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Modeling and Analysis of Dynamic Waveforms in Nonlinear Fractional Models of Fifth Order

Muhammad Ashraf[1], Noor Muhammad[2], Md. Abdullah Bin Masud[3], Md. Nur Alam[4,5,*], and Md. Jakir Hossen[6,*]

[1]Department of Mathematics, CUI Vehari Campus, Vehari, Pakistan.

[2]School of Mathematics, Sichuan University, Chengdu 610065, China.

[3]Department of Mathematics, Jagannath University, Dhaka-1100, Bangladesh.

[4]Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India.

[5]Department of Mathematics, Pabna University of Science and Technology, Pabna-6600, Bangladesh.

[6]Faculty of Engineering and Technology (FET), Multimedia University, Melaka-75450, Malaysia.

Abstract

The overarching purpose of this work is to derive new exact traveling wave solutions for a fifth-order generalized nonlinear fractional differential equation (5th-order GNFDE) by applying the Improved Auxiliary equation method. This equation is characterized by M-fractional derivatives (M-FD), which offer a larger basis for modeling complex dynamical systems with memory effects. The proposed methodology enables various types of solutions designed in the shape of traveling wave solutions, solitary wave solutions, and other prominent solution types, indicating the robustness and versatility of the approach in dealing with nonlinear fractional differential equations. Some investigated solutions are demonstrated in 2D and 3D graphics by smearing definite values to the parameters under constrained conditions to boost the key propagating features. The results contribute significantly to the development of analytical techniques for solving high-order nonlinear fractional differential equations (NLFDE). In addition, the method is efficient and applicable to various non-linear systems, further enhancing its practical efficacy.

Keywords. Improved Auxiliary equation method (IAE), M-fractional derivative (M-FD), Nonlinear fractional differential equations, Exact traveling wave solutions; Solitary wave solutions, Fifth-order generalized equation.

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1. Introduction

The study of wave phenomena has significant relevance across various applied scientific disciplines. Fractional derivatives in nonlinear differential equations extend the idea of classical differential equations with integer-order derivatives. Nonlinear fractional partial differential equations (NLPDEs) have emerged as crucial tools in modeling complex phenomena across a broad spectrum of scientific and engineering disciplines. Their applications span fields such as plasma physics, solid state physics, fluid dynamics, chemical kinetics and mathematical biology, theoretical and applied physics, biomechanics, chemical dynamics, biological systems, power-law non-local effects, relativistic models, nonlinear optical systems, engineering design, signal processing, electrical systems, and solid mechanics, among others. Due to their ability to describe complex systems with both deterministic and random behaviors, PDEs have become a foundation in applied mathematics and physical sciences [9, 10, 26, 27, 46]. The single key aspect of the non-linear physical phenomenon is the determination of the precise solutions of non-linear fractional PDEs. Exact solutions deliver treasured physical insights and help to understand the principles of various physical models, such as those in solid-state physics, chemical physics, biology, optical fibers, and plasma physics. Over the past 200 years, many

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* Corresponding author. Email: jakir.hossen@mmu.edu.my, nuralam23@pust.ac.bd.

academics have been paying attention to fractional calculus. Frequent non-linear elements, such as fluid mechanics, chemical reactions, biological processes, etc., are used as models. Commanding by fractional-order PDEs is the classic integer-order PDEs generalization. A variety of advanced mathematical methodologies have been formulated to derive precise analytical solutions for non-linear partial differential equations. Among these techniques are solitary wave solutions, traveling wave solutions, and periodic wave solutions, all providing valuable insights into the underlying characteristics of the solutions to these equations. A variety of algebraic methods that have been applied to the nonlinear PDEs to generate exact solutions such as [11, 14, 20, 41, 44]. Recently, Wazwaz [49] introduced a fifth-order nonlinear evolution equation, expressed as

$$U_{ttt} - U_{xxxxx} - 4\left(U_x U_t\right)_{xx} - 4\left(U_x U_{xt}\right)_x = 0, \tag{1.1}$$

and its generalized form is given by:

$$U_{ttt} - U_{xxxxx} - \mu\left(U_x U_t\right)_{xx} - \nu\left(U_x U_{xt}\right)_x = 0, \tag{1.2}$$

where μ and ν are arbitrary parameters.

