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A robust analytical approach to study the impact of fuzzy initial conditions on time-fractional Fisher's equation

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Abstract

This manuscript investigates generalized time-fractional Fisher's equation with fuzzy initial conditions, formulated in the Caputo sense. This problem holds significant importance, as many real-world systems—particularly in biology, ecology, and epidemiology are inherently influenced by memory-dependent dynamics and uncertainties in initial data arising from imprecise measurements or incomplete information. To realistically model this uncertainty, initial conditions are represented using triangular fuzzy numbers, offering a flexible and intuitive approach. The Reduced Differential Transform Method (RDTM) is employed to address the challenges posed by fractional derivatives and fuzzy initial conditions, owing to its computational efficiency, effectiveness in handling nonlinearities, and capability to generate approximate solutions in the form of a convergent series. Two illustrative examples are presented to validate the proposed approach, with results compared against exact solutions and established numerical methods. These comparisons demonstrate the accuracy, efficacy, and robustness of RDTM in capturing the dynamics of the fuzzy time-fractional Fisher's equation. Furthermore, graphical analyzes, including surface plots for various fractional orders, illustrate how the solution evolves over time and space, and how different fuzzy initial conditions influence the behavior of the system. Overall, the findings highlight the importance of integrating fractional calculus and fuzzy logic in modeling complex systems under uncertainty.

Keywords. Burger Fisher's equation, Fuzzy fractional differential equations, Reduced differential transform method .

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Real-world phenomena are often represented by physical models that contain some uncertainty due to various factors. Moreover, fuzzy sets act as an excellent technique for modeling the ambiguity and impreciseness that characterize real-world phenomena. We apply it to domains where data contains uncertainty, such as economics, medicine, environmental sciences, physical sciences, and social sciences. In set theory the study of fuzziness was introduced by Zadeh in 1965 in an effort to resolve these challenges.

Over the past years, fractional calculus has gained popularity due to its wide range of applications in scientific and engineering fields. A variety of applications in engineering and science have made fractional calculus increasingly popular over the years. A fractional differential equation (FDE) incorporates derivatives of non-integer orders into traditional differential equations. For example, anomalous diffusion, viscoelastic materials, and complicated dynamic systems where previous states affect future behaviour are all described by FDEs. Despite their versatility and enhanced modelling capabilities, FDEs have certain limitations. One significant drawback is their inability to handle uncertainty and imprecision in the data and parameters effectively. There is often an uncertainty, vagueness, or incomplete information in real-world systems, which FDEs, based on crisp values, have difficulty incorporating. As a result, models used to deal with inherently uncertain systems may be less accurate or incomplete.

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To address this challenge, researchers have increasingly turned to fuzzy fractional differential equations (FFDEs), which combine the flexibility of fractional derivatives with the capacity of fuzzy logic to handle uncertainty. By integrating the memory-preserving properties of fractional calculus with the uncertainty-handling ability of fuzzy logic, FFDEs offer a comprehensive modeling framework for dynamic systems influenced by vagueness and ambiguity. This hybrid approach is particularly valuable in fields such as engineering, economics, and environmental science, where precise measurements are often unattainable and systems are subject to uncertain parameters. Various analytical and numerical methods have been proposed in the literature to solve FFDEs, aiming to improve the representation of real-world phenomena under uncertain conditions [1–3, 13, 18, 21, 22, 25, 28, 36, 37, 39, 42, 43].

Recent developments focus not only on improving solution accuracy but also on reducing computational complexity, ensuring the methods remain practical for large-scale systems. The increasing availability of real-time and imprecise data from complex systems further emphasizes the need for robust FFDE-based models. Consequently, the development of efficient computational techniques for solving FFDEs remains a central challenge in applied mathematics and systems modeling.

Most fractional differential equations (FDEs) in a fuzzy environment are challenging to solve accurately, making the use of analytical and numerical computational methods essential. S. Rashid, R. Ashraf, and Z. Hammouch investigate the behaviour of magneto-acoustic waves with uncertainty in plasma using the homotopy perturbation method for transformation (HPTM) [34]. To solve differential equations of both integer and non-integer order under uncertainty, Ahmad et al. present the fuzzy natural transform method (FNTM) for the first time and provide examples to show how efficiently it works [4]. M. Mossa Al-Sawalha et al. present a novel analysis of fuzzy fractional Emden-Fowler equations using a new iterative transform method [6]. Arfan et al. study establishes new results for natural transforms under uncertainty and provides an analytical solution with test cases for the 2D fuzzy wave equation with a force term utilizing the fuzzy natural transform method and homotopy perturbation approach [8]. The use of FFDEs in economic modeling and the significance of integrating uncertainty into model analysis have been highlighted by F. Babakordi et al. [10]. Narayanamoorthy and Thangapandi study a modified fractional Runge-Kutta method to solve FFDEs in the Caputo sense and provide an example to show the applicability, efficiency, and accuracy of this method [26]. Nadeem et al. apply the Sawi homotopy perturbation transform scheme (SHPTS) to study two-dimensional fuzzy fractional-order heat equations with an external diffusion source and demonstrate the efficiency by using examples and contour/surface plots [24]. With the use of the generalized integral transform and the Adomian decomposition method, Rashmid et al. investigate atmospheric internal waves using numerical solutions for the fuzzy 4^{th}-order time-fractional Boussinesq equation [35]. They highlight the applicability of this method in hydraulic engineering and provide illustrated results for fractional derivatives and uncertainty parameters. Azhar and Iqbal study the fuzzy fractional Mellin transform for fuzzy valued functions. It suggests methods, illustrated with examples, for solving FFDEs with the left-sided Riemann–Liouville fractional derivative [9]. Singh et al. presents a fuzzy power series approach combined with the Taylor exponential method to solve fuzzy fractional differential equations under the Caputo derivative. The method's validity and convergence are demonstrated through a numerical example [38]. Zhang and Niu develops a stable analytical approach for solving initial value problems in fuzzy differential equations using fuzzy system and differential envelope theories. The method is validated through dynamic system applications, showing high accuracy and sensitivity to physical parameters [44].While these methods have contributed significantly to the field, challenges remain—particularly in solving highly nonlinear FFDEs with fuzzy initial or boundary conditions. Many of the existing approaches either involve complex computations or lack general applicability across different types of nonlinear equations. Furthermore, these methods often fail to balance computational simplicity with the need for accuracy and robustness.

This study is motivated by the need for a computationally efficient, simple, and accurate approach to solving highly nonlinear fuzzy fractional differential equations. The key objective is to develop a method that can simultaneously address nonlinearity, memory effects, and uncertainty. To this end, we propose the Reduced Differential Transform Method (RDTM), a semi-analytical technique known for reducing computational complexity and achieving high accuracy without requiring discretization, linearization, or perturbation. While the classical Differential Transform Method (DTM) has been widely applied in solving differential equations, the reduced version improves upon it by minimizing unnecessary computations and enhancing convergence. By extending RDTM to the fuzzy fractional domain, this

research introduces a novel computational framework for solving FFDEs. This method offers a practical alternative to existing techniques by simplifying the solution process and preserving the accuracy required in modeling real-world systems characterized by both fuzziness and memory effects. The following overview provides a brief roadmap of the paper, outlining the structure and main contributions of each section. Section 2 presents the basic definitions and preliminaries of fuzzy set theory, fuzzy numbers, and fractional calculus, which are essential for understanding the fuzzy fractional framework. Section 3 explains the key ideas and how to use the proposed Reduced Differential Transform Method (RDTM) for solving FFDEs, including how to deal with fuzzy initial conditions. In section 4, a numerical example is provided to illustrate the applicability and accuracy of the proposed method. Section 5 discusses the results obtained, including comparisons with existing techniques to highlight the advantages of RDTM. Finally, section 6 concludes the paper with a summary of findings and future research directions. The key achievements of this study are summarized below:

- Developed a robust application of the Reduced Differential Transform Method (RDTM) to solve fuzzy fractional differential equations.
- Addressed the challenges of highly nonlinear systems with fuzzy initial conditions.
- Demonstrated the computational efficiency and accuracy of the method through a practical numerical example.
- Provided a methodological advancement for modeling uncertainty and memory effects simultaneously.
- Contributed to the broader field of fuzzy fractional calculus with an adaptable and efficient solution technique.

2. Preliminaries

The following section describes the definition, fundamental properties, and attributes of fuzzy valued functions, fuzzy numbers, and fuzzy derivatives that can be applied to later calculations.

Definition 2.1. Let $F : \Re \to [0\ 1]$ be a function defined on set of real number $\Re$. A fuzzy set is defined as set of ordered pairs of the form $\widetilde{F} = \{(x,\, F(x)) :\ x \in \Re\}$ where denotes membership value and $F \in [0\ 1]$.

