*Research Paper*
*Computational Methods for Differential Equations*
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-18
DOI:10.22034/cmde.2025.67759.3240

# A hybrid approach to solving the Lane-Emden differential equation using generalized hat functions and hermite interpolation

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**Abstract**

This paper presents a novel and highly effective numerical method for solving the Lane-Emden equation, significantly expanding its applicability to a broad class of nonlinear differential equations. Our approach leverages the power of Generalized Hat functions and their operational matrix of integration to transform the target equation into a manageable, block-structured nonlinear system, which is then efficiently solved via forward substitution. Critically, a key innovation of our method is the use of quintic Hermite interpolation for constructing the final solution. This departure from the typical reliance on primary function bases results in a markedly more accurate approximation. A key strength of this method lies in its remarkable robustness. Unlike many existing techniques, its accuracy remains consistent regardless of the length of the solution interval. Furthermore, its adaptability is exceptional: with minimal modifications, it can be readily extended to tackle fractional-order Lane-Emden equations and a wide variety of other nonlinear ordinary differential equations. While several solution approximation methods are possible, we demonstrate the superior accuracy of Hermite interpolation. The paper provides a thorough analysis, including detailed error assessments, and showcases the method's accuracy, efficiency, and versatility through compelling numerical examples. We believe this innovative approach offers a significant advancement in the numerical solution of these important equations.

**Keywords.** Lane-Emden Equation, Generalized Hat Functions, Hermite Interpolation, Error Analysis, Operational Matrix of Integration, Numerical Approximation.
**2010 Mathematics Subject Classification.** 65L05, 34K06, 34K28.



## 1. Introduction

The Lane-Emden equation is a fundamental tool in astrophysics used to understand and model the structure and evolution of stars. It describes the balance between gravity and the pressure gradient within a star, providing insights into key properties like temperature, density, and pressure distribution [12, 23]. This equation emerges from several basic principles [12]:

- Hydrostatic equilibrium: Stars are massive objects, but their internal forces balance each other, preventing them from collapsing under their own gravity.
- Newton's law of gravitation: Every particle within a star exerts gravitational attraction on others.
- Ideal gas law: The pressure within a star is related to its temperature and density.

By combining these principles, mathematicians Jonathan Homer Lane and Robert Emden derived the Lane-Emden equation, expressing the relationship between density $\rho$, radius $r$, and a polytropic index $n$ representing the pressure-density relation within the star. The Lane-Emden equation has numerous applications in understanding stars: [12]

- Modeling stellar structure: By solving the equation with different boundary conditions and polytropic indices, scientists can create mathematical models for various types of stars, like main-sequence stars, white dwarfs, and neutron stars.

Received: 09 June 2025 ; Accepted: 19 October 2025.
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- Studying stellar evolution: Understanding how the density and pressure vary within a star helps predict its behavior and evolution over time, including stages like burning fuel, collapsing, or exploding.
- Interpreting observations: Comparing theoretical models based on the Lane-Emden equation with observations of stars (e.g., luminosity, spectra) helps astronomers infer their internal properties and processes.

While powerful, the Lane-Emden equation has limitations: [12]

- Simplified assumptions: It assumes spherical symmetry and constant temperature within specific regions, which isn't always true for real stars.
- Numerical solutions: Analytical solutions often exist only for specific cases, requiring numerical methods for other scenarios.

The Lane-Emden equation, despite its limitations, remains a cornerstone for understanding stellar structure and a valuable tool for astrophysicists to explore the complexities of stars. Its applications range from modeling basic star types to unraveling the mysteries of stellar evolution and interpreting observational data. By studying this equation, scientists gain deeper insights into the fascinating world of these celestial giants. The equation is given by:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

where $\xi$ is a dimensionless radius, $\theta$ is a dimensionless function related to the density, and $n$ is the polytropic index that determines the equation of state of the fluid. The equation has exact solutions for some values of $n$, such as $n = 0, 1, 5$, and numerical solutions for others [1, 7, 9, 19, 21]. The solutions, known as polytropes, can be used to approximate the density, pressure, temperature, and mass profiles of stars with different compositions and evolutionary stages. For example, a polytrope with $n = 3$ can model a white dwarf, while a polytrope with $n = 1.5$ can model a main-sequence star. The Lane-Emden equation is important because it provides a simple way to understand the basic properties and stability of stars, without solving the full set of equations of stellar structure. It also reveals some universal features of self-gravitating systems, such as the mass-radius relation and the critical mass for collapse. The equation can also be applied to other astrophysical phenomena, such as planetary atmospheres, neutron stars, dark matter halos, and galaxy clusters.

A wide range of numerical techniques have been applied to this equation. An extensive array of numerical techniques has been devised and implemented for the solution of this equation. Notable examples of these include the following papers: The Taylor wavelet method is utilized to solve the equation as described in [7]. Studies [1], [11] and [16] propose numerical methods for Lane-Emden equations. These methods utilize a collocation approach based on various types of B-splines, coupled with the quasilinearization technique to handle the nonlinearities. A novel machine learning approach for the solution of Lane-Emden type differential equations is introduced in [9], employing Rational Chebyshev polynomials. The authors develop and improved a higher-order compact finite difference scheme specifically designed to solve systems of equations of the Lane-Emden-Fowler type [4, 21]. The authors proposed an Artificial Neural Network (ANN) approach for the solution of the Equation in [19].

This paper presents a novel numerical approach for solving the Lane-Emden equation, broadening its applicability to a diverse array of nonlinear differential equations. The proposed method leverages Generalized Hat functions in combination with their operational matrix of integration to transform the Lane-Emden equation into a structured nonlinear system of equations. This resulting system is then effectively solved through forward substitution. To further enhance the precision of the solution, quintic Hermite interpolation is applied, yielding an accurate approximation of the equation's solution.

One of the standout features of this method is its exceptional flexibility. Unlike many traditional techniques, the accuracy of the proposed approach remains consistent regardless of the length of the solution interval, making it highly robust. Additionally, with minor adjustments, the method can be extended to address fractional-order Lane-Emden equations, showcasing its adaptability. Beyond Lane-Emden equations, the approach is versatile enough to handle a broad spectrum of nonlinear ordinary differential equations.

The method provides multiple pathways for deriving approximate solutions, with Hermite interpolation proving particularly effective in achieving superior accuracy. The paper offers a thorough analysis of these methods, supported by detailed error assessments, to highlight their comparative strengths and weaknesses. To validate the proposed approach, numerical examples are included, demonstrating its accuracy, efficiency, and adaptability across various problem settings. This innovative method opens new avenues for tackling complex differential equations with improved precision and computational efficiency.

This paper is organized as follows. Section 2 introduces Generalized Hat Functions (GHFs) of degrees 1 to 4. It also delves into their properties, operational matrices of integration, and provides a comprehensive discussion of Hermite interpolation and its associated error bounds. Section 3 presents the proposed method in detail, outlining its methodology and implementation. Section 4 analyzes the convergence properties of the proposed method, providing theoretical support for its accuracy and reliability. Section 5 demonstrates the accuracy, efficiency, and convergence of the proposed method through several numerical examples. It also conducts comparative analyses with other existing methods to highlight the strengths and potential advantages of the proposed approach. Section 6 concludes the paper by summarizing the key findings, discussing the significance of the research, and outlining potential avenues for future work.

## 2. Generalized Hat Functions (GHFs)

In this section we recall the generalized hat functions and their properties. Generalaized Hat functions (GHFs) are continuous functions with shape of hats (in case of degree one). Suppose that the interval $[0, T]$, for $T > 0$, is divided into $N$ subintervals $[ih, (i+1)h]$, $i = 0, 1, \cdots, N-1$ of equal lengths $h$ where $h = \frac{T}{N}$.

