4}$$

The iterative method outlined in Equation (5.4) *is employed to produce numerical solutions for Example 5.2. The maximum absolute errors, denoted as* $|\zeta_n(\tau)-\zeta(\tau)|$*, corresponding to various iteration counts, are detailed in Table 3. Additionally, Table 4 provides the values of the approximate solution* ζ_{12}*, which represents the twelfth iteration, evaluated at several points. Figure 2 shows the numerical and exact solutions for this problem .*

Example 5.3. *We consider the non-linear equation*

TABLE 3. Maximum absolute error of Example 5.2 using (5.4) for various iterations.

Number of iterations	***1***	***3***	***5***	***7***
Max.abs.error	1.5×10^{-1}	9.2×10^{-3}	5.7×10^{-4}	3.6×10^{-5}

TABLE 4. Numerical results of Example 5.2 using (5.4).

τ	*Numerical solution* ζ_{12}	*Max.Abs.error*
0.0	*0*	*0*
0.1	0.124812	4.6×10^{-9}
0.2	0.219574	1.2×10^{-8}
0.3	0.289674	2.1×10^{-8}
0.4	0.334803	2.9×10^{-8}
0.5	0.353553	3.4×10^{-8}
0.6	0.344259	3.5×10^{-8}
0.7	0.305345	3.2×10^{-8}
0.8	0.235515	2.5×10^{-8}
0.9	0.133873	1.4×10^{-8}
1.0	*0*	1.2×10^{-22}

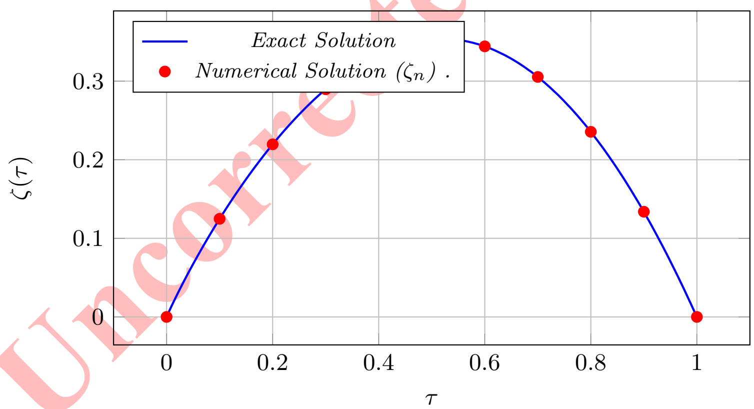


FIGURE 2. Exact vs Numerical Solutions For Example 5.2.

$$-\mathcal{T}_{4/3_3}\zeta(\tau) = \phi(\tau, \zeta(\tau)) + \psi(\tau, \zeta(\tau)), \quad \zeta(0) = 0, \quad \zeta(1) = 0, \tag{5.5}$$

where

$$\phi(\tau, \zeta(\tau)) = \pi\tau^{\frac{2}{3}} \sin\left(\frac{\pi\tau}{2}\right) + \frac{\pi^2}{4}\tau^{\frac{2}{3}}\zeta(\tau) - \frac{\pi^2}{4}\tau^{\frac{2}{3}}\left[\log(\tau + 1 - \tau^2) + \tau - \tau^{\frac{4}{3}}\right] + \frac{1}{9},$$

$$\psi(\tau, \zeta(\tau)) = \frac{(-2\tau + 2\tau^2 + 3)\tau^{\frac{2}{3}}}{e^{2\zeta(\tau) - 2\tau\cos\left(\frac{\pi\tau}{2}\right) + 2\tau^{\frac{4}{3}} - 2\tau}} + \frac{1}{3}.$$

Table 5. Maximum absolute error of Example 5.3 using (5.7)-(5.8) for various iterations.

Number of iterations	*5*	*10*	*15*	*20*
Max abs error for ζ_n	1.6×10^{-2}	1.3×10^{-3}	1.1×10^{-4}	1.4×10^{-4}
Max abs error for ξ_n	1.7×10^{-2}	1.4×10^{-3}	1.9×10^{-4}	1.4×10^{-4}

One can see easily that all conditions of Theorem 4.2 are held for the functions ϕ and ψ, so the conformable fractional boundary valuie problem (5.5) has a unique positive solution. On the other hand it is easy to cheack that the analytic solution for the conformable fractional boundary valuie problem (5.5) is

$$\zeta(\tau) = \log(\tau + 1 - \tau^2) + \tau\cos\left(\frac{\pi\tau}{2}\right) + \tau - \tau^{4/3}. \tag{5.6}$$

Now we try to reach the solution by the sequences (4.10) and (4.11).