Definition 2.2. Fuzzy number are defined as a fuzzy sets of the form $\widetilde{\nu} : \Re \to [0\ 1]$ satisfying the following properties:

(1) $\widetilde{\nu}(\omega)$ denotes upper semi continuous and is equal to zero outside some interval $[c\ d]$.
(2) $\exists$ real number υ_1, υ_2 such that $\varepsilon_1 \leq \upsilon_1 \leq \upsilon_2 \leq \varepsilon_2$ and $\widetilde{\nu}(\omega)$ is monotonic increasing and decreasing on $[\varepsilon_1\ \upsilon_1]$ and $[\upsilon_2\ \varepsilon_2]$ respectively.
(3) $\widetilde{\nu}(\omega) = 1$, $\upsilon_1 \leq \omega \leq \upsilon_2$.

Definition 2.3. The β level set of a fuzzy number $\widetilde{\nu}$ is defined as $\widetilde{\nu}_\beta = \{\omega \in X :\ \widetilde{\nu}(\omega) \geq \beta\}$.

Definition 2.4. Let E^1 be the fuzzy number space and $\widetilde{\nu} \in E^1$ be a fuzzy number which is represented in the parametric form $\left[\underline{\nu_\beta}\,,\, \overline{\nu_\beta}\right]$ of function $\underline{\nu_\beta}$ and ν_β for any $\beta \in [0\ 1]$ which satisfied the properties that $\underline{\nu_\beta}$ and ν_β is a bounded non-decreasing and increasing left continuous function in $(0\ 1]$, $\underline{\nu_\beta} \leq \overline{\nu_\beta}$ and $\overline{\nu_\beta}$ are right continuous at $\beta = 0$.
Conversely, if the pair of functions $\upsilon_1(\beta)$ and $\upsilon_2(\beta)$ satisfy the above condition then $\exists$ unique $\widetilde{\nu_\beta} \in E^1$ such that $[\widetilde{\nu}]_\beta = [\upsilon_1(\beta), \upsilon_2(\beta)]$ for each $\beta \in [0\ 1]$. Define $D : E^1 \times E^1 \to R_+ \cup \{0\}$ by $D(\widetilde{\nu_1}, \widetilde{\nu_2}) = \sup \max\left\{\left|\underline{\nu_1}_\beta - \underline{\nu_2}_\beta\right|, \left|\overline{\nu_1}_\beta - \overline{\nu_2}_\beta\right|\right\}$, where $[\widetilde{u}]_\beta = \left[\underline{u_\beta}\,,\, \overline{u_\beta}\right], [\widetilde{v}]_\beta = \left[\underline{\nu_\beta}\,,\, \overline{\nu_\beta}\right]$. $D(\widetilde{\nu_1}, \widetilde{\nu_2})$ is called the distance between fuzzy numbers $\widetilde{\nu_1}$ and $\widetilde{\nu_2}$. From the result, we know that

(1) $\left(E^1, D\right)$ represent a complete metric space.
(2) $D(\widetilde{\nu}_1 + \widetilde{\nu}_2, \widetilde{\nu}_3 + \widetilde{\nu}_2) = D(\widetilde{\nu}_1, \widetilde{\nu}_3)$.
(3) $D(\kappa.\widetilde{\nu}_1, \kappa.\widetilde{\nu}_2) = |\kappa| D(\widetilde{\nu}_1, \widetilde{\nu}_2)$, $\kappa \in R$ where $\widetilde{\nu}_1, \widetilde{\nu}_2$ and $\widetilde{\nu}_3 \in E^1$.

Definition 2.5. As discussed in Definition 2.4, fuzzy numbers can be converted into an interval form. Therefore, for any arbitrary fuzzy number $\widetilde{\wp_1} = \left[\underline{\wp_1}(\beta)\,,\, \overline{\wp_1}(\beta)\right]$ and $\widetilde{\wp_2} = \left[\underline{\wp_2}(\beta)\,,\, \overline{\wp_2}(\beta)\right]$, the following fundamental properties holds:

(1) $\widetilde{\wp_1} = \widetilde{\wp_2}$ iff $\underline{\wp_1}(\beta) = \underline{\wp_2}(\beta)$ and $\overline{\wp_1}(\beta) = \overline{\wp_2}(\beta)$.
(2) $\widetilde{\wp_1} + \widetilde{\wp_2} = \left[\underline{\wp_1}(\beta) + \underline{\wp_2}(\beta)\,,\, \overline{\wp_1}(\beta) + \overline{\wp_2}(\beta)\right]$.

Definition 2.6. A function $\zeta : [a\, b] \to E$ is called to be fuzzy continuous at point $\omega_0 \in [a\ b]$ if for any $\varepsilon > 0$ and $\exists\ \delta > 0$ such that $D(\zeta(\omega), \zeta(\omega_0)) < \varepsilon$ whenever $|\omega - \omega_0| < \delta$.

Definition 2.7. For any two fuzzy numbers i.e. $\wp_1, \wp_2 \in E$. If there exist $\gamma \in E$ such that $\wp_1 = \wp_2 + \gamma$. Then γ is called H difference of $\wp_1$ and $\wp_2$, is denoted by $\wp_1 \Theta \wp_2$.

Definition 2.8. A function $\zeta : [a\, b] \to E$ is called to be generalized differentiable at point $\upsilon_0 \in (a\ b)$. If there exist an element $\zeta(\upsilon_0) \in E$ such that

(1) For $\forall\, \ell > 0$ (sufficiently small values) ,$\exists\ \zeta(\upsilon_0 + \ell) \Theta \zeta(\upsilon_0)$ and $\zeta(\upsilon_0) \Theta \zeta(\upsilon_0 - \ell)$ following holds :

$$\lim_{\ell \to 0} \frac{\zeta(\upsilon_0 + \ell) \Theta \zeta(\upsilon_0)}{\ell} = \lim_{\ell \to 0} \frac{\zeta(\upsilon_0) \Theta \zeta(\upsilon_0 - \ell)}{\ell} = \zeta'(\upsilon_0).$$

(2) For $\forall\, \ell > 0$ (sufficiently small values) ,$\exists\ \zeta(\upsilon_0) \Theta \zeta(\upsilon_0 + \ell))$ and $\zeta(\upsilon_0 - \ell) \Theta \zeta(\upsilon_0)$ following holds :

$$\lim_{\ell \to 0} \frac{\zeta(\upsilon_0) \Theta \zeta(\upsilon_0 + \ell)}{-\ell} = \lim_{\ell \to 0} \frac{\zeta(\upsilon_0 - \ell) \Theta \zeta(\upsilon_0)}{-\ell} = \zeta'(\upsilon_0).$$

(3) For $\forall\, \ell > 0$ (sufficiently small values), $\exists\ \zeta(\upsilon_0 + \ell) \Theta \zeta(\upsilon_0)$ and $\zeta(\upsilon_0 - \ell) \Theta \zeta(\upsilon_0)$ following holds :

$$\lim_{\ell \to 0} \frac{\zeta(\upsilon_0 + \ell) \Theta \zeta(\upsilon_0)}{\ell} = \lim_{\ell \to 0} \frac{\zeta(\upsilon_0 - \ell) \Theta \zeta(\upsilon_0)}{-\ell} = \zeta'(\upsilon_0).$$

(4) For $\forall\, \ell > 0$ (sufficiently small values), $\exists\ \zeta(\upsilon_0) \Theta \zeta(\upsilon_0 + \ell)$ and $\zeta(\upsilon_0) \Theta \zeta(\upsilon_0 - \ell)$ following holds:

$$\lim_{\ell \to 0} \frac{\zeta(\upsilon_0) \Theta \zeta(\upsilon_0 + \ell)}{-\ell} = \lim_{\ell \to 0} \frac{\zeta(\upsilon_0 - \ell) \Theta \zeta(\upsilon_0)}{\ell} = \zeta'(\upsilon_0).$$

Definition 2.9. Consider a fuzzy valued function $\zeta : [a\, b] \to E$ and $\zeta(\omega_0, \beta) = \left[\underline{\zeta}(\omega_0, \beta), \overline{\zeta}(\omega_0, \beta)\right]$ where $\omega_0 \in [a\, b]$ and $\beta \in [0\ \ 1]$,then

(1) ζ is (1) differentiable function, then $\underline{\zeta}(\omega_0, \beta)$ and $\overline{\zeta}(\omega_0, \beta)$ are differentiable function, and

$$\zeta'(\omega_0, \beta) = \left[\underline{\zeta'}(\omega_0, \beta), \overline{\zeta'}(\omega_0, \beta)\right].$$

(2) ζ is (1) differentiable function, then $\underline{\zeta}(\omega_0, \beta)$ and $\overline{\zeta}(\omega_0, \beta)$ are differentiable function, and

$$\zeta'(\omega_0, \beta) = \left[\overline{\zeta'}(\omega_0, \beta), \underline{\zeta'}(\omega_0, \beta)\right].$$

Definition 2.10. The Riemann-Liouville fractional integral of order χ of fuzzy number valued function ζ, based on its $\beta-$ cut representations, can be expressed as $\left[J^{\chi}\widetilde{\zeta}(\omega;\beta)\right] = \left[J^{\chi}\underline{\zeta}(\omega;\beta)\ \ J^{\chi}\overline{\zeta}(\omega;\beta)\right]$, $\omega > 0$ where $J^{\chi}\underline{\zeta}(\omega;\beta) = \frac{1}{\Gamma(\chi)} \int_0^{\omega} (\omega - \tau)^{\chi - 1} \underline{\zeta}(\tau)\, d\tau$, $\omega > 0$, and $J^{\chi}\overline{\zeta}(\omega;\beta) = \frac{1}{\Gamma(\chi)} \int_0^{\omega} (\omega - \tau)^{\chi - 1} \overline{\zeta}(\tau)\, d\tau$, $\omega > 0$.