The approaches to solving non-linear PDEs significantly improving the tools available for the mathematical modeling of physical phenomena. His work, along with contributions from others in the field, has endorsed the advancement of more effective and systematic approaches for obtaining exact analytical solutions to complex equations. Riemann-Liouville fractional derivative [33], Caputo fractional derivative [34, 48], Caputo-Fabrizio [17], conformable fractional derivative [31, 40, 42], and others are among the definitions of fractional derivatives that originate in the literature. In 2017, Sousa and Oliveira [47] introduced an M-fractional derivative (M-FD) that integrates the Mittag-Leffler function with a single parameter [**?**], ensuring its alignment with the characteristics of classical integer-order calculus. Building on this idea, we present a truncated M-FD that consolidates previously defined fractional derivative types, while upholding the fundamental properties of integer-order calculus.

Non-integer order models can be solved by using the truncated M-FD. Incorporating fractional derivatives in NLPDEs provides a more accurate representation of memory and hereditary effects, commonly observed in physical and biological systems. In recent years, a variety of robust methods have been developed and widely applied to determine exact solutions of nonlinear fractional differential equations (FDEs). These include homotopy analysis Sumudu transform method [29], the generalized (G'/G)-expansion technique and the generalized tanh-coth method [37], the improved $tan\frac{\phi(\eta)}{2}$-expansion method and the semi inverse variational principle method [28, 32], Hirota's bilinear method [18, 24, 25], the homotopy analysis method [19], the modified expansion function method [4], the sin-cos method [50], the tan(h) method [36], and method of the exponential rational function [21]. Additional significant methods including the sine-Gordon [23] expansion technique, the extended $\left(\frac{G'}{G}\right)$-expansion method [57], the tanh-coth expansion method [55], and the first integral approach [56]. Further advancements have led to techniques such as the $\exp(-\phi(\eta))$ method [15], new auxiliary equation methods [54], the Riccati equation method [51], and various extended direct algebraic and exp-function methods [43, 58]. Moreover, the Jacobi elliptic function expansion [35], $\exp(-F(\eta))$-expansion method [5], generalized algebraic methods [12], and several other techniques [6–8, 52] have proven to be valuable tools for the analysis and solution of non-linear FDEs. Specifically, various methods have been employed to derive exact solutions for fifth-order generalized nonlinear fractional PDEs. These methodologies include the generalized Kudryashov method and the sub-equation technique [45], the (G'/G^2)-expansion approach for obtaining exact traveling wave solutions to nonlinear conformable evolution equations [30], as well as the application of the (G'/G^2)-expansion method for deriving analytical solutions to the time-fractional prolonged Zakharov–Kuznetsov equation [16]. Furthermore, analogous techniques have been utilized to derive novel traveling wave solutions for a range of nonlinear fractional partial differential equations [13]. Recent improvements to tanh-function algorithms, F'/F-expansion strategies [53], and other innovative methods [22, 38, 39] have further enhanced the precision and applicability of these techniques in the literature. These advancements highlight the ongoing advancement of techniques used in the study of nonlinear fractional differential equations such as [1–3]. Our goal in this study is to examine the exact solutions of novel fifth-order generalized nonlinear fractional differential equations (NLPDEs) using the M-fractional derivative and the Improved Auxiliary Equation (IAE) method. Founded on the work of Wazwaz , we are motivated to introduce the first extended nonlinear equation in the context of the first truncated M-fractional conformable as:

$$D_t^{3\alpha} H + \beta_1 D_t^{\alpha} H_{xxxx} + \beta_2 \left(H_x D_t^{\alpha} H\right)_{xx} + \beta_3 \left(H_x D_t^{\alpha} H_x\right)_x = 0, \tag{1.3}$$

where, $\beta_1, \beta_2, \beta_3$ are nonzero real parameters and α is the fractional order such that $0 < \alpha \leq 1$.

Our work contributes significantly to the frame of knowledge by developing the IAE method to solve 5th-order generalized nonlinear fractional DEs. The use of the M-FD develops the accuracy and efficiency of the solutions, consenting for a deeper understanding of the complex behavior characteristics in such systems. Through this article, we aim to provide new insights and analytical solutions that can assist as a foundation for forthcoming studies and applications of fractional calculus in nonlinear PDEs.

The organization of this paper unfolds by way of: section 2 outlines the essential preliminaries of fractional calculus, particularly the M-FD, which form the foundation of the study. Section 3 describes the key principles and steps of the Improved Auxiliary Equation (IAE) Method. In section 4, we derive the exact and solitary wave solutions for the proposed equations using the IAE method. Section 5 presents the results with comprehensive graphical discussions, displaying 2-dim. and 3-dim. representations of the derived results, including their physical interpretations. Lastly, the section **??** concludes the paper with a summary of the findings and insights into the potential applications of the results.