Let the starting point be chosen as $\zeta_0(\tau) = 0$ and $\zeta_0(\tau) = \tau - \tau^2$. The higher iterates are given by the following iterative procedure:

$$\begin{aligned}\zeta_{n+1}(\tau) = &\int_0^\tau (1-\tau\varsigma)^{\nu-1}\Bigg[\pi\varsigma^{2/3}\sin\left(\frac{\pi\varsigma}{2}\right) + \frac{\pi^2}{4}\varsigma^{2/3}\zeta_n - \frac{\pi^2}{4}\varsigma^{2/3}[\ln(\varsigma+1-\varsigma^2)+\varsigma-\varsigma^{4/3}]\\ &+\frac{1}{9} + \frac{(-2\varsigma+2\varsigma^2+3)\varsigma^{2/3}}{\exp\left(2\xi_n - 2\varsigma\cos\left(\frac{\pi\varsigma}{2}\right) + 2\varsigma^{4/3} - 2\varsigma\right)}\Bigg]d\varsigma + \int_\tau^1 \tau(1-\varsigma)\varsigma^{\nu-2}\\ &\times\Bigg[\pi\varsigma^{2/3}\sin\left(\frac{\pi\varsigma}{2}\right) + \frac{\pi^2}{4}\varsigma^{2/3}\zeta_n - \frac{\pi^2}{4}\varsigma^{2/3} - \frac{\pi^2}{4}\varsigma^{2/3}[\ln(\varsigma+1-\varsigma^2)+\varsigma-\varsigma^{4/3}]\\ &+\frac{1}{9} + \frac{(-2\varsigma+2\varsigma^2+3)\varsigma^{2/3}}{\exp\left(2\xi_n - 2\varsigma\cos\left(\frac{\pi\varsigma}{2}\right) + 2\varsigma^{4/3} - 2\varsigma\right)}\Bigg]d\varsigma,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\xi_{n+1}(\tau) = &\int_0^\tau (1-\tau\varsigma)^{\nu-1}\Bigg[\pi\varsigma^{2/3}\sin\left(\frac{\pi\varsigma}{2}\right) + \frac{\pi^2}{4}\varsigma^{2/3}\xi_n - \frac{\pi^2}{4}\varsigma^{2/3}[\ln(\varsigma+1-\varsigma^2)+\varsigma-\varsigma^{4/3}]\\ &+\frac{1}{9} + \frac{(-2\varsigma+2\varsigma^2+3)\varsigma^{2/3}}{\exp\left(2\zeta_n - 2\varsigma\cos\left(\frac{\pi\varsigma}{2}\right) + 2\varsigma^{4/3} - 2\varsigma\right)}\Bigg]d\varsigma + \int_\tau^1 \tau(1-\varsigma)\varsigma^{\nu-2}\\ &\times\Bigg[\pi\varsigma^{2/3}\sin\left(\frac{\pi\varsigma}{2}\right) + \frac{\pi^2}{4}\varsigma^{2/3}\xi_n - \frac{\pi^2}{4}\varsigma^{2/3} - \frac{\pi^2}{4}\varsigma^{2/3}[\ln(\varsigma+1-\varsigma^2)+\varsigma-\varsigma^{4/3}]\\ &+\frac{1}{9} + \frac{(-2\varsigma+2\varsigma^2+3)\varsigma^{2/3}}{\exp\left(2\zeta_n - 2\varsigma\cos\left(\frac{\pi\varsigma}{2}\right) + 2\varsigma^{4/3} - 2\varsigma\right)}\Bigg]d\varsigma.\end{aligned} \tag{5.8}$$