The GHFs from first through forth degrees are defined as the following, respectively:

**First degree (Linear) GHFs definition.** First-degree GHFs were the building blocks upon which the broader concept of GHFs was constructed. The GHFs of degree one are defined as: [25]

$$\psi_0(t) = \begin{cases} \frac{h-t}{h}, & 0 \le t < h, \\ 0, & \text{Otherwise.} \end{cases}$$

$$\psi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \le t < ih, \\ \frac{(i+1)h-t}{h}, & ih \le t < (i+1)h, \qquad i = 1, 2, \cdots, N-1 \\ 0, & \text{Otherwise.} \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{t-(T-h)}{h}, & T-h \le t \le T, \\ 0, & \text{Otherwise.} \end{cases}$$

**Second degree (Quadratic) GHFs definition.** [14] For an even integer $N \ge 2$

$$\psi_0(t) = \begin{cases} \frac{1}{2h^2}(t-h)(t-2h), & 0 \le t < 2h, \\ 0, & \text{Otherwise.} \end{cases}$$

For odd $i$, $\quad 1 \le i \le N-1$

$$:\psi_i(t) = \begin{cases} \frac{-1}{h^2}(t-(i-1)h)(t-(i+1)h), & (i-1)h \le t < (i+1)h, \\ 0, & \text{Otherwise.} \end{cases}$$

For even $i$, $\quad 2 \le i \le N-2$

$$:\psi_i(t) = \begin{cases} \frac{1}{2h^2}(t-(i-1)h)(t-(i-2)h), & (i-2)h \le t < ih, \\ \frac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & ih \le t < (i+2)h, \\ 0, & \text{Otherwise.} \end{cases}$$

and

$$\psi_N(t) = \begin{cases} \frac{1}{2h^2}(t-(T-h))(t-(T-2h)), & T-2h \le t \le T, \\ 0, & \text{Otherwise.} \end{cases}$$

**Third degree (Cubic) GHFs definition.** [5] Suppose $N$ is a positive integer of multiple three $(N = 3m,\ m \in \mathbb{N})$ and $h = \frac{T}{N}$. A set of adjust GHFs of degree 3 is defined on $[0, T]$ as:

$$\psi_0(t) = \begin{cases} \frac{-1}{6h^3}(t-h)(t-2h)(t-3h), & 0 \le t < 3h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 3k+1,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h), & (i-1)h \le t < (i+2)h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 3k+2,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{-1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h), & (i-2)h \le t < (i+1)h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 3k,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h), & (i-3)h \le t < ih, \\ \frac{-1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h), & ih \le t < (i+3)h, \\ 0, & \text{Otherwise.} \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{1}{6h^3}(t-(T-h))(t-(T-2h))(t-(T-3h)), & T-3h \le t \le T, \\ 0, & \text{Otherwise.} \end{cases}$$

**Forth degree (Quartic) GHFs definition.** [17] Suppose $N$ is a positive integer of multiple three $(N = 4m,\ m \in \mathbb{N})$ and $h = \frac{T}{N}$. A set of adjust GHFs of degree 4 is defined on $[0, T]$ as:

$$\psi_0(t) = \begin{cases} \frac{(t-h)(t-2h)(t-3h)(t-4h)}{24h^4}, & 0 \le t < 4h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 4k+1,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{(t-(i-1)h)(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)}{6h^4}, & (i-1)h \le t < (i+3)h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 4k+2,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{(t-(i-2)h)(t-(i-1)h)(t-(i+1)h)(t-(i+2)h)}{4h^4}, & (i-2)h \le t < (i+2)h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 4k+3,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{-(t-(i-3)h)(t-(i-2)h)(t-(i-1)h)(t-(i+1)h)}{6h^4}, & (i-3)h \le t < (i+1)h, \\ 0, & \text{Otherwise.} \end{cases}$$

If $i = 4k,\ k = 0, 1, \cdots, m-1$:

$$\psi_i(t) = \begin{cases} \frac{(t-(i-4)h)(t-(i-3)h)(t-(i-2)h)(t-(i-1)h)}{24h^4}, & (i-4)h \le t < ih, \\ \frac{(t-(i+1)h)(t-(i+2)h)(t-(i+3)h)(t-(i+4)h)}{24h^4}, & ih \le t < (i+4)h, \\ 0, & \text{Otherwise.} \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{24h^4}, & T-4h \le t \le T, \\ 0, & \text{Otherwise.} \end{cases}$$

2.1. **GHFs properties and their operational matrix of integration.** It is easy to see that $\psi_i$'s belong to $L^2[0,T]$ and are linearly independent, also have the following properties:

$$\psi_i(kh) = \delta_{ik}, \quad i,k = 0,1,\dots,N, \qquad \sum_{i=0}^{N} \psi_i(t) = 1, \tag{2.1}$$

where $\delta_{ik}$ is the Kronecker delta.

Any function $y(t) \in L^2[0,T]$ can be approximated in terms of GHFs as:

$$y(t) \simeq y_N(t) = \sum_{k=0}^{N} c_k \psi_k(t) = \mathbf{C}_y^T \mathbf{\Psi}(t) = \mathbf{\Psi}^T(t)\mathbf{C}_y, \tag{2.2}$$

where, $\mathbf{\Psi}(t) = [\psi_0(t), \psi_1(t), \cdots, \psi_N(t)]^T$ and $\mathbf{C}_y = [c_0, c_1, \cdots, c_N]^T$ and the coefficients $c_k$'s in (2.2) are given by:

$$c_k = y(kh), \quad k = 0,1,\cdots,N. \tag{2.3}$$

According to (2.2) and (2.3), the product of the two functions can be approximated in terms of GHFs as follows:

$$f(t)g(t) \simeq \sum_{k=0}^{N} f(t_k)g(t_k)\psi_k(t) = (C_f \odot C_g)^T \mathbf{\Psi}(t), \tag{2.4}$$

where $\odot$ is Hadamard product (elementwise product), therefore, for $r \ge 0$,

$$(f(t))^r \simeq \sum_{k=0}^{N} (f(t_k))^r \psi_k(t) = ((C_f)_{\odot}^{r})^T \mathbf{\Psi}(t). \tag{2.5}$$

Using GHFs, one can approximate integral of GHFs, hence

$$\int_0^t \mathbf{\Psi}(s)\, ds \simeq \mathbf{P}\mathbf{\Psi}(t),$$

where, $\mathbf{P}$ is a $(N+1)\times(N+1)$ matrix called operational matrix of integration that is given for various degrees of GHFs in sequel.

**Operation matrix of integration for linear GHFs.** [25] For $N \in \mathbb{N}$, $\mathbf{P}$ is obtained as follows:

$$\mathbf{P} = \frac{h}{2} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

**Operation matrix of integration for quadratic GHFs.** [14] For $N = 2m, \quad m \in \mathbb{N}$, $\mathbf{P}$ is obtained as:

$$\mathbf{P} = \frac{h}{12}\begin{bmatrix} 0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \dots & \mathbf{p}_2 & \mathbf{p}_2 \\ \mathbf{0}_{2\times 1} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 2} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{D} & \mathbf{N}_1 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \dots & \mathbf{D} & \mathbf{N}_1 \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \dots & \mathbf{0}_{2\times 2} & \mathbf{D} \end{bmatrix}_{(N+1)\times(N+1)},$$

where, $\mathbf{p}_1 = [5, 4]$, $\mathbf{p}_2 = [4, 4]$,

$$\mathbf{D} = \begin{bmatrix} 8 & 16 \\ -1 & 4 \end{bmatrix},$$

and

$$\mathbf{N}_1 = \begin{bmatrix} 16 & 16 \\ 9 & 8 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} 16 & 16 \\ 8 & 8 \end{bmatrix}.$$