Definition 2.11. Consider a fuzzy valued function $\widetilde{\zeta}(\omega;\beta)$ and $\widetilde{\zeta}(\omega,\beta) = \left[\underline{\zeta}(\omega,\beta), \overline{\zeta}(\omega,\beta)\right]$, for $\beta \in [0\ 1]$,$0 < \chi < 1$ and $\omega \in (a\ b)$.

(1) If $\widetilde{\zeta}(\omega,\beta)$ is a Caputo-type fuzzy differential function in the first form , then $D^{\chi}\widetilde{\zeta}(\omega;\beta) = \left[D^{\chi}\underline{\zeta}(\omega;\beta)\ \ \ D^{\chi}\overline{\zeta}(\omega;\beta)\ \right]$.

(2) If $\widetilde{\zeta}(\omega,\beta)$ is a Caputo-type fuzzy differential function in the second form , then $D^{\chi}\widetilde{\zeta}(\omega;\beta) = \left[D^{\chi}\overline{\zeta}(\omega;\beta)\ \ D^{\chi}\underline{\zeta}(\omega;\beta)\right]$, where $D^{\chi}\underline{\zeta}(\omega;\beta) = \frac{1}{\Gamma(m-\chi)} \int_0^{\omega} (\omega - \tau)^{m-\chi-1} \underline{\zeta^m}(\tau)\, d\tau,\ m - 1 < \chi < m,$

$D^{\chi}\widetilde{\zeta}(\omega;\beta) = \frac{1}{\Gamma(m-\chi)} \int_0^{\omega} (\omega - \tau)^{m-\chi-1} \overline{\zeta}(\tau)\, d\tau,\ m - 1 < \chi < m.$

3. Reduced Differential Transform Method for Fuzzy Problems

Some crucial mathematical definitions and preliminaries of the proposed computational method required for better comprehension is discussed in this section 3. FRDTM concept has been derived from the two-dimensional Taylor's series expansion with respect to specific variable ω and τ. Consider a function $\nu(\omega,\tau)$ and assume that $\nu(\omega,\tau)=h(\omega)\,l(\tau)$ [30]. One-dimensional differential transform method (ODDTM) definition states that the function $\nu(\omega,\tau)$ can be expressed as:

$$\nu(\omega,\tau)=\sum_{\kappa=0}^{\infty}H_{\kappa}\omega^{\kappa}\sum_{\eta=0}^{\infty}L_{\eta}\tau^{k}=\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}V_{\kappa,\eta}(\kappa,\eta)\,\omega^{\kappa}\tau^{\eta}, \tag{3.1}$$

where $V_{\kappa,\eta}=H_{\kappa}L_{\eta}$ is called the spectrum of $\nu(\omega,\tau)$. The following describes the fundamental concept behind RDTM [29, 31, 32]:

Definition 3.1. If $\nu(\omega,\tau)$ is a continuously differentiable function in the domain of interest, then the transform or spectrum of $\nu(\omega,\tau)$ with respect to τ at τ_0 is defined as

$$V_{\kappa}(\omega)=\frac{1}{\kappa!}\left[\frac{\partial^{\kappa}}{\partial\tau^{\kappa}}\nu(\omega,\tau)\right]_{\tau=\tau_0}, \tag{3.2}$$

where $\nu(\omega,\tau)$ is the original function and $V_{\kappa}(\omega)$ is the reduced differential transform of $\nu(\omega,\tau)$ respectively.

Definition 3.2. The inverse Reduced differential transform of $V_{\kappa}(\omega)$ is defined as:

$$\nu(\omega,\tau)=\sum_{\kappa=0}^{\infty}V_{\kappa}(\omega)(\tau-\tau_0)^{\kappa}. \tag{3.3}$$

From Eqs.(3.2) and (3.3), we get

$$\nu(\omega,\tau)=\sum_{\kappa=0}^{\infty}\left[\frac{1}{\kappa!}\left[\frac{\partial^{\kappa}}{\partial\tau^{\kappa}}\nu(\omega,\tau)\right]_{\tau=\tau_0}\right](\tau-\tau_0)^{\kappa}. \tag{3.4}$$

Table 1 below lists the basic RDTM operations that can be obtained using Eqs. (3.2) and (3.3).

Table 1. Basic Operation of RDTM.

Original Function	**Transformed Function**
$\theta_1(\xi,\Upsilon)\pm\theta_2(\xi,\Upsilon)$	$\Theta_1(\xi)\pm\Theta_2(\xi)$
$\lambda\theta(\xi,\Upsilon)$	$\lambda\Theta_{\rho}(\xi)$
$\frac{\partial\theta(\xi,\Upsilon)}{\partial\xi}$	$\frac{\partial\Theta_{\rho}(\xi)}{\partial\xi}$
$\frac{\partial\theta(\xi,\Upsilon)}{\partial t}$	$(\vartheta+1)\,\Theta_{\vartheta+1}(\xi)$
$\frac{\partial\theta(\xi,\Upsilon)}{\partial\xi\,\partial\Upsilon}$	$(\vartheta+1)\,\frac{\partial\Theta_{\vartheta+1}(\xi)}{\partial\xi}$
$\theta_1(\xi,\Upsilon)\,\theta_2(\xi,\Upsilon)$	$\sum_{\eta=0}^{\ell}\Theta_{1,\eta}(\xi)\,\Theta_{2,\ell-\eta}(\xi)$
$\xi^{\mathrm{A}}\Upsilon^{\mathrm{B}}$	$\xi^{\mathrm{A}}\delta(h-\mathrm{B})$ where $\delta(h-\mathrm{B})=\begin{cases}1 & ,h=\mathrm{B}\\ 0 & ,h\neq\mathrm{B}\end{cases}$

3.1. **Convergence Analysis of Proposed Method.** In terms of operator, consider the nonlinear PDE

$$\Im\left[\theta\left(\chi,\Upsilon\right)\right]+\Re\left[\theta\left(\chi,\Upsilon\right)\right]+\mathrm{N}\left[\theta\left(\chi,\Upsilon\right)\right]=\phi\left(\chi,\Upsilon\right), \tag{3.5}$$

with initial condition

$$\theta\left(x,0\right)=\varphi\left(x\right), \tag{3.6}$$

where $\phi\left(\chi,\Upsilon\right)$ is an inhomogeneous term, N represents nonlinear operator, $L=\frac{\partial}{\partial\Upsilon}$, R is a first-order operator with a partial derivative. In reference [23], the convergence of the series solution derived by RDTM is explored. We revisit the [23] theorems that guarantee the convergence of the series solution of Eq. (3.5).

Theorem 3.3. *If* $\zeta_k\left(\wp,\ell\right)=V_k\left(\wp\right)\left(\ell-\ell_0\right)^k$, *then the series solution* $\sum_{i=0}^{n}\zeta_i\left(\wp,\ell\right)$ *for Eq. (??),* $\forall\, i\in N\cup\{0\}$ *(I) is convergent if there exist* $0<\eta<1$ *such that* $\left\|\zeta_{i+1}\right\|\leq\eta\left\|\zeta_i\right\|$,*(ii) is divergent if there exist* $\eta>1$ *such that* $\left\|\zeta_{i+1}\right\|\geq\eta\left\|\zeta_i\right\|$. *The truncation error of the series Eq. (3.5), which is a specific case of Banach's fixed point theorem.*

Theorem 3.4. *Suppose* $\sum_{i=0}^{n}\zeta_i\left(\wp,\ell\right)$ *is required series solution, where* $\zeta_l\left(\wp,\ell\right)=V_k\left(\wp\right)\left(\ell-\ell_0\right)^l$, *converges to* $\varepsilon\left(\wp,\ell\right)$.*If* $\sum_{i=0}^{n}\zeta_i\left(\wp,\ell\right)$ *is the truncated series used to approximate the solution and then estimated maximum absolute truncated error is as* $\left\|\varepsilon\left(\wp,\ell\right)-\sum_{i=0}^{m}\zeta_i\left(\wp,\ell\right)\right\|\leq\frac{1}{1-\eta}\eta^{m+1}\left\|\zeta_0\right\|$.

In [23], the Theorems 3.3 and 3.4 can be verified. It is determined from Theorems 3.3 and 3.4 that when there exists $0<\eta<1$ such that $\left\|\zeta_{l+1}\right\|\leq\eta\left\|\zeta_l\right\|$, for $\forall\, l\in N\cup\{0\}$, the series solution for non-linear Eq. (3.5) derived using RDTM converges to an exact solution. Moreover, $\left\|\varepsilon\left(\omega,t\right)-\sum_{i=0}^{m}\zeta_i\left(\omega,\tau\right)\right\|\leq\frac{1}{1-\eta}\eta^{m+1}\left\|\zeta_0\right\|$ represents the maximum estimated absolute truncated error.