2. Preliminaries

M-Fractional Derivative (M-FD) and Its Fundamental Properties The M-fractional derivative, also known as the truncated M-fractional derivative, is a type of fractional derivative that was introduced to address some limitations of existing fractional derivative definitions. It aims to preserve certain properties of classical integer-order derivatives, like linearity, the product rule, and the chain rule, while generalizing other fractional derivatives.

Let $u(t) : (0, \infty) \to \mathbb{R}$, and the truncated M-FD of u of order α is expressed as [47]:

$$D_{M,t}^{\alpha,\beta} u(t) = \lim_{\tau \to 0} \frac{u\left(t\, E_\beta\left(\tau^{1-\alpha}\right)\right) - u(t)}{\tau}, \tag{2.1}$$

where $0 < \alpha < 1$ and $\beta > 0$.

Here, $E_\beta(\cdot)$ represents the truncated Mittag-Leffler function with a single parameter, which is defined as:

$$E_\beta(m) = \sum_{j=0}^{\infty} \frac{m^j}{\Gamma(\beta j + 1)}, \quad \beta > 0,\, m \in \mathbb{C}. \tag{2.2}$$

Characteristics of the Truncated M-Fractional Derivative (TM-FD) Consider parameters $0 < \alpha \leq 1, \beta > 0, r, s \in \mathbb{R}$. Suppose the functions u and v are α-differentiable at a point $t > 0$. Based on the principles outlined in [18], the following properties can be established:

(i) The truncated M-fractional derivative of a linear combination of two functions u(t) and v(t) is given by:

$$D_{M,t}^{\alpha,\gamma}\left(su(t) + rv(t)\right) = sD_{M,t}^{\alpha,\gamma} u(t) + rD_{M,t}^{\alpha,\gamma} v(t), \tag{2.3}$$

(ii) The truncated M-fractional derivative of the product of two functions u(t) and v(t) can be expressed as:

$$D_{M,t}^{\gamma}\left(u(t) \cdot v(t)\right) = u(t) \cdot D_{M,t}^{\gamma} v(t) + v(t) \cdot D_{M,t}^{\gamma} u(t), \tag{2.4}$$

(iii) The truncated M-fractional derivative of the ratio of two functions u(t) and v(t) is defined as:

$$D_{M,t}^{\gamma}\left(\frac{u(t)}{v(t)}\right) = \frac{v(t) \cdot D_{M,t}^{\gamma} u(t) - u(t) \cdot D_{M,t}^{\gamma} v(t)}{(v(t))^2}, \tag{2.5}$$

(iv) The truncated M-fractional derivative of a constant function u(t) = c is:

$$D_{M,t}^{\gamma}(c) = 0, \tag{2.6}$$

(v) The truncated M-fractional derivative of a function u(t) is represented as:

$$D_{M,t}^{\gamma} u(t) = \frac{t^{1-\alpha}}{\Gamma(\gamma + 1)} \cdot \frac{du(t)}{dt}, \tag{2.7}$$

Significance of Truncted M-Fractional Derivative.
(i) It provides a simple, non-singular kernal.
(ii) Avoids some inconsistencies of other fractional derivatives.
(iii) It is very useful in modeling physical systems with memory or non-local dynamics.

3. Description of IAE Method

In this part of the paper, we delve into the fundamental steps and core principles underlying the IAE method. We consider traveling wave equation, expressed as a partial differential equation (PDE):

$$H[H, H_x, H_y, H_t, H_{xx}, H_{xy}, H_{yt}, ...] = 0. \tag{3.1}$$

Let $H = H(x, y, t, \dots)$, where the function H is expressed as a transformation dependent on ϕ. The parameter ϕ, in turn, is defined as a function of the independent variables $x, y, t, \dots$, such that:

$$H = H(\phi), \quad \text{with } \phi = \phi(x, y, t, \dots). \tag{3.2}$$

By substituting Equation (2) into Equation (1), the transformation yields a nonlinear ordinary differential equation (NODE). This resulting equation can be represented in a generalized form, where O is a functional that depends on H and its derivatives, such that:

$$O[H, H', H'', H''', \dots] = 0, \tag{3.3}$$

where H represents the dependent variable and the terms H', H'', H''', etc., represent its derivatives.