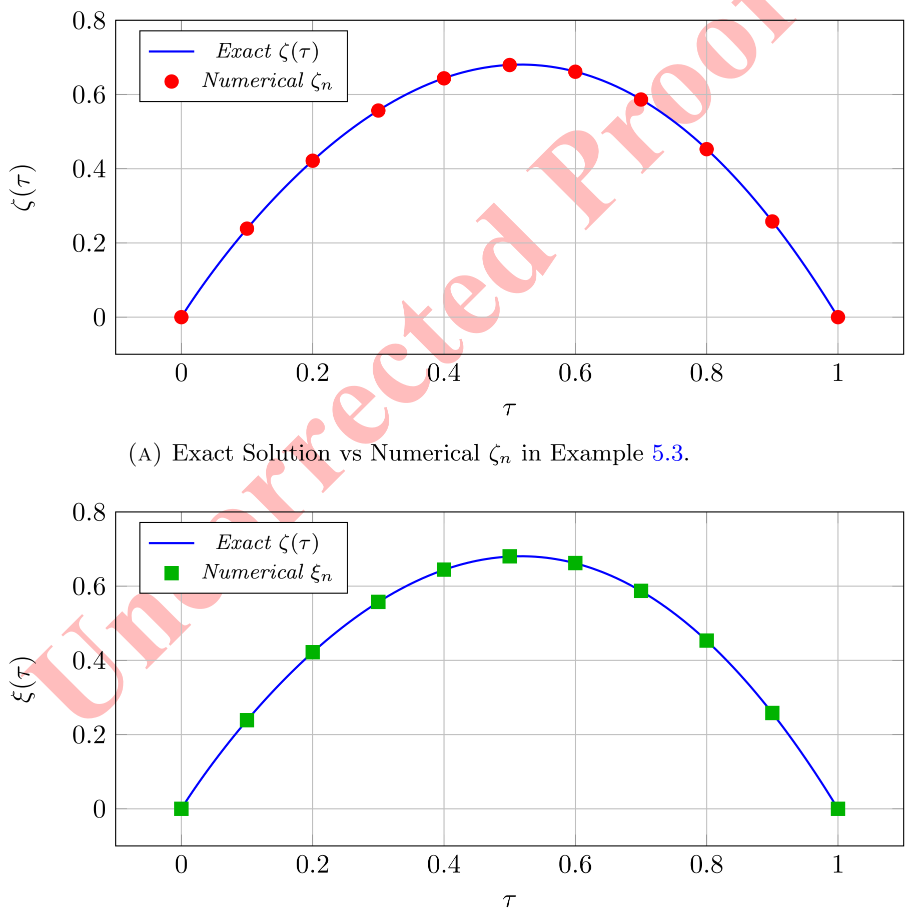
The iterative scheme (5.7)-(5.8) is used to generate numerical solutions for Example 5.3. The maximum absolute errors $|\zeta_n(\tau)-\zeta(\tau)|$ and $|\xi_n(\tau)-\xi(\tau)|$ for various numbers of iterations are reported in Table 5. In Table 6, we present the values of the approximate solution ζ_{12} and ξ_{12}, namely the twelfth iterate, at different points, also Figure 3 shows the numerical solutions ξ_n, ζ_n and exact solutions for this problem .

6. Conclusion

Boundary value problems that incorporate fractional differential equations with conformable derivatives have not been extensively explored through numerical methods, primarily due to the singular nature of Green's function. This study establishes the existence and uniqueness of positive solutions for the Dirichlet problem related to conformable fractional differential equations by applying specific conditions to the right-hand side function and utilizing various fixed point theorems. Additionally, corresponding iterative schemes have been developed to approximate the solution,

TABLE 6. Numerical results of Example 5.3 using (5.7)-(5.8).

t	*Numerical solution* ζ_n	*Max. Abs. error*	*Numerical solution* ξ_n	*Max. Abs. error*
0.0	*0*	*0*	*0*	*0*
0.1	*0.238517*	1.4×10^{-5}	*0.238824*	2.9×10^{-4}
0.2	*0.421501*	1.7×10^{-4}	*0.422081*	4.1×10^{-4}
0.3	*0.556800*	2.9×10^{-4}	*0.557591*	5.0×10^{-4}
0.4	*0.643621*	3.7×10^{-4}	*0.644544*	5.5×10^{-4}
0.5	*0.679434*	4.1×10^{-4}	*0.680402*	5.6×10^{-4}
0.6	*0.661317*	4.1×10^{-4}	*0.662240*	5.2×10^{-4}
0.7	*0.586527*	3.5×10^{-4}	*0.587317*	4.4×10^{-4}
0.8	*0.452716*	2.6×10^{-4}	*0.453296*	3.2×10^{-4}
0.9	*0.257887*	1.4×10^{-4}	*0.258195*	1.7×10^{-4}
1.0	*0*	*0*	*0*	4.7×10^{-10}



(A) Exact Solution vs Numerical ζ_n in Example 5.3.

(B) Exact Solution vs Numerical ξ_n in Example 5.3.

FIGURE 3. Exact Solution and Numerical Solutions ζ_n and ξ_n

which converges to the unique solution. Several numerical examples are presented to illustrate the effectiveness of the proposed method

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