**Operation matrix of integration for cubic GHFs.** [5] For $N = 3m, \quad m \in \mathbb{N}$, $\mathbf{P}$ is obtained as:

$$\mathbf{P} = \frac{h}{144}\begin{bmatrix} 0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \dots & \mathbf{p}_2 & \mathbf{p}_2 \\ \mathbf{0}_{3\times 1} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{D} & \mathbf{N}_1 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \dots & \mathbf{D} & \mathbf{N}_1 \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \dots & \mathbf{0}_{3\times 3} & \mathbf{D} \end{bmatrix}_{(N+1)\times(N+1)},$$

where, $\mathbf{p}_1 = [54, 48, 54]$, $\mathbf{p}_2 = [54, 54, 54]$,

$$\mathbf{D} = \begin{bmatrix} 114 & 192 & 162 \\ -30 & 48 & 162 \\ 6 & 0 & 54 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 162 & 162 & 162 \\ 162 & 162 & 162 \\ 108 & 102 & 108 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_2 = \begin{bmatrix} 162 & 162 & 162 \\ 162 & 162 & 162 \\ 108 & 108 & 108 \end{bmatrix}.$$

**Operation matrix of integration for quartic GHFs.** [17] For $N = 4m, \quad m \in \mathbb{N}$, $\mathbf{P}$ is obtained as:

$$\mathbf{P} = \frac{h}{720}\begin{bmatrix} 0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_2 & \dots & \mathbf{p}_2 & \mathbf{p}_2 \\ \mathbf{0}_{4\times 1} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{4\times 1} & \mathbf{0}_{4\times 4} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{4\times 1} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{D} & \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \mathbf{0}_{4\times 1} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{D} & \mathbf{N}_1 & \dots & \mathbf{N}_2 & \mathbf{N}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{4\times 1} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \dots & \mathbf{D} & \mathbf{N}_1 \\ \mathbf{0}_{4\times 1} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \dots & \mathbf{0}_{4\times 4} & \mathbf{D} \end{bmatrix}_{(N+1)\times(N+1)},$$

where, $\mathbf{p}_1 = [251, 232, 243, 224]$, $\mathbf{p}_2 = [224, 224, 224, 224]$,

$$\mathbf{D} = \begin{bmatrix} 646 & 992 & 918 & 1024 \\ -264 & 192 & 648 & 384 \\ 106 & 32 & 378 & 1024 \\ -19 & -8 & -27 & 224 \end{bmatrix}, \ \mathbf{N}_1 = \begin{bmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 475 & 456 & 467 & 448 \end{bmatrix} \ \text{and } \mathbf{N}_2 = \begin{bmatrix} 1024 & 1024 & 1024 & 1024 \\ 384 & 384 & 384 & 384 \\ 1024 & 1024 & 1024 & 1024 \\ 448 & 448 & 448 & 448 \end{bmatrix}.$$

2.2. **Hermite Interpolation.** Hermite interpolation, named after Charles Hermite, is a method of polynomial interpolation, which generalizes Lagrange interpolation. Lagrange interpolation allows computing a polynomial of degree less than $n$ that takes the same value at $n$ given points as a given function. Instead, Hermite interpolation computes a polynomial of degree less than $n$ such that the polynomial and its first few derivatives have the same values at $m$ (fewer than $n$) given points as the given function and its first few derivatives at those points [10, 15]. The number of pieces of information, function values and derivative values, must add up to $n$. Hermite interpolation consists of computing a polynomial of degree as low as possible that matches an unknown function both in observed value, and the observed value of its first $m$ derivatives. This means that $n(m+1)$ values

$$\begin{array}{cccc}(x_0,y_0) & (x_1,y_1) & \dots & (x_{n-1},y_{n-1})\\ (x_0,y_0') & (x_1,y_1') & \dots & (x_{n-1},y_{n-1}')\\ \vdots & \vdots & \ddots & \vdots\\ (x_0,y_0^{(m)}) & (x_1,y_1^{(m)}) & \dots & (x_{n-1},y_{n-1}^{(m)})\end{array}$$

must be known. The resulting polynomial has a degree less than $n(m+1)$.

2.2.1. *Quintic Hermite interpolation.* The quintic Hermite interpolation based on the function $(f)$, its first $(f')$ and second derivatives $(f'')$ at two different points ($x_i$ and $x_{i+1}$) can be used for example to interpolate the position of an object based on its position, velocity and acceleration [13, 24]. The general form is given by

$$\begin{aligned} H(x) &= f_i + (x-x_i)f_i' + \frac{f_i''}{2}(x-x_i)^2 \\ &+ \frac{f_{i+1} - f_i - hf_i' - \frac{h^2}{2}f_i''}{h^3}(x-x_i)^3 \\ &+ \frac{3f_i - 3f_{i+1} + 2h(f_i' + \frac{1}{2}f_{i+1}') + \frac{h^2}{2}f_i''}{h^4}(x-x_i)^3(x-x_{i+1}) \\ &+ \frac{6f_{i+1} - 6f_i - 3h(f_i' + f_{i+1}') + \frac{h^2}{2}(f_{i+1}'' - f_i'')}{h^5}(x-x_i)^3(x-x_{i+1})^2, \end{aligned}$$

where $f_i = f(x_i)$, $f_{i+1} = f(x_{i+1})$, $f_i' = f'(x_i)$, $f_{i+1}' = f'(x_{i+1})$, $f_i'' = f''(x_i)$, $f_{i+1}'' = f''(x_{i+1})$ and $h = x_{i+1} - x_i$, is the distance between the two knots.

For quintic Hermite interpolation, which involves a fifth-degree polynomial, the error term involves the sixth derivative of the function $f$ [24]. The error bound can be expressed as:

$$|f(x) - H(x)| \leq \frac{M_6}{6!}|x-x_i|^3|x-x_{i+1}|^3 \leq \frac{M_6}{6!}\frac{h^6}{2^6}, \tag{2.6}$$

where $M_6 = \max_{x\in[x_i,x_{i+1}]|f^{(6)}(x)|}$.



## 3. The Method

Consider the following second order singular ordinary differential equation with initial conditions

$$y''(x) + \frac{\alpha}{t}y'(x) + y^n(x) = 0, \qquad 0 \leq x \leq T, \tag{3.1}$$

$$y(0) = \beta, \quad y'(0) = \eta, \tag{3.2}$$

where $\alpha, T, \beta$ and $\eta$ are given numbers and $n$ is a known nonnegative integer. This second order singular ordinary differential equation is known as the Lane-Emden equation[18]. We will find the approximate solution using the numerical methods that we discussed earlier as well as another one. For the cases where $n = 0, 1, 5$, this equation possesses a closed-form solution while for other values of $n$, a closed-form solution is not known. With $\alpha = 2,\ \beta = 1,\ \eta = 0$ the exact solution of the equation is:

$$n = 0, \qquad y(x) = 1 - \frac{x^2}{6},$$

$$
\begin{aligned}
n &= 1, \qquad y(x) = \frac{\sin x}{x},\\
n &= 5, \qquad y(x) = \frac{1}{\sqrt{1+\frac{1}{3}x^2}}.
\end{aligned}
$$

Initially, by taking the limit of the Equation (3.1) and using L'Hopital's rule, we have:

$$
\begin{aligned}
&\lim_{x\to 0}(y''(x) + \frac{\alpha}{x}y'(x) + (y(x))^n) = 0,\\
&\Rightarrow \quad y''(0) + \alpha y''(0) + (y(0))^n = 0,\\
&\Rightarrow \quad y''(0) = \frac{-\beta^n}{1+\alpha}. \qquad (3.3)
\end{aligned}
$$