To explore soliton solutions, we integrate Eq. (3.3) repeatedly, setting the integration constants to zero. Herein, we provide a comprehensive list of the core functions of the tools suggested: Assume the solution to Eq. (3.1) takes the form:

$$H(\phi) = \frac{\sum_{i=0}^{n} c_i a^{if(\phi)}}{\sum_{i=0}^{n} d_i a^{if(\phi)}}, \tag{3.4}$$

where, c_i and d_i, $(i = 0, 1, 2, 3, \dots, n)$ are the free parameters, and both of these can not be zero at the same time. Here, a^i are the real constants. where n is the positive integer taken by balancing the highest-order linear term with the highest-order nonlinear term in the equation. Eq. (3.3) and for this purpose we follow the expression $D\left[\frac{d^g H(\psi)}{d\psi^g}\right] = n + g, D\left[H^g\left(\frac{d^h H(\psi)}{d\psi^h}\right)^k\right] = \text{ng} + k(n + h)$. So, expanding the Eq. (3.4) for the value of n and the Eq. (3.4) satisfy the following equation

$$f'(\phi) = \frac{pa^{-f(\phi)} + ra^{f(\phi)} + q}{\log(a)}. \tag{3.5}$$

4. Derivation of Solutions via Improved Auxiliary Equation Method

Rewriting the first truncated M-fractional conformable first extended nonlinear equation as:

$$D_t^{3\alpha} H + \beta_1 D_t^{\alpha} H_{xxxx} + \beta_2 \left(H_x D_t^{\alpha} H\right)_{xx} + \beta_3 \left(H_x D_t^{\alpha} H_x\right)_x = 0. \tag{4.1}$$

Consider the wave transformation, which can be described as follows:

$$H(x, t) = H(\phi), \quad \phi = \delta x + \frac{\Gamma(\gamma + 1)\rho t^{\alpha}}{\alpha}. \tag{4.2}$$

The nonlinear partial differential equation (PDE) presented in Equation (4.1) can be reduced to a nonlinear ordinary differential equation (ODE) through the application of the wave transformation outlined in Equation (4.2). By substituting this transformation into Equation (4.1), we obtain the following ODE:

$$\rho^3 H^{(3)}(\phi) + (2\beta_2 + \beta_3)\,\delta^3 \rho \left[H'(\phi) U^{(3)}(\phi) + H''(\phi)^2\right] + \beta_1 \delta^4 \rho H^{(5)}(\phi) = 0. \tag{4.3}$$

By analyzing the Equation (3.4), we define that the parameter n is equal to 1, which is taken by equating the highest-order linear term with the highest-order nonlinear term. Equation (3.4) simplifies to the following form:

$$H(\phi) = \frac{c_0 + c_1 a^{\ f(\phi)}}{d_0 + d_1 a^{\ f(\phi)}}, \tag{4.4}$$

where, ϕ represents the wave transformation variable, defined as a function of space and time such that

$$\phi = \phi(x, t).$$

By substituting Equation (4.4) into Equation (4.3) using Equation (3.5) and its required derivatives, we apply Mathematica software to determine the necessary parameter values. By combining all the coefficients of $a^{f(\phi)}$, we derive a system of algebraic equations. Solving this system yields various solution sets for Equation (4.3), as discussed in [47]. The following solution sets are derived from this process:

$$\begin{aligned} &\left\{ c_1 = -\frac{12\beta_1 \delta d_0 r}{2\beta_2 + \beta_3}, d_1 = 0, \rho = \pm i\sqrt{\beta_1}\delta^2\sqrt{q^2 - 4pr} \right\}, \\ &\left\{ c_0 = \frac{12\beta_1 \delta d_1 p}{2\beta_2 + \beta_3}, d_0 = 0, \rho = \pm i\sqrt{\beta_1}\delta^2\sqrt{q^2 - 4pr} \right\}, \end{aligned} \tag{4.5}$$

we apply the initial solution set of the Equation (4.5) to the Equation (4.4), which results in the following types of exact solutions:

Case 1: if $q^2 - 4pr < 0$ with $r \neq 0$ following as;

$$H_1(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta\left(q - \sqrt{4pr - q^2}\tan\left(\frac{1}{2}\sqrt{4pr - q^2}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)\right)}{2\beta_2 + \beta_3}, \tag{4.6}$$

or

$$H_2(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta\left(\sqrt{4pr - q^2}\cot\left(\frac{1}{2}\sqrt{4pr - q^2}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right) + q\right)}{2\beta_2 + \beta_3}. \tag{4.7}$$