4. Illustrations

In this section, we demonstrate the applicability of the proposed method to a class of general fractional Fisher equations with fuzzy initial conditions and obtain the series solution using the MATLAB software package.

Example 4.1. Consider the nonlinear time-fractional Fisher's equation

$$\frac{\partial^{\gamma}\widetilde{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\xi^{\gamma}}=\frac{\partial^2\widetilde{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\zeta^2}+6\widetilde{\nu}\left(\zeta,\xi,\varsigma\right)\left(1-\widetilde{\nu}\left(\zeta,\xi,\varsigma\right)\right), \tag{4.1}$$

with fuzzy initial condition

$$\widetilde{\nu}\left(\zeta,0\right)=\widetilde{\mathrm{A}}\frac{1}{\left(1+e^{\zeta}\right)^2}, \tag{4.2}$$

where $\widetilde{\mathrm{A}}$ is a fuzzy triangular number given by $[-1,0,1]$.

It is also represented as a $\varsigma-$ cut, i.e., $\left[\underline{\mathrm{A}},\overline{\mathrm{A}}\right]=[\varsigma-1,1-\varsigma]$. The parametric form of Eq. (??) is

$$\frac{\partial^{\gamma}\underline{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\xi^{\gamma}}=\frac{\partial^2\underline{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\zeta^2}+6\underline{\nu}\left(\zeta,\xi,\varsigma\right)\left(1-\underline{\nu}\left(\zeta,\xi,\varsigma\right)\right), \tag{4.3}$$

$$\frac{\partial^{\gamma}\overline{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\xi^{\gamma}}=\frac{\partial^2\overline{\nu}\left(\zeta,\xi,\varsigma\right)}{\partial\zeta^2}+6\overline{\nu}\left(\zeta,\xi,\varsigma\right)\left(1-\overline{\nu}\left(\zeta,\xi,\varsigma\right)\right), \tag{4.4}$$

for $\varsigma\in[0\ 1]$. Applying RDTM to Eqs. (4.1)–(4.2), we get the following recurrence formula

$$\frac{\Gamma\left(\gamma\left(\iota+1\right)+1\right)}{\Gamma\left(\gamma\iota+1\right)}\underline{V}_{\iota+1}\left(\zeta\right)=\frac{\partial^2\underline{V}_{\iota}\left(\zeta,\varsigma\right)}{\partial\zeta^2}+6\underline{V}_{\iota}\left(\zeta,\varsigma\right)-6\sum_{m=1}^{\iota}\underline{V}_m\left(\zeta,\varsigma\right)\underline{V}_{\iota-m}\left(\zeta,\xi,\varsigma\right), \tag{4.5}$$

$$\frac{\Gamma(\gamma(\iota+1)+1)}{\Gamma(\gamma\iota+1)}\overline{V}_{\iota+1}(\zeta)=\frac{\partial^2\overline{V}_{\iota}(\zeta,\varsigma)}{\partial\zeta^2}+6\overline{V}_{\iota}(\zeta,\varsigma)-6\sum_{m=1}^{\iota}\overline{V_m}(\zeta,\varsigma)\overline{V}_{\iota-m}(\zeta,\varsigma)\,, \tag{4.6}$$

where $\left[\underline{V_l}(\zeta;\varsigma),\overline{V_\iota}(\zeta;\varsigma)\right]$ is a transform function of $\widetilde{\nu}(\zeta,\xi,\varsigma)$. Applying RDTM to the initial condition, transform initial conditions are given by

$$\underline{V}_0(\zeta,\varsigma)=(\varsigma-1)\frac{1}{(1+e^{\varsigma})^2}\,, \tag{4.7}$$

$$\overline{V}_0(\zeta,\varsigma)=(1-\varsigma)\frac{1}{(1+e^{\varsigma})^2}\,. \tag{4.8}$$

The coefficient of the series solution for $\iota=0,1,2,3,.....$ can be found by using the recurrence formula Eqs. (4.3)-(4.4) with the transformed initial condition in Eqs. (4.5)-(4.6). To facilitate the reader's understanding, selected coefficients from the series solution are computed and provided, demonstrating the application of the proposed approach.

For $\iota=0$, Eqs.(4.3)-(4.4) reduces to

$$\frac{\Gamma(\gamma+1)}{\Gamma(1)}\underline{V}_1(\zeta)=\frac{\partial^2\underline{V}_0(\zeta,\varsigma)}{\partial\zeta^2}+6\underline{V}_0(\zeta,\xi,\varsigma)-6\underline{V}_0(\zeta,\xi,\varsigma)\underline{V}_0(\zeta,\xi,\varsigma)\,, \tag{4.9}$$

$$\frac{\Gamma(\gamma+1)}{\Gamma(1)}\overline{V_1}(\zeta)=\frac{\partial^2\overline{V}_0(\zeta,\xi,\varsigma)}{\partial\zeta^2}+6\overline{V}_0(\zeta,\xi,\varsigma)-6\overline{V_0}(\zeta,\xi,\varsigma)\overline{V_0}(\zeta,\xi,\varsigma)\,. \tag{4.10}$$

Substituting value from Eqs.(4.5)-(4.6) in Eqs.(4.7)-(4.8), we get

$$\underline{V_1}(\zeta,\varsigma)=\frac{\Gamma(1)}{\Gamma(\gamma+1)}\left[\frac{2(\varsigma-1)\left(5e^{2\zeta}-3\varsigma+5e^{\zeta}+6\right)}{(e^{\zeta}+1)^4}\right], \tag{4.11}$$

$$\overline{V_1}(\zeta,\varsigma)=-\frac{\Gamma(1)}{\Gamma(\gamma+1)}\left[\frac{2(\varsigma-1)\left(5e^{2\zeta}+3\varsigma+5e^{\xi}\right)}{(e^{\xi}+1)^4}\right]. \tag{4.12}$$

For $\iota=1$, Eqs.(4.3)-(4.4) reduces to

$$\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}\underline{V_2}(\zeta)=\frac{\partial^2\underline{V_1}(\zeta,\xi,\varsigma)}{\partial\zeta^2}+6\underline{V_1}(\zeta,\xi,\varsigma)-6\sum_{m=1}^{1}\underline{V_l}(\zeta,\xi,\varsigma)\underline{V_{1-m}}(\zeta,\xi,\varsigma)\,, \tag{4.13}$$

$$\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}\overline{V_2}(\zeta)=\frac{\partial^2\overline{V_1}(\zeta,\xi,\varsigma)}{\partial\zeta^2}+6\overline{V_1}(\zeta,\xi,\varsigma)-6\sum_{m=1}^{1}\overline{V_m}(\zeta,\xi,\varsigma)\overline{V}_{1-m}(\zeta,\xi,\varsigma)\,. \tag{4.14}$$

Substituting value from Eqs.(4.5), (4.6), (4.11), and (4.12) in Eqs.(4.13)-(4.14), we get

$$\underline{V_2}(\zeta,\varsigma)=\frac{2(\varsigma-1)}{\Gamma(2\gamma+1)}\left[\frac{252e^{2\zeta}-126m+75e^{3\zeta}+50e^{4\zeta}+143e^{\zeta}-126me^{2\zeta}-84me^{\zeta}+36m^2+108}{(e^{\zeta}+1)^6}\right], \tag{4.15}$$

$$\overline{V_2}(\zeta,\varsigma)=\frac{2(\varsigma-1)}{\Gamma(2\gamma+1)}\left[\frac{75e^{3\zeta}-18m+50e^{4\zeta}-25e^{\zeta}-126me^{2\zeta}+84me^{\zeta}+36m^2}{(e^{\zeta}+1)^6}\right]. \tag{4.16}$$

Similarly, MATLAB software can be employed to compute the remaining coefficients of the series solution efficiently. By applying the inverse differential transform, as given in Eq. (3.4), the analytical approximate solution is obtained in the form of a power series as follows:

$$\underline{\nu}(\zeta,\xi,\varsigma)=(\varsigma-1)\frac{1}{(1+e^{\varsigma})^2}+\frac{\Gamma(1)}{\Gamma(\gamma+1)}\left[\frac{2(\varsigma-1)\left(5e^{2\varsigma}-3\varsigma+5e^{\varsigma}+6\right)}{(e^{\varsigma}+1)^4}\right]\xi$$

$$+\frac{2(\varsigma-1)}{\Gamma(2\gamma+1)}\left[\frac{252e^{2\varsigma}-126m+75e^{3\varsigma}+50e^{4\varsigma}+143e^{\varsigma}-126me^{2\varsigma}-84me^{\varsigma}+36m^2+108}{(e^{\varsigma}+1)^6}\right]\xi^2+\ldots \quad (4.17)$$

$$\overline{\nu}(\zeta,\xi,\varsigma)=(1-\varsigma)\frac{1}{(1+e^{\varsigma})^2}-\frac{\Gamma(1)}{\Gamma(\gamma+1)}\left[\frac{2(\varsigma-1)\left(5e^{2\zeta}+3\varsigma+5e^{\xi}\right)}{(e^{\xi}+1)^4}\right]\xi$$
$$+\frac{2(\varsigma-1)}{\Gamma(2\gamma+1)}\left[\frac{75e^{3\zeta}-18m+50e^{4\zeta}-25e^{\zeta}-126me^{2\zeta}+84me^{\zeta}+36m^2}{(e^{\zeta}+1)^6}\right]\xi^2+\ldots \quad (4.18)$$

Example 4.2. Consider the nonlinear time-fractional Fisher's equation

$$\frac{\partial^{\gamma}\widetilde{\nu}(\zeta,\xi,\varsigma)}{\partial\xi^{\gamma}}=\frac{\partial^2\widetilde{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\widetilde{\nu}(\zeta,\xi,\varsigma)\frac{\partial\widetilde{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta}-\widetilde{\nu}(\zeta,\xi,\varsigma)(1-\widetilde{\nu}(\zeta,\xi,\varsigma)),\ 0<\gamma\leq 1, \quad (4.19)$$

with fuzzy initial condition

$$\widetilde{\nu}(\zeta,0)=\widetilde{\mathrm{A}}\left(\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\zeta}{4}\right)\right), \quad (4.20)$$

where $\widetilde{\mathrm{A}}$ is a fuzzy triangular number given by $[-1,0,1]$.