Then, we rewrite the Equation (3.1) as $xy'' + \alpha y' + xy^n = 0$ and consider an approximate solution for $y''(x)$ based on the given basis, i.e. GHFs, as follows:

$$
y''(x) \simeq \sum_{i=0}^{N} c_i \psi(x) = C_{y''}^T \mathbf{\Psi}(x), \qquad (3.4)
$$

therefore, we will have the followings facts about $y'(x)$ and $y(x)$,

$$
\begin{aligned}
y'(x) &= \int_0^x y''(t)\,dt + y'(0) && (3.5)\\
&\simeq C_{y''}^T \mathbf{P}\mathbf{\Psi}(x) + \eta, && (3.6)\\
y(x) &= \int_0^x y'(t)\,dt + y(0) && (3.7)\\
&\simeq C_{y''}^T \mathbf{P}^2\mathbf{\Psi}(x) + \eta x + \beta, && (3.8)
\end{aligned}
$$

where $\mathbf{P}$ is the operational matrix of integral of the selected GHFs. Besides, we expand the functions $f(x) = 1$ and $g(x) = x$ in terms of GHFs. So, we will have $1 = J^T\mathbf{\Psi}(x)$ and and $x = E^T\mathbf{\Psi}(x)$ where $J = [1, 1, \dots, 1]^T$ an $N+1$-vector and $E = [x_0, x_1, \dots, x_N]^T$ where $x_i = ih,\ i = 0, 1, \dots, N$ are the knots of GHFs. By substituting them in (3.6) and (3.8) we have:

$$
\begin{aligned}
y'(x) &\simeq (C_{y''}^T\mathbf{P} + \eta J^T)\mathbf{\Psi}(x) = C_{y'}^T\mathbf{\Psi}(x), && (3.9)\\
y(x) &\simeq (C_{y''}^T\mathbf{P}^2 + \eta E^T + \beta J^T)\mathbf{\Psi}(x) = C_{y}^T\mathbf{\Psi}(x), && (3.10)
\end{aligned}
$$

where $C_{y'} = \mathbf{P}^T C_{y''} + \eta J$ and $C_y = (\mathbf{P}^T)^2 C_{y''} + \eta E + \beta J$ then substituting in (3.1) using (2.5) we will have:

$$
(E \odot C_{y''})^T\mathbf{\Psi}(x) + \alpha(C_{y''}^T\mathbf{P} + \eta J^T)\mathbf{\Psi}(x) + E^T \odot (C_{y''}^T\mathbf{P}^2 + \eta E^T + \beta J^T)_{\odot}^{n}\mathbf{\Psi}(x) = 0,
$$

or equivalently,

$$
(E \odot C_{y''}) + \alpha(\mathbf{P}^T C_{y''} + \eta J) + E \odot ((\mathbf{P}^T)^2 C_{y''} + \eta E + \beta J)_{\odot}^{n} = 0. \qquad (3.11)
$$

System resulting from the Equation (3.11) is a nonlinear block system. It is a nonlinear system of multi-variable polynomial equations, and its solution can be determined by employing a suitable numerical method. The number of equations within each block corresponds to the degree of the selected Generalized Hat Functions (GHFs). Furthermore, each equation within a block is a multi-variable polynomial whose degree corresponds to that of the specific Lane-Emden equation under consideration, $n$. The number of unknowns in the first block equals the degree of the selected GHFs. Subsequently, the number of unknowns in each successive block increases by the degree of the GHFs compared to the preceding block. This means that in the first block, the number of equations and the number of unknowns are the same, and the block equations can be solved independently from the others. Hence, the system can be solved using forward substitution. For instance, with $\alpha = 2,\ \beta = 1,\ \eta = 0,\ n = 3,\ N = 12$ and using forth-degree GHFs, resulting system is as follows:

$$\begin{aligned}
&\frac{5(2430c_1 - 1665c_2 + 840c_3 - 180c_4 + 7351)^3}{2^{16}\cdot 3^{16}} + \frac{(3018c_1 - 792c_2 + 318c_3 - 57c_4 - 251)}{2^4\cdot 3^4} = 0,\\
&\frac{5(555c_1 - 135c_2 + 75c_3 - 15c_4 + 421)^3}{2^3\cdot 3^{16}} + \frac{(372c_1 + 342c_2 + 12c_3 - 3c_4 - 29)}{2\cdot 3^4} = 0,\\
&\frac{5}{2^{16}\cdot 3^3}(200c_1 + 15c_2 + 30c_3 - 5c_4 + 76)^3 + \frac{1}{2^4}(34c_1 + 24c_2 + 54c_3 - c_4 - 3) = 0,\\
&\frac{10}{3^{16}}(720c_1 + 180c_2 + 240c_3 + 173)^3 + \frac{2}{3^4}(96c_1 + 36c_2 + 96c_3 + 156c_4 - 7) = 0,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\frac{25}{2^{16}\cdot 3^{16}}(30720c_1 + 8640c_2 + 15360c_3 + 2955c_4 + 2430c_5 - 1665c_6 + 840c_7 - 180c_8 + 4976)^3\\
&+ \frac{1}{2^4\cdot 3^4}(3072c_1 + 1152c_2 + 3072c_3 + 1425c_4 + 7338c_5 - 792c_6 + 318c_7 - 57c_8 - 224) = 0,\\
&\frac{5}{2^3\cdot 3^{12}}(800c_1 + 240c_2 + 480c_3 + 135c_4 + 185c_5 - 45c_6 + 25c_7 - 5c_8 + 92)^3\\
&+ \frac{1}{2\cdot 3^4}(384c_1 + 144c_2 + 384c_3 + 171c_4 + 372c_5 + 882c_6 + 12c_7 - 3c_8 - 28) = 0,\\
&\frac{35}{2^{16}\cdot 3^{16}}(46080c_1 + 14400c_2 + 30720c_3 + 9900c_4 + 16200c_5 + 1215c_6 + 2430c_7 - 405c_8 + 3856)^3\\
&+ \frac{1}{2^4\cdot 3^4}(3072c_1 + 1152c_2 + 3072c_3 + 1401c_4 + 2754c_5 + 1944c_6 + 8694c_7 - 81c_8 - 224) = 0,\\
&\frac{20}{3^{16}}(1680c_1 + 540c_2 + 1200c_3 + 420c_4 + 720c_5 + 180c_6 + 240c_7 + 103)^3\\
&+ \frac{2}{3^4}(96c_1 + 36c_2 + 96c_3 + 42c_4 + 96c_5 + 36c_6 + 96c_7 + 291c_8 - 7) = 0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\frac{5}{2^{16}\cdot 3^{11}}(20480c_1 + 6720c_2 + 15360c_3 + 5600c_4 + 10240c_5 + 2880c_6 + 5120c_7 + 985c_8\\
&+ 810c_9 - 555c_{10} + 280c_{11} - 60c_{12} + 912)^3\\
&+ \frac{1}{2^4\cdot 3^4}(3072c_1 + 1152c_2 + 3072c_3 + 1344c_4 + 3072c_5 + 1152c_6 + 3072c_7 + 1425c_8\\
&+ 11658c_9 - 792c_{10} + 318c_{11} - 57c_{12} - 224) = 0,\\
&\frac{25}{2^3\cdot 3^{16}}(4320c_1 + 1440c_2 + 3360c_3 + 1260c_4 + 2400c_5 + 720c_6 + 1440c_7 + 405c_8\\
&+ 555c_9 - 135c_{10} + 75c_{11} - 15c_{12} + 136)^3\\
&+ \frac{1}{2\cdot 3^4}(384c_1 + 144c_2 + 384c_3 + 168c_4 + 384c_5 + 144c_6 + 384c_7 + 171c_8\\
&+ 372c_9 + 1422c_{10} + 12c_{11} - 3c_{12} - 28) = 0,\\
&\frac{55}{2^{16}\cdot 3^{16}}(76800c_1 + 25920c_2 + 61440c_3 + 23520c_4 + 46080c_5 + 14400c_6 + 30720c_7 + 9900c_8\\
&+ 16200c_9 + 1215c_{10} + 2430c_{11} - 405c_{12} + 1616)^3\\
&+ \frac{1}{2^4\cdot 3^4}(3072c_1 + 1152c_2 + 3072c_3 + 1344c_4 + 3072c_5 + 1152c_6 + 3072c_7 + 1401c_8\\
&+ 2754c_9 + 1944c_{10} + 13014c_{11} - 81c_{12} - 224) = 0,\\
&\frac{10}{3^{12}}(880c_1 + 300c_2 + 720c_3 + 280c_4 + 560c_5 + 180c_6 + 400c_7 + 140c_8 + 240c_9 + 60c_{10} + 80c_{11} + 11)^3
\end{aligned}$$

$$+\frac{2}{3^4}(96c_1+36c_2+96c_3+42c_4+96c_5+36c_6+96c_7+42c_8+96c_9+36c_{10}+96c_{11}+426c_{12}-7)=0 \quad (3.14)$$