Case 2: when the condition $q^2 - 4pr > 0$ holds, and $r \neq 0$, as a result;

$$H_3(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta\left(\sqrt{q^2 - 4pr}\tanh\left(\frac{1}{2}\sqrt{q^2 - 4pr}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right) + q\right)}{2\beta_2 + \beta_3}, \tag{4.8}$$

or

$$H_4(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta\left(\sqrt{q^2 - 4pr}\coth\left(\frac{1}{2}\sqrt{q^2 - 4pr}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right) + q\right)}{2\beta_2 + \beta_3}. \tag{4.9}$$

Case 3: in the scenario where $q^2 + 4pr < 0$, $r \neq 0$, and $r = -p$, consequently, the transformation holds;

$$H_5(x,t) = \frac{c_0}{d_0} - \frac{6\beta_1\delta r\left(q - \sqrt{-4p^2 - q^2}\tan\left(\frac{1}{2}\sqrt{-4p^2 - q^2}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)\right)}{(2\beta_2 + \beta_3)\,p}, \tag{4.10}$$

or

$$H_6(x,t) = \frac{c_0}{d_0} - \frac{6\beta_1\delta r\left(\sqrt{-4p^2 - q^2}\cot\left(\frac{1}{2}\sqrt{-4p^2 - q^2}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right) + q\right)}{(2\beta_2 + \beta_3)\,p}. \tag{4.11}$$

Case 4: in the case where $q^2 + 4p^2 > 0$, $r \neq 0$, and $r = -p$, it follows that;

$$H_7(x,t) = \frac{c_0}{d_0} - \frac{6\beta_1\delta r\left(\sqrt{4p^2 + q^2}\tanh\left(\frac{1}{2}\sqrt{4p^2 + q^2}\left(\delta x + \frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right) + q\right)}{(2\beta_2 + \beta_3)\,p}, \tag{4.12}$$

or

$$H_8(x,t) = \frac{c_0}{d_0} - \frac{6\beta_1\delta r\left(\sqrt{4p^2+q^2}\coth\left(\frac{1}{2}\sqrt{4p^2+q^2}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)+q\right)}{(2\beta_2+\beta_3)\,p}. \tag{4.13}$$

Case 5: when $q^2-4p^2<0$ and $r=p$, hence;

$$H_9(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta r\left(q-\sqrt{4p^2-q^2}\tan\left(\frac{1}{2}\sqrt{4p^2-q^2}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)\right)}{(2\beta_2+\beta_3)\,p}, \tag{4.14}$$

or

$$H_{10}(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta r\left(\sqrt{4p^2-q^2}\coth\left(\frac{1}{2}\sqrt{4p^2-q^2}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)+q\right)}{(2\beta_2+\beta_3)\,p}. \tag{4.15}$$

Case 6: when $q^2-4p^2>0$ and $r=p$, resulting as;

$$H_{11}(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta r\left(\sqrt{q^2-4p^2}\tanh\left(\frac{1}{2}\sqrt{q^2-4p^2}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)+q\right)}{(2\beta_2+\beta_3)\,p}, \tag{4.16}$$

or

$$H_{12}(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta r\left(\sqrt{q^2-4p^2}\coth\left(\frac{1}{2}\sqrt{q^2-4p^2}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)\right)+q\right)}{(2\beta_2+\beta_3)\,p}. \tag{4.17}$$

Case 7: in the case When $q^2=4pr$, as a result;

$$H_{13}(x,t) = \frac{c_0}{d_0} + \frac{6\beta_1\delta(q\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)+2)}{(2\beta_2+\beta_3)\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)}, \tag{4.18}$$

Case 8: under the condition that $rp<0$, $q=0$, also $r\neq 0$, consequently;

$$H_{14}(x,t) = \frac{c_0}{d_0} + \frac{12\beta_1\delta r\sqrt{-\frac{p}{r}}\tanh(\sqrt{-pr}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right))}{2\beta_2+\beta_3}, \tag{4.19}$$

or

$$H_{15}(x,t) = \frac{c_0}{d_0} + \frac{12\beta_1\delta r\sqrt{-\frac{p}{r}}\coth(\sqrt{-pr}\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right))}{2\beta_2+\beta_3}. \tag{4.20}$$

Case 9: in this case $q=0$ and holds $p=-r$, therefore;

$$H_{16}(x,t) = \frac{c_0}{d_0} + \frac{12\beta_1\delta r\left(e^{2r\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)}+1\right)}{(2\beta_2+\beta_3)\left(e^{2r\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)}-1\right)}, \tag{4.21}$$