It is also represented as $\varsigma-$ cut i.e., $\left[\underline{\mathrm{A}},\overline{\mathrm{A}}\right]=[\varsigma-1,1-\varsigma]$. The parametric form of Eq. (4.19) is

$$\frac{\partial^{\gamma}\underline{\nu}(\zeta,\xi,\varsigma)}{\partial\xi^{\gamma}}=\frac{\partial^2\underline{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\underline{\nu}(\zeta,\xi,\varsigma)\frac{\partial\underline{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta}-\underline{\nu}(\zeta,\xi,\varsigma)(\underline{\nu}(\zeta,\xi,\varsigma)-1),\ 0<\gamma\leq 1, \quad (4.21)$$

$$\frac{\partial^{\gamma}\overline{\nu}(\zeta,\xi,\varsigma)}{\partial\xi^{\gamma}}=\frac{\partial^2\overline{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\overline{\nu}(\zeta,\xi,\varsigma)\frac{\partial\overline{\nu}(\zeta,\xi,\varsigma)}{\partial\zeta}+6\overline{\nu}(\zeta,\xi,\varsigma)(\overline{\nu}(\zeta,\xi,\varsigma)-1), \quad (4.22)$$

for $\varsigma\in[0\ 1]$. Applying RDTM to Eqs. (4.21) and (4.22), we get the following recurrence formula:

$$\frac{\Gamma(\gamma(\iota+1)+1)}{\Gamma(\gamma\iota+1)}\underline{V}_{\iota+1}(\zeta)=\frac{\partial^2\underline{V}_{\iota}(\zeta,\varsigma)}{\partial\zeta^2}-\sum_{m=1}^{\iota}\underline{V}_m(\zeta,\varsigma)\frac{\partial^2\underline{V}_{\iota-m}(\zeta,\varsigma)}{\partial\zeta^2}+\underline{V}_{\iota}(\zeta,\varsigma)-\sum_{m=1}^{\iota}\underline{V}_m(\zeta,\varsigma)\underline{V}_{\iota-m}(\zeta,\varsigma), \quad (4.23)$$

$$\frac{\Gamma(\gamma(\iota+1)+1)}{\Gamma(\gamma\iota+1)}\overline{V}_{\iota+1}(\zeta)=\frac{\partial^2\overline{V}_{\iota}(\zeta,\varsigma)}{\partial\zeta^2}-\sum_{m=1}^{\iota}\overline{V}_m(\zeta,\varsigma)\frac{\partial\overline{V}_{\iota-m}(\zeta,\varsigma)}{\partial\zeta}+\overline{V}_{\iota}(\zeta,\varsigma)-\sum_{m=1}^{\iota}\overline{V}_m(\zeta,\varsigma)\overline{V}_{\iota-m}(\zeta,\varsigma), \quad (4.24)$$

where $\left[\underline{V}_l(\zeta;\varsigma),\overline{V}_{\iota}(\zeta;\varsigma)\right]$ is transform function of $\widetilde{\nu}(\zeta,\xi,\varsigma)$. Applying RDTM to initial condition, transform initial condition is given by

$$\underline{V}_0(\zeta,\varsigma)=(\varsigma-1)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\zeta}{4}\right)\right), \quad (4.25)$$

$$\overline{V}_0(\zeta,\varsigma)=(1-\varsigma)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\zeta}{4}\right)\right). \quad (4.26)$$

Using the transformed initial condition Eqs. (4.25)-(4.26) and the recurrence formula Eqs. (4.23)-(4.24), the coefficient of the series solution can be obtained for $\iota=0,1,2,3,\ldots$. For Eqs. (4.23)-(4.24) reduces to

$$\frac{\Gamma(\gamma+1)}{\Gamma(1)}\underline{V}_1(\zeta)=\frac{\partial^2\underline{V}_0(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\underline{V}_0(\zeta,\xi,\varsigma)\frac{\partial^2\underline{V}_0(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\underline{V}_0(\zeta,\xi,\varsigma)\underline{V}_0(\zeta,\xi,\varsigma)+\underline{V}_0(\zeta,\xi,\varsigma), \quad (4.27)$$

$$\frac{\Gamma(\gamma+1)}{\Gamma(1)}\underline{V}_1(\zeta)=\frac{\partial^2\underline{V}_0(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\underline{V}_0(\zeta,\xi,\varsigma)\frac{\partial^2\underline{V}_0(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\underline{V}_0(\zeta,\xi,\varsigma)\underline{V}_0(\zeta,\xi,\varsigma)+\underline{V}_0(\zeta,\xi,\varsigma). \quad (4.28)$$

Substituting value from Eqs. (4.25)-(4.26) in Eqs. (4.23)-(4.24), we get

$$\underline{V_1}(\zeta,\varsigma)=(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)\left[\frac{3\xi+3\tanh\left(\frac{\zeta}{4}\right)-4\xi\tanh\left(\frac{\zeta}{4}\right)+\xi\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2-2\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2-11}{16\Gamma(\gamma+1)}\right], \tag{4.29}$$

$$\overline{V_1}(\zeta,\varsigma)=-\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)(\varsigma-1)\left[\frac{3\xi+5\tanh\left(\frac{\zeta}{4}\right)-4\xi\tanh\left(\frac{\zeta}{4}\right)+\xi\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2+5}{16\Gamma(\gamma+1)}\right]. \tag{4.30}$$

For $\iota=1$, Eqs. (4.23)-(4.24) reduces to

$$\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}\underline{V_2}(\zeta)=\frac{\partial^2\underline{V_1}(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\sum_{m=0}^{1}\underline{V_m}(\zeta,\xi,\varsigma)\frac{\partial^2\underline{V_{1-m}}(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\sum_{m=0}^{1}\underline{V_m}(\zeta,\xi,\varsigma)\underline{V_{1-m}}(\zeta,\xi,\varsigma)+\underline{V_1}(\zeta,\xi,\varsigma), \tag{4.31}$$

$$\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}\bar{V}_2(\zeta)=\frac{\partial^2\bar{V}_1(\zeta,\xi,\varsigma)}{\partial\zeta^2}-\sum_{m=0}^{1}\overline{V_m}(\zeta,\xi,\varsigma)\frac{\partial\bar{V}_{\iota-m}(\zeta,\xi,\varsigma)}{\partial\zeta}-\sum_{m=0}^{1}\overline{V_m}(\zeta,\xi,\varsigma)\bar{V}_{\iota-m}(\zeta,\xi,\varsigma)+\bar{V}_{\iota}(\zeta,\xi,\varsigma). \tag{4.32}$$

Substituting value from Eqs. (4.25), (4.26), (4.29), and (4.30) in Eqs. (4.31)-(4.32), we get

$$\underline{V_2}(\zeta,\varsigma)=\frac{(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)}{64\Gamma(2\gamma+1)}\begin{bmatrix}2\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^4-11\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^3+23\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2\\-21\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2+7\varsigma^2-9\varsigma\tanh\left(\frac{\zeta}{4}\right)^4\\+30\varsigma\tanh\left(\frac{\zeta}{4}\right)^3-50\varsigma\tanh\left(\frac{\zeta}{4}\right)^2\\+82\varsigma\tanh\left(\frac{\zeta}{4}\right)-53\varsigma+10\tanh\left(\frac{\zeta}{4}\right)^4\\-16\tanh\left(\frac{\zeta}{4}\right)^3+33\tanh\left(\frac{\zeta}{4}\right)^2-55\tanh\left(\frac{\zeta}{4}\right)+78\end{bmatrix}, \tag{4.33}$$

$$\overline{V_2}(\zeta,\varsigma)=\frac{(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)}{64\Gamma(2\gamma+1)}\begin{bmatrix}2\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^4-11\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^3+23\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2\\-21\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2+7\varsigma^2+\varsigma\tanh\left(\frac{\zeta}{4}\right)^4\\+14\varsigma\tanh\left(\frac{\zeta}{4}\right)^3-42\varsigma\tanh\left(\frac{\zeta}{4}\right)^2+2\varsigma\tanh\left(\frac{\zeta}{4}\right)\\+25\varsigma+25\tanh\left(\frac{\zeta}{4}\right)^2+25\tanh\left(\frac{\zeta}{4}\right)\end{bmatrix}. \tag{4.34}$$