As can be seen, the nonlinear system of equations possesses a block structure. Each block consists of four equations (corresponding to the selected GHF degree), and each equation is a cubic polynomial (matching the degree of the Lane–Emden equation under consideration). The equations of the first block (3.12) involve four unknowns (equal to the chosen GHF degree). The equations of the second block (3.13) involve eight unknowns (the unknowns from the previous block plus four new ones). Similarly, the third block (3.14) introduces four additional unknowns on top of those from the second block, and this pattern continues.

The first equation of the system (3.11) is given by $0 \cdot c_0 = 0$, from which the value of $c_0$ cannot be determined. To evaluate $c_0$, we use the relation (2.3), which implies that $c_0 = y''(0)$. Thus, by employing Equation (3.3), we obtain

$$c_0 = y''(0) = \frac{-\beta^n}{1+\alpha}, \quad (3.15)$$

where $c_0$, $\beta$, and $\alpha$ are defined in (3.4) and (3.2), respectively.

As an illustration, solving the first block (3.12) yields

$$c_1 = -0.17120567670917203, \qquad c_2 = 0.0281110844432213,$$
$$c_3 = 0.07975584653620632, \qquad c_4 = 0.07028833734919199.$$

Substituting this solution into the second block (3.13) and solving gives

$$c_5 = 0.048372063669963355, \qquad c_6 = 0.03154452660495657,$$
$$c_7 = 0.020675495938253192, \qquad c_8 = 0.013920512043363673.$$

Finally, substituting the solutions from (3.12) and (3.13) into the third (last) block (3.14) yields

$$c_9 = 0.009800170181348163, \qquad c_{10} = 0.007291774134836329,$$
$$c_{11} = 0.005780375143477206, \qquad c_{12} = 0.00488650320421807.$$

Moreover, as derived in (3.15), we obtain $c_0 = -\frac{1}{3}$. Having determined $C_{y''}$, and using (3.9) and (3.10), we can subsequently construct $C_{y'}$ and $C_y$, respectively.

As mentioned, we are dealing with a nonlinear system of multi-variable polynomial equations with a block structure, and the first block is as follows:

$$\begin{cases} p_{1,1}(c_1,\ldots,c_k)=0,\\ p_{1,2}(c_1,\ldots,c_k)=0,\\ \vdots\\ p_{1,k}(c_1,\ldots,c_k)=0, \end{cases}$$

where $k$ is the degree of GHFs, $N+1$ is the number of GHF knots $(x_0, x_1, \ldots, x_N)$ and $p_{1,i}(c_1,\ldots,c_k)$, $\quad i=1,\ldots,k$ are plynomials of degree $n$ where $n$ is the degree of Lane-Emden Equation (3.1).

The above system of nonlinear equations can be solved numerically using various methods such as the Newton-Raphson method, Broyden's method, Fixed-Point iteration method, Homotopy and continuation methods, the Levenberg-Marquardt algorithm, Trust-Region Methods, or others.

When solutions are determined, substitute them into the next block which has the following form which consists of $k$ multi-variable polynomials equations each of degree $n$ and has $2k$ variables $c_1,\ldots,c_k,c_{k+1},\ldots,c_{2k}$,

$$\begin{cases} p_{2,1}(c_1,\ldots,c_k,c_{k+1},\ldots,c_{2k})=0,\\ p_{2,2}(c_1,\ldots,c_k,c_{k+1},\ldots,c_{2k})=0,\\ \vdots\\ p_{2,k}(c_1,\ldots,c_k,c_{k+1},\ldots,c_{2k})=0, \end{cases}$$

resulting in a system of $k$ polynomial equations with $k$ unknowns $c_{k+1},\ldots,c_{2k}$. By solving this system similarly to the previous step, the values of $c_{k+1},\ldots,c_{2k}$ will be determined. In the same manner, by substituting the solutions obtained from solving the previous blocks into the next block, a system with $k$ equations in $k$ unknowns will be obtained. By solving this system block by block, its solutions will eventually be determined.

The significant advantage of this method for solving the system (3.11) (which can be a large system, $N \gg k$ with nonlinear multi-variable polynomial equations in $N$ unknowns), is that the solution of a very large system of equations can be obtained by solving much smaller systems, without the need for very time-consuming and complex calculations.

After determining $C_{y''} = [c_0, c_1, \dots, c_N]^T$, (3.9) and (3.10) yield $C_{y'}$ and $C_y$ respectively.

By using Eqs. (3.8), we can produce the approximate solution of the equation. We call this method the **usual approach**.

Another approach can be made by using the spline method. For more information on this topic, see the references [2, 3, 6, 8, 20]. Since we have approximated values of $y''(x)$, $y'(x)$, and $y(x)$ at the nodes, i.e. $\mathbf{C}_{y''}$, $\mathbf{C}_{y'}$ and $\mathbf{C}_y$ respectively, we use a spline interpolation function to produce a piecewise-polynomial function to approximate the solution of the equation. We call this method the **spline approach**.

Another approach can be made in this way: we use (3.4) to yield an approximation to $y''(x)$, and then use (3.5) and (3.7) to reach an approximate solution to the problem. We call this approach the **integration approach**.

Finally, the **main approach**, the coefficients $C_{y''}$, $C_{y'}$ and $C_y$ are treated as the values of their corresponding functions at the knots $x_i$, $i = 0, \dots, N$, as defined in (2.3). Consequently, instead of employing (3.8) or (3.10) to derive the approximate $y(x)$, solution for the Equation (3.1), in terms of GHFs, we can utilize Hermite interpolation to produce an approximate solution for the equation.

In addition to the theoretical analysis of error, solved examples also support the fact that this method yields a more accurate solution than the conventional method (resulting (3.8)) and even splines of the same degree.

## 4. Notes on Convergence

In this section, we present some topics on the convergence of the proposed method. For proofs and further details, you can refer to [22] and its references.

**Corollary 4.1.** *[22] Every Volterra operator,*

$$Tf(x) = \int_0^x k(x,y) f(y)\, dy,$$

*where $k(x,y)$ is continuous on $\Delta = \{(x,y) \in [a,b] \times [a,b]\,; a \le y \le x \le b\}$ is a compact operator.*

**Lemma 4.2.** *[22] Suppose $V$ be a Banach space, and assume $\{\mathcal{P}_n\}$ be a collection of bounded projections on $V$ with*

$$\mathcal{P}_n u \to u \text{ as } n \to \infty, u \in V.$$

*If operator $\mathcal{K} : V \to V$ is compact, then*

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \text{ as } n \to \infty.$$

Therefore, by Lemma 4.2, it can be proven that the Hermite interpolant, as an operator, is a projection and compact. Additionally, for the convergence of the Hermite interpolation method, you can refer to references [10, 13, 24].