Case 10: if $p=0$ and $r=0$,
Case 11: when $p=q=k$, and additionally, $r=0$,
Case 12: if $r=0$ and $q=0$,
Case 13: when $r=0$, in these cases, we have constant solution thats why we are omitting due to lack of physical significance of constant solution.
Case 14: If $q=K, r=K$, and $p=0$, it follows that;

$$H_{17}(x,t) = \frac{c_0}{d_0} - \frac{12\beta_1\delta r e^{K\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right)}}{(2\beta_2+\beta_3)\,(1-K\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^\alpha}{\alpha}\right))}. \tag{4.22}$$

Case 15: assuming $q = p + r$, which provide result as;

$$H_{18}(x,t) = \frac{c_0}{d_0} - \frac{12\beta_1\delta r\left(pe^{\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)(p-r)}-1\right)}{(2\beta_2+\beta_3)\left(re^{\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)(p-r)}-1\right)}. \tag{4.23}$$

Case 16: assuming that $q = -(p + r)$, we get results as;

$$H_{19}(x,t) = \frac{c_0}{d_0} - \frac{12\beta_1\delta r\left(e^{\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)(p-r)}-p\right)}{(2\beta_2+\beta_3)\left(e^{\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)(p-r)}-r\right)}. \tag{4.24}$$

Case 17: if $p = 0$ which follows;

$$H_{20}(x,t) = \frac{c_0}{d_0} + \frac{12\beta_1\delta qre^{q\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)}}{(2\beta_2+\beta_3)\left(re^{q\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)}-1\right)}. \tag{4.25}$$

Case 18: assuming that r, p, q are all nonzero, result as;

$$H_{21}(x,t) = \left\{\frac{c_0}{d_0} - \frac{6\beta_1\delta r\left(\sqrt{3}\tan\left(\frac{1}{2}\sqrt{3}p\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)\right)-1\right)}{2\beta_2+\beta_3}\right\}. \tag{4.26}$$

Case 19: if $p = 0$ and $q = 0$

$$H_{22}(x,t) = \frac{12\beta_1\delta}{(2\beta_2+\beta_3)\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right)} + \frac{c_0}{d_0}. \tag{4.27}$$

Case 20: assuming $r = p$, and $q = 0$, then;

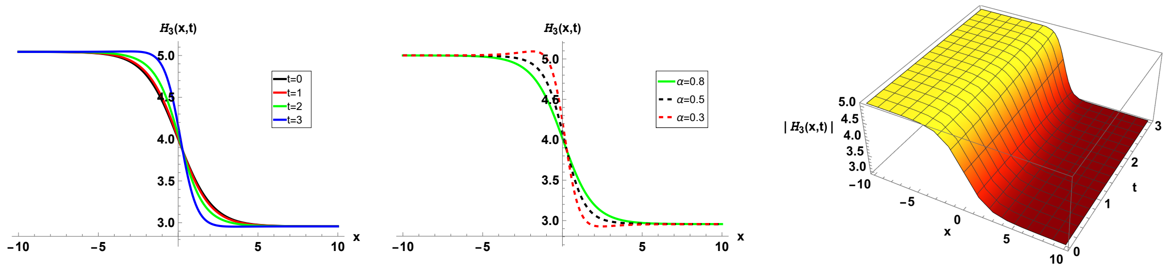
$$H_{23}(x,t) = \frac{c_0}{d_0} - \frac{12\beta_1\delta r\tan(p\left(\delta x+\frac{\Gamma(\gamma+1)\rho t^{\alpha}}{\alpha}\right))}{2\beta_2+\beta_3}. \tag{4.28}$$

The comprehensive analysis of the obtained exact solutions reveals that the nonlinear partial differential equation admits a rich variety of wave structures, depending on the specific parameter regimes. Notably, the solutions include: Hyperbolic function solutions which are characteristic of solitary wave phenomena. These solutions describe localized, non-dispersive waveforms that retain their shape and velocity over time. Trigonometric function solutions which correspond to periodic wave behaviors, reflecting oscillatory patterns inherent in the underlying nonlinear dynamics. Rational function solutions, which emerge under particular parameter constraints, represent special or degenerate cases of the equation where the nonlinear terms simplify, yielding algebraic expressions.

These results demonstrate the efficiency of the improved auxiliary equation method in gevorning the wide range of analytical wave solutons.