Similarly, MATLAB software can be employed to compute the remaining coefficients of the series solution efficiently. By applying the inverse differential transform, as given in Eq. (3.4), the analytical approximate solution is obtained in the form of a power series as follows:

$$\begin{aligned}\underline{v}(\zeta,\xi,\varsigma)&=(\varsigma-1)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\zeta}{4}\right)\right)\\&+\frac{(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)}{16\Gamma(\gamma+1)}\begin{bmatrix}3\xi+3\tanh\left(\frac{\zeta}{4}\right)-4\xi\tanh\left(\frac{\zeta}{4}\right)+\xi\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2\\-2\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2-11\end{bmatrix}\xi\end{aligned}$$

$$+\frac{(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)}{64\Gamma(2\gamma+1)}\begin{bmatrix}2\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^4-11\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^3+23\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2\\-21\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2+7\varsigma^2-9\varsigma\tanh\left(\frac{\zeta}{4}\right)^4+30\varsigma\tanh\left(\frac{\zeta}{4}\right)^3\\-50\varsigma\tanh\left(\frac{\zeta}{4}\right)^2-53\varsigma+82\varsigma\tanh\left(\frac{\zeta}{4}\right)+10\tanh\left(\frac{\zeta}{4}\right)^4\\-16\tanh\left(\frac{\zeta}{4}\right)^3+33\tanh\left(\frac{\zeta}{4}\right)^2-55\tanh\left(\frac{\zeta}{4}\right)+78\end{bmatrix}\xi^2+\ldots$$

$$\begin{aligned}\overline{\nu}(\zeta,\xi,\varsigma)&=(1-\varsigma)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\zeta}{4}\right)\right)\\&-\frac{\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)(\varsigma-1)}{16\Gamma(\gamma+1)}\left[3\xi+5\tanh\left(\frac{\zeta}{4}\right)-4\xi\tanh\left(\frac{\zeta}{4}\right)+\xi\left(\tanh\left(\frac{\zeta}{4}\right)\right)^2+5\right]\xi\\&+\frac{(\varsigma-1)\left(\tanh\left(\frac{\zeta}{4}\right)-1\right)}{64\Gamma(2\gamma+1)}\begin{bmatrix}2\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^4-11\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^3+23\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2\\-21\varsigma^2\tanh\left(\frac{\zeta}{4}\right)^2+7\varsigma^2+\varsigma\tanh\left(\frac{\zeta}{4}\right)^4+14\varsigma\tanh\left(\frac{\zeta}{4}\right)^3\\-42\varsigma\tanh\left(\frac{\zeta}{4}\right)^2+2\varsigma\tanh\left(\frac{\zeta}{4}\right)+25\varsigma+25\tanh\left(\frac{\zeta}{4}\right)^2\\+25\tanh\left(\frac{\zeta}{4}\right)\end{bmatrix}\xi^2+\ldots\end{aligned}$$

To verify the accuracy and efficiency of the proposed algorithm, we perform the numerical simulation. The results clearly show that the proposed technique offers significantly higher accuracy as compared to other computational approaches documented in the literature. Table 2 shows the comparison of the proposed method with exact solution for different values of ζ, considering $\gamma = 1$, $\varsigma = 0$,$\xi = 0.001$ for Example 4.1.

TABLE 2. Comparison of RDTM and exact for different values of ζ and $\gamma = 1$,$\varsigma = 0$, $\xi = 0.001$ for Example 4.1.

ζ	RDTM	Exact	$\lvert\nu_{exact} - \nu_{RDTM}\rvert$
0.01	0.2487515651	0.2487515651	0
0.02	0.24626413220	0.2462641322	0
0.03	0.24378938390	0.2437893839	0
0.04	0.24132743990	0.2413274399	0
0.05	0.23887841780	0.2388784178	0
0.06	0.23644243230	0.2364424323	0
0.07	0.23401959570	0.2340195957	0
0.08	0.23161001730	0.2316100173	0
0.09	0.22921380430	0.2292138043	0
0.1	0.22683106100	0.2268310610	0

The absolute difference between the exact solution and the RDTM results is zero for all values of ζ presented in Table 2, demonstrating the high accuracy of the proposed method. From Table 2, it is clear that the obtained result shows an excellent agreement with the exact solution, and the zero absolute error highlights the consistency and effectiveness of RDTM in approximating the solution of the given reaction-diffusion model. Table 3 shows the comparison of absolute error for analytical approaches existing in the literature and clearly shows that the approximate solution obtained using RDTM shows an excellent agreement with exact solution in comparison with HPSTM, Q-HATM, ADM, Q-HASTM for different values of ξ, considering fixed parameter value ζ and $\gamma = 1$, $\varsigma = 0$, $\xi = 0.001$ for Example 4.1. From Tables 1 and 2 it is evident that the RDTM method is highly accurate and efficient for solving nonlinear reaction-diffusion equations with fuzzy initial conditions.

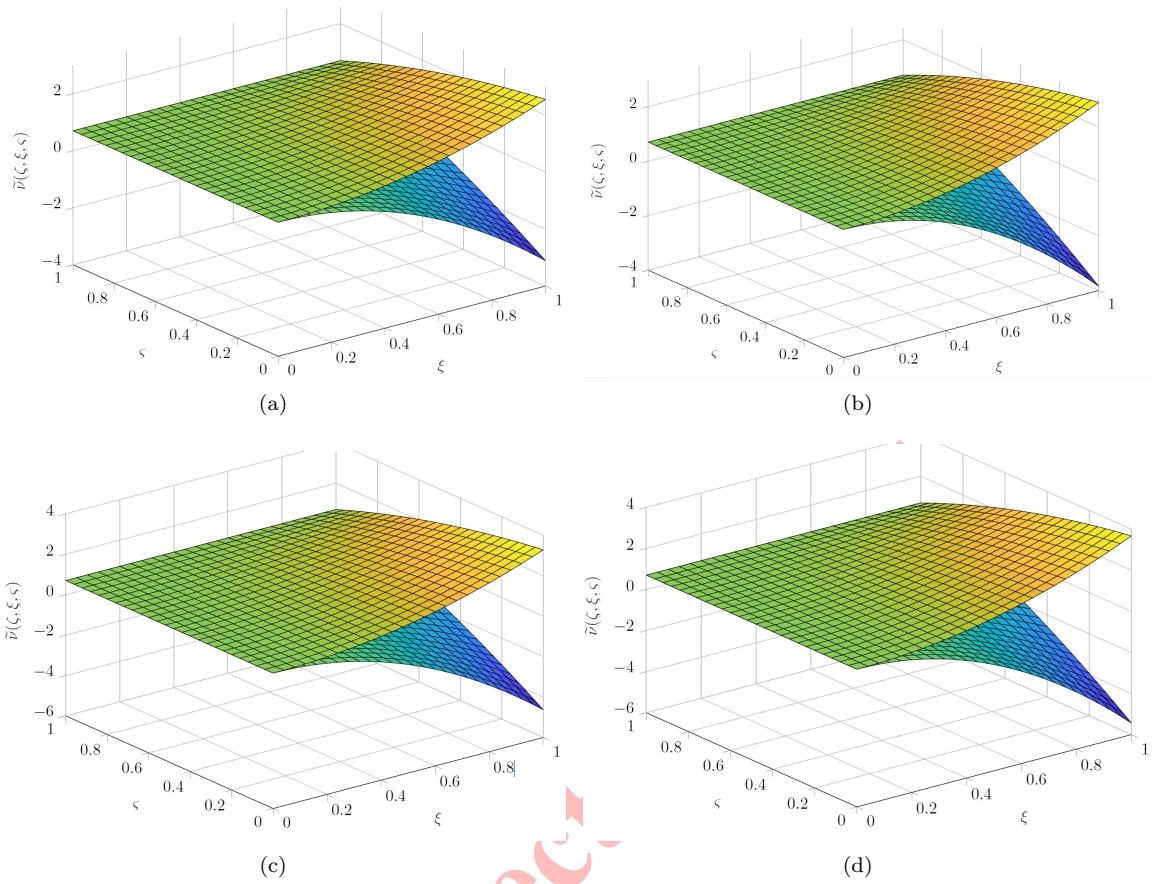

FIGURE 1. Surface plot for different value of ξ and ς with fixed value of (a) $\gamma = 1$, (b) $\gamma = 0.9$ (c)$\gamma = 0.8$, (d) $\gamma = 0.7$ for Example 4.1.