**Theorem 4.3.** *The presented method in section 3 to solve ODE (3.1) is numerically convergent.*

*Proof.* First, we define operator

$$\mathcal{P}_n : L^2[0,1] \to W_n := \operatorname{span}\{\psi_0, \dots, \psi_n\} \subseteq L^2[0,1],$$

as the orthogonal projection operator that maps every $f \in L^2[0,1]$ to its approximation in $W_n$, i.e. $\mathcal{P}_n f(x) = \mathbf{C}^T \boldsymbol{\Psi}(x)$. Observing that $\mathcal{P}_n$ is a bounded linear operator is uncomplicated. Now, we define operator $\mathcal{K} : L^2[0,1] \to L^2[0,1]$ as $\mathcal{K}f = \int_0^x f(t)\, dt$. By Corollary 4.1 $\mathcal{K}$ is compact.

We know that:

$$\mathcal{P}_n y'' = \mathbf{C}^T \boldsymbol{\Psi}(x) \to y'', \text{ as } n \to \infty,$$

for $y'' \in L^2[0,1]$. Also, verifying that,

$$\mathcal{P}_n\mathcal{K}y'' = \mathcal{P}_n \int_0^x y''(t)\,dt = \mathbf{C}^T\mathbf{P}\mathbf{\Psi}(x),$$

is uncomplicated. Here $\mathbf{P}$ is the operational matrix of integration for $\mathbf{\Psi}(x)$, (GHFs). So, by Lemma 4.2 we have:

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0, \text{ as } n \to \infty.$$

Therefore, $y' \to \mathbf{C}^T\mathbf{P}\,\mathbf{\Psi}(x) + y_0'$.

And

$$\mathcal{K}^2 f = \int_0^x \int_0^t f(u)\,du = \int_0^x (x-t)f(t)\,dt,$$

agian, by Corollary 4.1, in a similar way as above, can deduce that

$$y \to \mathbf{C}^T\mathbf{P}^2\mathbf{\Psi}(x) + y_0'x + y_0,$$

and this complets the proof. □

Therefore, both the Hermite interpolation method and the approximation of a function using GHFs, as operators, and consequently the composition of these two operators, is compact and, according to the above theorem, the proposed method converges.

## 5. Illustrative Examples

In this section, various examples are provided to demonstrate the convergence, efficiency, and accuracy of the proposed method. As you will see, the theoretical discussions presented in the previous sections confirm the presented examples.

All examples in this section have been implemented and solved using Wolfram Mathematica version 14.1 on a desktop computer with a CPU model AMD A8-9600 RADEON R7, 10 COMPUTE CORES 4C+6G 3.10 GHz and 16 GB of RAM.

**Example 5.1.** Consider the following Lane-Emden equation with $n = 0, 1, 5$ for which the exact solution is available,

$$y''(x) + \frac{2}{x}y'(x) + (y(x))^n = 0, \quad 0 \le x \le 10,$$
$$y(0) = 1, \quad y'(0) = 0,$$

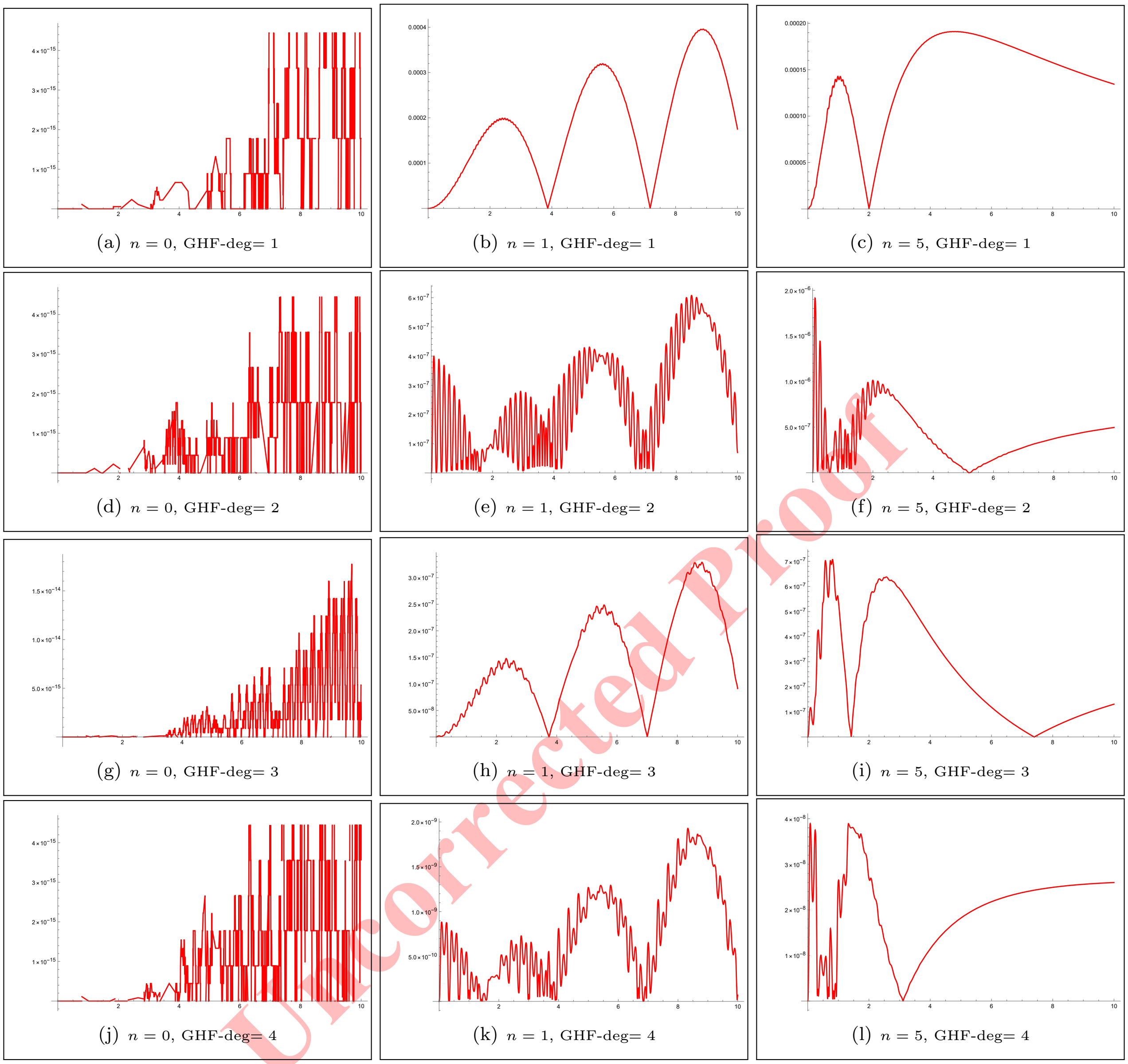
where the exact solution of the equation is:

$$y(x) = \begin{cases} 1 - \frac{x^2}{6}, & n = 0, \\ \frac{\sin x}{x}, & n = 1, \\ \frac{1}{\sqrt{1+\frac{1}{3}x^2}}, & n = 5. \end{cases}$$

The equation is solved with the mentioned values of $n$ by the proposed method with $N = 120$ (number of knots), and the GHFs of degree $1, 2, 3, 4$. The results are shown in Figure 1.

Compared to methods [1, 7, 9], our proposed method not only provides good accuracy over arbitrary interval but also achieves superior accuracy on $[0, 1]$,as demonstrated in Tables 1 and 2. A machine learning-based method using 100 test points and 10 equally spaced points in $[0, 1]$ for Padé approximation learning is presented in [9]. Ref. [1] introduces a B-splines and collocation-based method while Ref. [7] proposes a Taylor wavelet-based method. Furthermore, as shown in the examples, our method can solve the problem over any desired interval length.