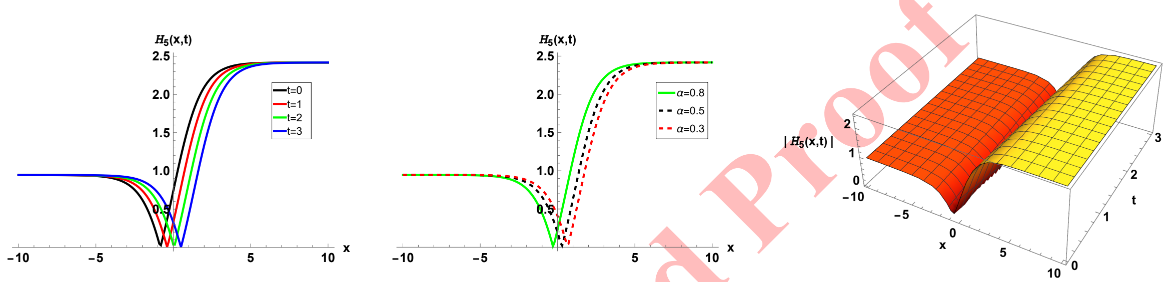
5. Graphical Discussion and Results

This study systematically derives a range of exact and solitary wave solutions the work also illustrates 2-Dim. (2D) and 3-Dim. (3D) graphical representations of selected key soultions, highlighting significant features. The graphical representation of these solutions is presented to visually highlight their distinct physical characteristics. Various types of waveforms are illustrated, including kink-shaped solitons, periodic solitons, periodic kink type soliton solution, and bell-shaped solitons, each demonstrating unique features and behaviors. As shown in Figure 1, displays the 2-D and 3-D profiles the kink-shaped soliton solution $H_3(x,t)$ for selected parameter values such as: $q = 3, p = 0.2, r = -1, \delta = 0.3, \rho = 0.2, \beta_1 = 0.5, \beta_2 = 1.3, \beta_3 = 0.1, \gamma = 0.5, c_0 = -0.5, d_0 = 0.1$. The 2-D plot (left) shows a smooth kink-like transition at various time levels ($t = 0, 1, 2, 3$), the middle figure shows the influence of fractional operator while the 3-D surface plot (right) highlights the solitons propagation in space and time, confirming its stability and kink-shaped behavior. Figure 2, depicts the 2-D and 3-D profiles of the $H_5(x,t)$ with particular parameters such as



(a) (b) (c)

FIGURE 1. 2-D and 3-D graphical kink soliton profile of $H_3(x,t)$, (a) 2-D visuals for different time t, (b) 2-D graph with different fractional order α, (c) 3-D graph with fractional order $\alpha = 0.5$.



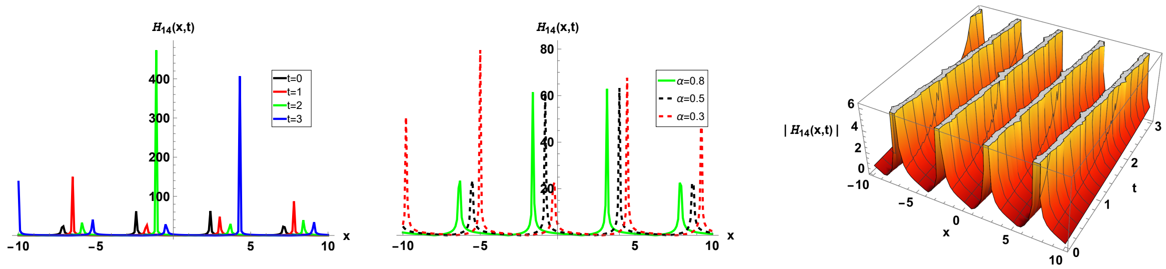
(b) (c)

FIGURE 2. 2-D and 3-D graphical profile of $H_5(x,t)$, (a) 2-D visuals for the different value of time t, (b) 2-D graphs with different value of fractional operator α, (c) 3-D graph with fractional order $\alpha = 1$.