Figure 1 illustrates the surface plot of the solution for Example 4.1 corresponding to different values of the fractional order γ. From Figure 1, it is clear that as the fractional order γ decreases, the fuzzy solution $\widetilde{\nu}(\zeta, \xi, \varsigma)$ exhibits greater spread, indicating amplified uncertainty. The fuzziness from the triangular fuzzy number $\widetilde{\mathrm{A}} = [-1, 0, 1]$ propagates more strongly in lower γ due to enhanced memory effects. For $\varsigma = 0$, the solution shows maximum uncertainty, which narrows as $\varsigma \to 1$. The surfaces become steeper and more nonlinear with decreasing γ. Thus, both fractional order and fuzzy level significantly influence the system's dynamic behavior. Figure 2 displays the upper and lower bound solutions for Example 4.1 with fixed parameter values $\xi = \varsigma = 0.5$, corresponding to different values of the fractional order $\gamma = 1, 0.9, 0.8$, and 0.7. As γ decreases from 1 to 0.7, the gap between the upper and lower bounds widens, indicating increased fuzziness and uncertainty. For $\gamma = 1$, the bounds are relatively tighter, reflecting a more deterministic behavior. At lower γ , the memory effect in the system causes greater divergence in fuzzy values. Thus, lower γ values lead to more pronounced uncertainty in the solution.

The surface plots in Figure 3 demonstrate the impact of the fractional order γ and the fuzzy initial condition on the solution behavior of the nonlinear time-fractional Fisher's equation for Example 4.2. As γ decreases from 1.0 to 0.7, the solution profile becomes more diffused and the propagation rate slows down, highlighting the increasing influence of memory effects in fractional-order systems. The fuzzy initial condition, shown as a triangular fuzzy number, introduces uncertainty in the initial state, which is reflected in the spread and variation of the solution surfaces over time. This uncertainty is maintained throughout the evolution, showing that the method effectively

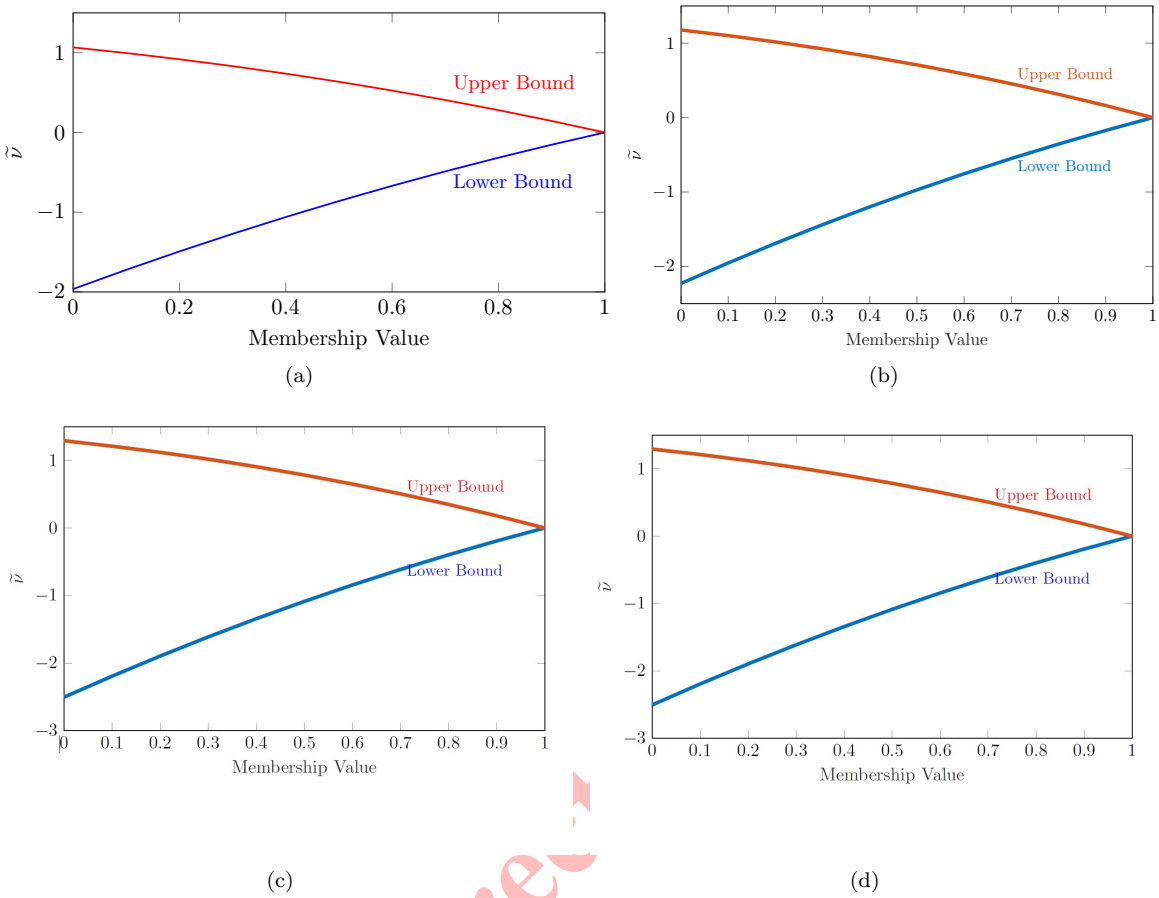

FIGURE 2. Surface plot for different value of ξ and ς with fixed value of (a) $\gamma = 1$, (b) $\gamma = 0.9$ (c) $\gamma = 0.8$, (d) $\gamma = 0.7$ for Example 4.1.

captures the effect of fuzziness on the system's dynamics. The combination of fractional derivatives and fuzzy inputs leads to a wider range of solution behaviors, enhancing the realism of the model. The numerical results are consistent and reliable, proving that the RDTM can effectively manage nonlinear fractional problems with fuzzy parameters. Overall, the plots offer straightforward explanations for how both fractional dynamics and fuzzy initial conditions shape the solution. Figure 4 illustrates how the fuzzy bounds of the solution evolve with varying membership values for different fractional orders γ. As γ decreases, the gap between the upper and lower bounds widens, highlighting increased uncertainty due to both the fuzzy initial condition and the memory effect of the fractional derivative.

Table 4 shows a clear comparison between the RDTM approximations—using the first, second, and third terms—and the exact solution for Example 4.2. The comparison is carried out for various values of the time variable ξ, considering fixed parameters $\gamma = 1$, $\varsigma = 0$, and $\zeta = 0.05$. It is clearly observed that as the number of approximated terms increases, the RDTM solution progressively converges to the exact solution. This experiment demonstrates the reliability and accuracy of RDTM in solving nonlinear time-fractional differential equations. The results exhibit excellent agreement between the higher-order RDTM approximations and the exact solution, thus validating the method's effectiveness and convergence behavior. Table 5 shows the absolute errors of the RDTM and Q-HASTM methods against the exact solution. The RDTM yields consistently low errors, especially for smaller values of ξ, confirming its reliability and numerical stability.

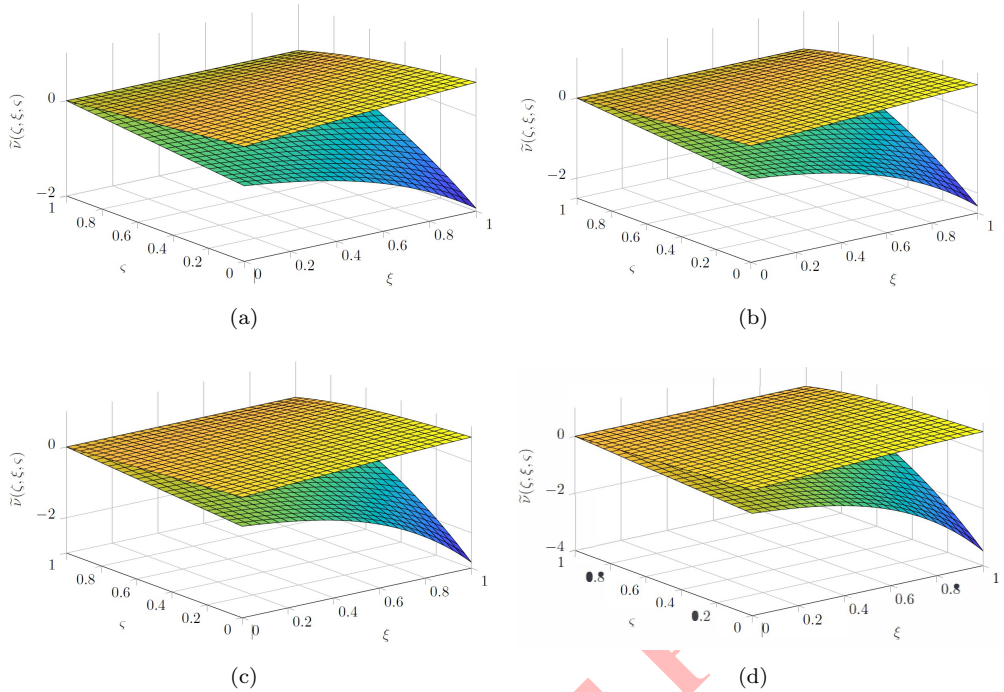

FIGURE 3. Surface plot for different value of ξ and ς with fixed value of (a) $\gamma = 1$, (b) $\gamma = 0.9$ (c) $\gamma = 0.8$, (d)$\gamma = 0.7$ for Example 4.2.