Also, in Figure 3 approximate solution obtained by the proposed method with $N = 30$ and $d = 1$ and the exact solution of the Lane-Emdan equation of degree $n = 1$ are shown.



(a) $n = 0$, GHF-deg$= 1$ (b) $n = 1$, GHF-deg$= 1$ (c) $n = 5$, GHF-deg$= 1$

(d) $n = 0$, GHF-deg$= 2$ (e) $n = 1$, GHF-deg$= 2$ (f) $n = 5$, GHF-deg$= 2$

(g) $n = 0$, GHF-deg$= 3$ (h) $n = 1$, GHF-deg$= 3$ (i) $n = 5$, GHF-deg$= 3$

(j) $n = 0$, GHF-deg$= 4$ (k) $n = 1$, GHF-deg$= 4$ (l) $n = 5$, GHF-deg$= 4$

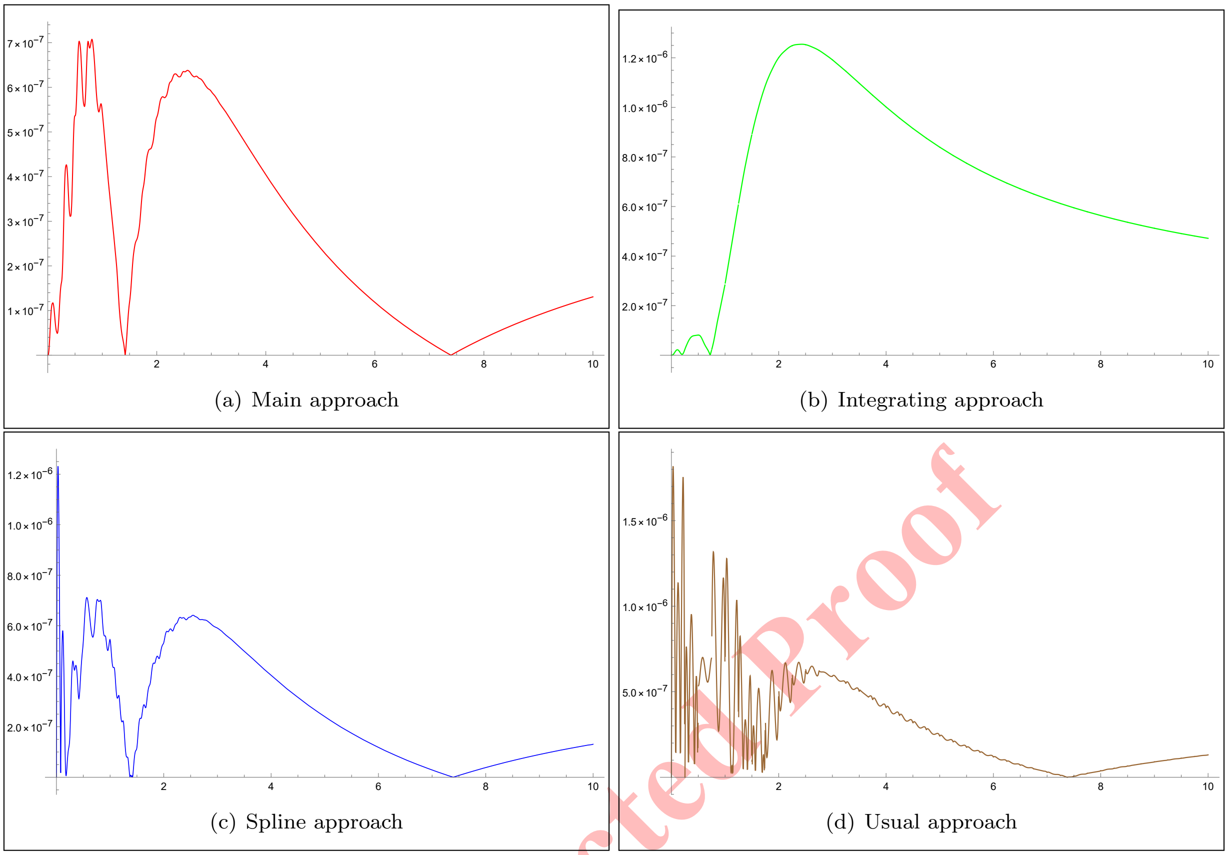


FIGURE 1. Error function for Lane-Emden equation of degree $n = 0, 1, 5$ (left to right) with GHF-degree$=1, 2, 3, 4$ (top-down) and $N = 120$ (number of knots) using the proposed method for the problems in the Example 5.1.

**Example 5.2.** In this example, we consider the Lane-Emden equations with $n = 2, 3, 4$,

$$y''(x) + \frac{2}{x}y'(x) + (y(x))^n = 0, \quad 0 \le x \le 10,$$
$$y(0) = 1, \quad y'(0) = 0,$$

which have no exact solution. The solution produced by the presented method with $N = 120$ is compared with the solution given by the `NDSOLVE` statement of WOLFRAM MATHEMATICA.



(a) Main approach (b) Integrating approach (c) Spline approach (d) Usual approach

Figure 2. Absolute value of Error function for Lane-Emden equation of degree $n = 5$ with GHF-degree=3 and $N = 120$ (number of knots) using the proposed method (main approach), spline approach of degree equal to GHFs, (3), integration approach and usual approach to produce $y(x)$.



Table 1. Comparison of the error in the proposed method with methods in [7, 9].

| $n$ | Proposed Method with $d = 3$ and $N = 12$ | Method in [9] with 10 points | Method in [7] with $M = 9$ |
|---|---|---|---|
| 0 | $1.11022 \times 10^{-16}$ | $5.1 \times 10^{-4}$ with $m = 3$, $n = 3$ | |
| 1 | $5.04346 \times 10^{-8}$ | $2.2 \times 10^{-4}$ with $m = 5$, $n = 1$ | |
| 5 | $7.0303 \times 10^{-7}$ | $6.2 \times 10^{-4}$ with $m = 5$, $n = 4$ | $6.52 \times 10^{-6}$ |

Table 2. Comparison of the error in the proposed method with methods in [1].

| $n$ | Proposed Method with $N = 84$ and $d = 4$ | Method in [1] with $N = 80$ |
|---|---|---|
| 5 | $4.29758 \times 10^{-13}$ | $7.474954 \times 10^{-11}$ |

The results, are shown in Figure 4, indicate that the solution derived from the proposed method, particularly towards the end of the solution interval, demonstrates a marked superiority over the alternative methods discussed

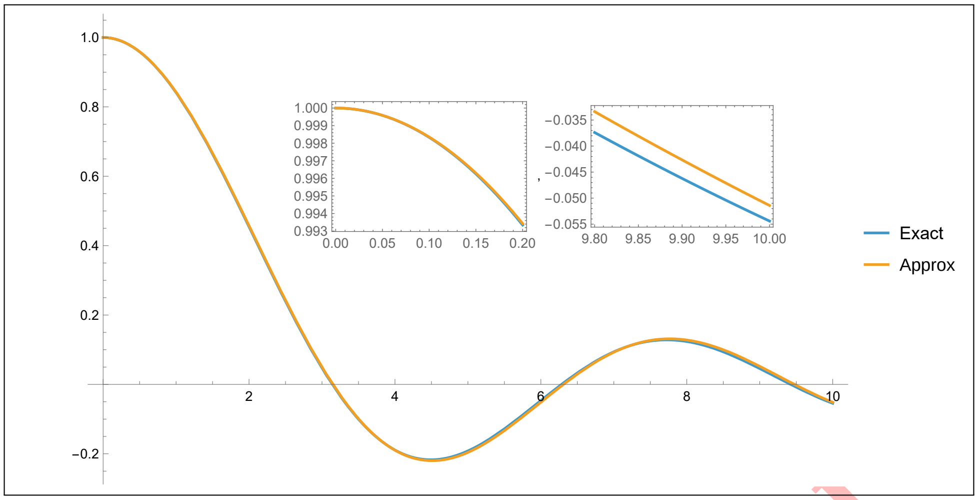



FIGURE 3. Exact and Obtained Solution for the equation with $n = 1$, $d = 1$, $N = 30$.