$q = 0.5, p = 0.9, \delta = 0.6, \rho = 0.9, \beta_1 = -0.2, \beta_2 = 0.1, \beta_3 = 0.6, \gamma = 0.5, c_0 = 0.2, d_0 = -0.7$. The 2-D plots shows the impact of time interval and the impact of fractional derivative operator respectively, while the 3-D surface plot highlights the temporal and spatial evolution of the solution, characterized by its a spatial type of soliton structure. The figure 3 presents the 2-D and 3-D graphical profiles of the periodic soliton solution $H_{14}(x,t)$ for specific parameter values: $q = 0, r = 2.3, p = 2.1, \delta = 0.3, \rho = 2.2, \beta_1 = 0.3, \beta_2 = 1.3, \beta_3 = 0.7, \gamma = 0.5, c_0 = 0.2, d_0 = 0.5$. The 2-D plot (left) reveals a sharp step-like transition at different time levels ($t = 0, 1, 2, 3$) and the middle 2-D plot shows the impact of fractional operator α, while the 3-D surface plot (right) demonstrates the propagation of this solution in both spatial and temporal dimensions, highlighting its periodic characteristics. As displayed in figure 4 the 2-D and 3-D graphical representations of the periodic kink type soliton solutions $H_{16}(x,t)$ for specific parameter values: $q = 0.7, \delta = 0.86, \rho = 0.2, \beta_1 = 1.5, \beta_2 = 1.5, \beta_3 = 0.1, \gamma = 1.3, c_0 = 1.3, d_0 = 0.1$. The 2-D plots highlights the impact of time and fractional derivative operator. The 3-D surface plot (right) further visualizes the localized behavior and propagation of the soliton solution over space and time. These solutions contribute vital insights into the dynamic behavior of complex systems that are modeled by such equations.

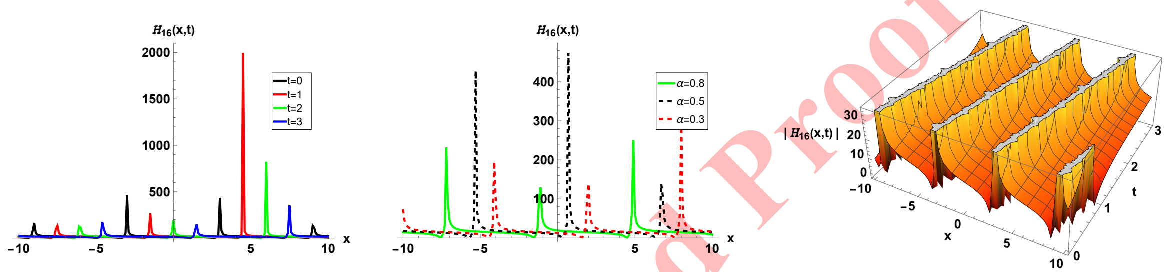
CONCLUSION

This work introduces a robust and efficient context for solving fifth-order generalized nonlinear fractional equation (5th-order GNFDE), operating improved Auxiliary equation method within the framework of M-fractional derivative. The method's capability to produce varied and exact solutions, including kink-shaped, singular kink-shaped, singular step-like, and bell-shaped solitons, imitates its substantial potential in advancing the analytical behavior of high-order



(a) (b) (c)

Figure 3. 2-D and 3-D graphical periodic soliton profile of $H_{14}(x,t)$, (a) 2-D visuals for the different value of time t, (b) 2-D graphs with different value of fractional operator α, (c) 3-D graph with fractional order $\alpha = 0.7$.



(a) (b) (c)

Figure 4. 2-D and 3-D graphical periodic-kink soliton profile of $H_{16}(x,t)$, (a) 2-D visuals for the different value of time t, (b) 2-D graphs with different value of fractional operator α, (c) 3-D graph with fractional order $\alpha = 0.5$.

fractional systems. Over and done with comprehensive 2-Dim. (2D) and 3-Dim. (3D) graphical visualizations, extracted all solutions highlight complex dynamical behaviors under specific parametric constraints, contributing deep insights into the interplay of nonlinearity and memory effects intrinsic in fractional systems. These outcomes accentuate the theoretical significance of the proposed approach and significantly improve its effectiveness in addressing real-world nonlinear phenomena. The findings offered in this study mark an extensive contribution to the area of fractional differential equations and nonlinear dynamic systems, providing a solid foundation for further exploration of mathematical models in complex systems. These findings serve as a powerful demonstration of how innovative analytical approaches can expose the rich underlying structure of nonlinear fractional systems, strengthening their relevance in modern mathematical and physical sciences.

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Muhammad Ashraf: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization, Software, Methodology.
Noor Muhammad: Project administration, Resources, Conceptualization, Writing – review & editing, Visualization,

Methodology, Investigation, Validation, Formal analysis, Software, Supervision.
Md. Nur Alam: Project administration, Resources, Conceptualization, Writing – review & editing, Visualization, Methodology, Investigation, Validation, Formal analysis, Software, Supervision.
Md. Abdullah Bin Masud: Writing – review & editing, Writing – original draft, Supervision, Investigation, Conceptualization.
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No data was used for the research described in the article.

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We confirm that we have reviewed and complied with the ethical standards required for the manuscript submitted to this journal. Additionally, we affirm that the manuscript has not been copyrighted, published, or submitted to any other publication.

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