TABLE 3. Comparison of absolute error HPSTM, Q-HATM, ADM, Q-HASTM, RDTM for different values of ζ and $\gamma = 1$,$\varsigma = 0$,$\xi = 0.001$ for Example 4.1.

	ζ	HPSTM [39]	Q-HATM [39]	ADM [39]	Q-HASTM [39]	RDTM
$\xi = 0.001$	0.01	1.853E-06	0	0.001480073	2.7E-09	0
	0.02	1.825E-06	0	0.001434179	2.8E-09	0
	0.03	1.798E-06	1E-09	0.001395353	3.4E-09	0
	0.04	1.771E-06	1E-09	0.001361605	3.5E-09	0
	0.05	1.743E-06	1E-09	0.001331592	3.3E-09	0

Table 6 provides a comparative evaluation of computational performance for Example 4.1 and 4.2 under symbolic-only and symbolic–numerical frameworks. The comparison includes computational time, run time, and memory usage for two discretization levels ($N = 3$ and $N = 5$). For Example 4.1, both methods exhibit low computational cost, with the symbolic-only approach demonstrating slightly better performance in terms of run time and memory efficiency. However, for Example 4.2, a substantial increase in computational resources is observed, especially for the symbolic–numerical method, where run time and memory usage are significantly higher. These results suggest that although the symbolic–numerical method may enhance flexibility or accuracy, the symbolic-only method remains more efficient and may be better suited for large-scale or real-time applications where computational efficiency is a priority.

For the classical Burgers–Fisher equation, we have carried out the convergence analysis of the analytical approximate

TABLE 4. Comparison of RDTM and exact solution for different value of ξ, $\gamma = 1$ $\varsigma = 0$, and $\zeta = 0.05$ for Example 4.2.

ξ	1^{st} term approx.	2^{nd} term approx.	3^{rd} term approx.	Exact
0	0.49375032550049	0.493750325500490	0.493750325500490	0.4937503255
0.1	0.524995443196571	0.525019852173483	0.524979187495003	0.5249791875
0.2	0.556240560892652	0.556338196800302	0.556012879372462	0.5560138906
0.3	0.587485678588733	0.587705359380945	0.586607413061984	0.5866175789
0.4	0.618730796284815	0.619121339915413	0.616518800492691	0.6165665045
0.5	0.649975913980896	0.650586138403707	0.645503053593702	0.6456563062
0.6	0.681221031676977	0.682099754845825	0.673316184294137	0.6737070994
0.7	0.712466149373058	0.713662189241767	0.699714204523115	0.7005671424
0.8	0.74371126706914	0.745273441591535	0.724453126209757	0.7261149812
0.9	0.774956384765221	0.776933511895128	0.747288961283182	0.7502601056
1	0.806201502461302	0.808642400152545	0.767977721672510	0.7729422594

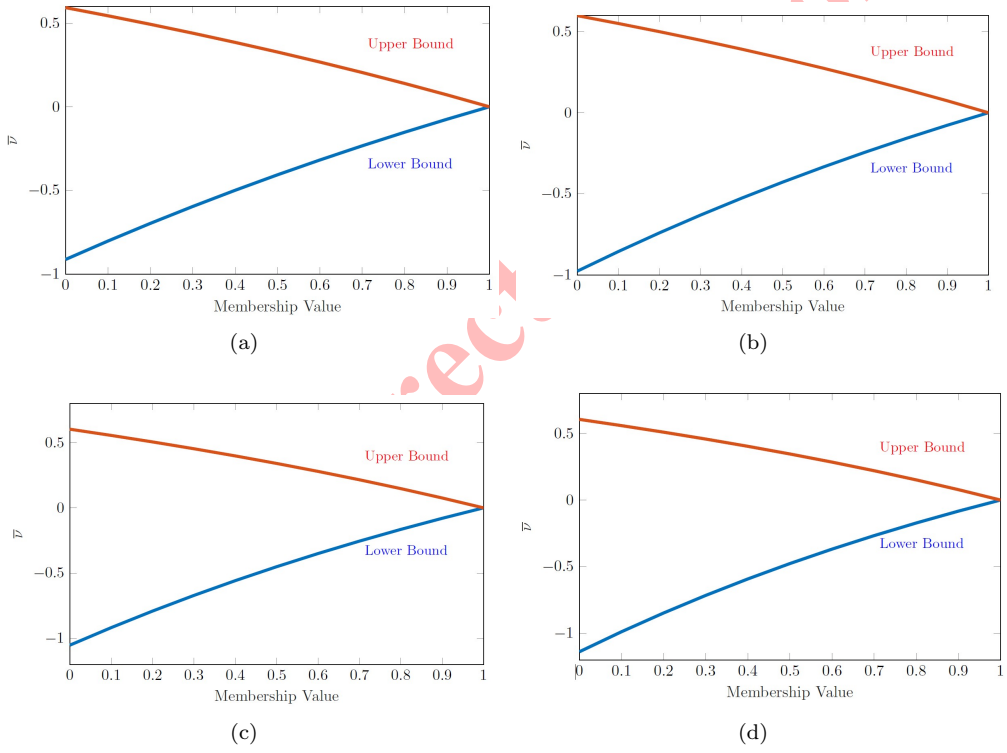

FIGURE 4. Surface plot for different value of ξ and ς with fixed value of (a) $\gamma = 1$, (b) $\gamma = 0.9$ (c)$\gamma = 0.8$, (d) $\gamma = 0.7$ for Example 4.2.

series solution of Eq. (4.1) in accordance with Theorem 3.3, as presented below.

$$\Psi_{0,v} = \frac{\|V_{1,0}\|}{\|V_{1,0}\|} < 1,$$

$$\Psi_{1,v} = \frac{\|V_{1,2}\|}{\|V_{1,1}\|} < 1,$$

$$\Psi_{2,v} = \frac{\|V_{1,3}\|}{\|V_{1,2}\|} < 1,$$

$$\vdots$$

TABLE 5. Comparison of absolute error Q-HASTM, and RDTM for different values of ξ, $\gamma = 1$ $\varsigma = 0$, and $\zeta = 0.05$ for Example 4.2.

ξ	Absolute Error Q-HASTM[39]	Absolute Error RDTM
0	0.0000000000	0.0000000000
0.1	0.0000001729	0.00000000000
0.2	0.0000021501	0.00000101123
0.3	0.0000131860	0.00001016584
0.4	0.0000529104	0.00004770401
0.5	0.0001596082	0.00015325261
0.6	0.0003953079	0.00039091511
0.7	0.0008494499	0.00085293788
0.8	0.0016410293	0.00166185499
0.9	0.0029192528	0.00297114432
1	0.00486285	0.004964538

TABLE 6. Comparison of run time, and memory usage for Examples 4.1 and 4.2.

Model	Computational Time	N	Run Time(s)	Memory usage (MB)
Example 1	Symbolic only	3	1.293319 seconds.	9.91821289062500e-05
		5	1.580027 seconds.	9.91821289062500e-05
	Symbolic + Numerical	3	1.362588 seconds.	0.000305175781250000
		5	1.760134 seconds.	0.000305175781250000
Example 2	Symbolic only	3	3.807369 seconds.	0.000129699707031250
		5	3.948282 seconds.	0.000129699707031250
	Symbolic + Numerical	3	12.375296 seconds.	0.0108642578125000
		5	14.309566 seconds.	0.0108642578125000

and so on. Hence, series solution obtained for Eq. (4.1) is convergent. For the classical Burgers–Fisher equation described, we have carried out the convergence analysis of the analytical approximate series solution of Eq. (4.19) in accordance with Theorem 3.3, as presented below.

$$\Psi_{0,v} = \frac{\|V_{1,0}\|}{\|V_{1,0}\|} < 1,$$
$$\Psi_{1,v} = \frac{\|V_{1,2}\|}{\|V_{1,1}\|} < 1,$$
$$\Psi_{2,v} = \frac{\|V_{1,3}\|}{\|V_{1,2}\|} < 1,$$
$$\vdots$$

and so on. Hence, series solution obtained for Eq. (4.19) is convergent.

5. Conclusion

In this study, we successfully applied the Reduced Differential Transform Method (RDTM) to solve fuzzy fractional differential equations with highly nonlinear dynamics and fuzzy initial conditions. The method demonstrated strong computational efficacy and accuracy, providing an effective approach for incorporating both memory effects and uncertainty in mathematical modeling.By simplifying nonlinear terms, reducing computational effort, and delivering rapidly convergent series solutions, RDTM effectively addresses fuzzy initial conditions, a common issue in conventional

methods. Through detailed numerical examples, the applicability and robustness of the proposed technique were clearly illustrated. We can explore several promising directions to extend the current work in the future. One potential avenue is the application of RDTM to other classes of fuzzy fractional differential equations, including systems with time delays, variable-order derivatives, or spatial dependence. As a result, developing adaptive versions of RDTM that may be able to dynamically adjust step sizes or transformation parameters could be an effective way to improve solution accuracy and compute efficiency. Integration with hybrid soft computing techniques or machine learning algorithms may also lead to further improvements in modeling complex, uncertain systems. Such advancements would strengthen the utility of RDTM as a flexible and powerful tool in the broader landscape of fuzzy fractional calculus.

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