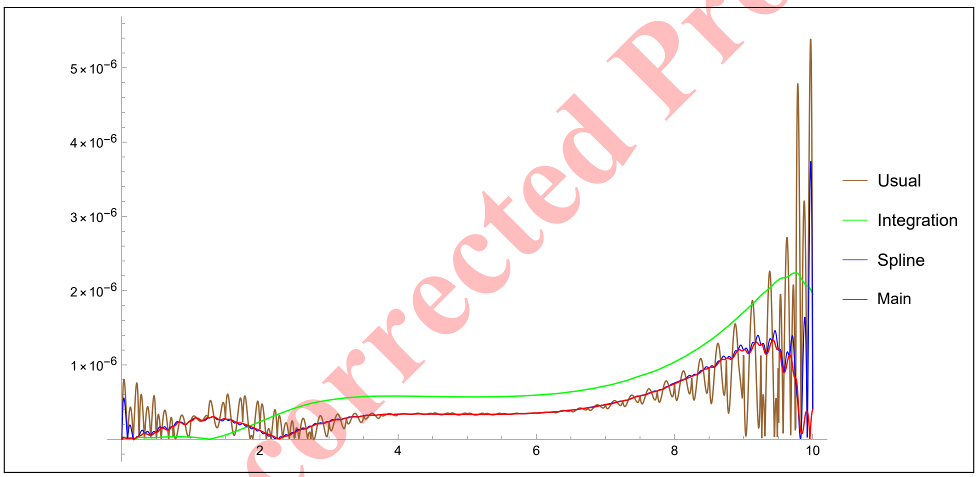



FIGURE 4. Log-plot for the absolute error of the proposed method with $N = 120$ applying to the Equation (3.1) with $n = 2$ and GHF-degree= 3 compared with the three other explained methods.

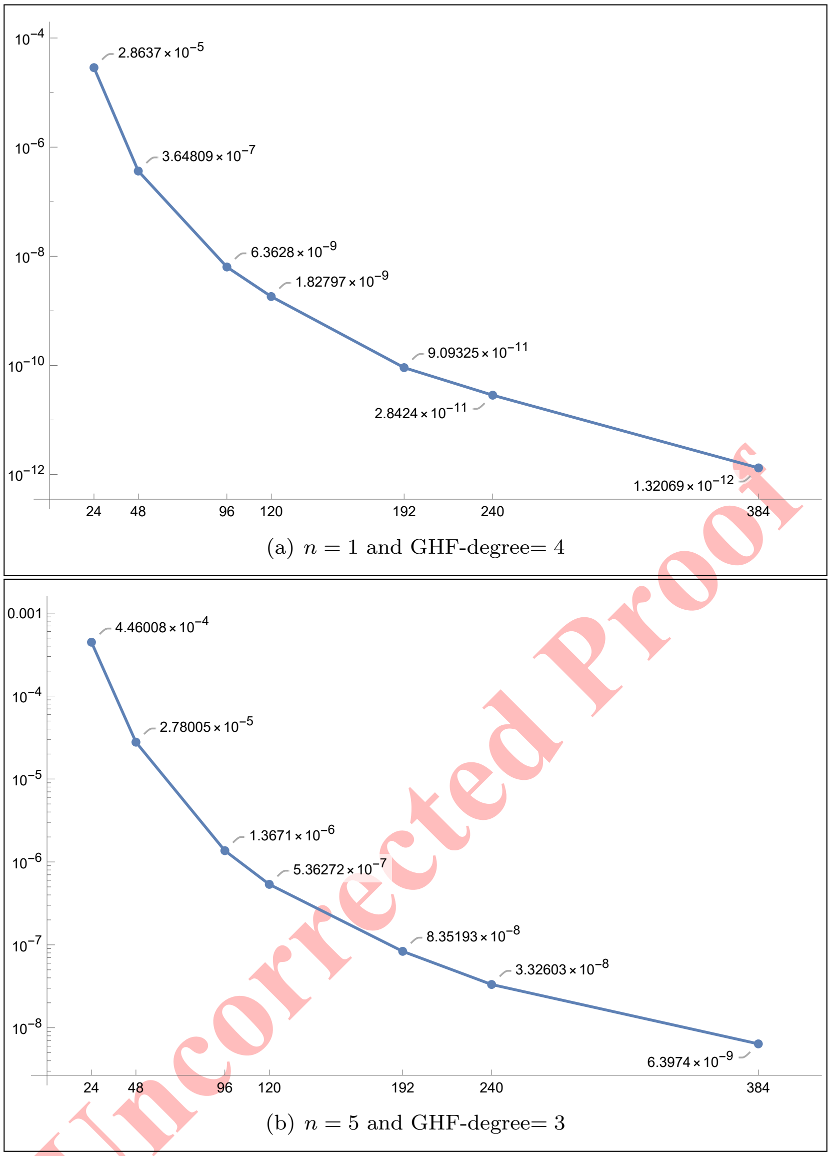


in the paper. This suggests that the method is both precise and efficient for long intervals. In contrast, the methods outlined in other publications tend to exhibit a significant increase in error when applied to longer intervals, which is why many of them restrict their focus to the interval $[0, 1]$ as the solution interval.

**Example 5.3.** In this example we solved the following equation with $n = 1, 5$ using the main approach with various number of nodes ($N = 24, 48, 96, 120, 192, 240, 384$), and the value of $\|L_\infty\|$-error in terms of $N$ are depicted in the Figure 5.

$$y''(x) + \frac{2}{x}y'(x) + (y(x))^n = 0, \quad 0 \le x \le 10,$$
$$y(0) = 1, \quad y'(0) = 0,$$

As expected and can be seen the error decreases as $N$ increases.




(a) $n = 1$ and GHF-degree= 4

(b) $n = 5$ and GHF-degree= 3



FIGURE 5. Log-plot of $\|\cdot\|_\infty$-error in terms of $N$, number of nodes, for the Equation (3.1) with 5(a) $n = 1$, GHF-degree= 4 and 5(b) $n = 5$, GHF-degree= 3 using the main approach.

## 6. Conclusion

One of the most significant advantages of the proposed method lies in its ability to calculate the solution of the equation over an interval of arbitrary length with the desired level of accuracy. This flexibility sets it apart from many previous studies [1, 9, 19, 21], where the interval $[0, 1]$ was often used due to limitations inherent in the methodologies employed. The proposed method eliminates this constraint, allowing for broader applications and accommodating problems with larger or more complex solution domains.

Another notable feature of the algorithm is its computational efficiency. Despite the potentially large values of $N$ used, which lead to the formation of sizable nonlinear systems, the method maintains a remarkably low runtime. This efficiency makes it practical and scalable for a wide range of applications. Moreover, the simplicity of the algorithm's structure ensures that it can be implemented with ease using a variety of programming languages or existing software tools, further broadening its accessibility to researchers and practitioners.

An additional noteworthy advantage of the proposed method is its adaptability to fractional-order Lane-Emden equations with only minor modifications. This adaptability extends its utility to a class of problems that are becoming increasingly important in various scientific and engineering fields. Furthermore, the method's general framework makes it a powerful tool for addressing the numerical solution of any type of nonlinear differential equation involving a single variable. This versatility highlights its potential as a universal approach in tackling challenging nonlinear problems.

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