Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-17
DOI:10.22034/cmde.2025.65742.3038

Analysis of a Prey-Predator model with an epidemic disease in predators

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Abstract

In this paper, a Prey-Predator model with an epidemic disease in predators, Holling functional response type II and the saturated incidence rate is studied. The system's equilibria and the basic reproduction number of the model R_0 are obtained. If $R_0 < 1$, the disease-free equilibrium is locally asymptotically stable and if $R_0 > 1$, the positive equilibrium is locally asymptotically stable. We studied Transcritical bifurcation by the Sotomayor theorem. As the infection rate increases, the asymptotic behavior of the system near the disease-free equilibrium approaches the positive equilibrium and the system has a transcritical bifurcation. We examined the sensitivity index for the basic reproduction number R_0. Finally, we perform numerical simulations to support our theoretical results.

Keywords. Prey-Predator model, Saturated incidence rate, Stability, Transcritical bifurcation, Holling functional response type II.
2010 Mathematics Subject Classification. 92D30, 34K20, 34A34.

1. INTRODUCTION

In recent years, many authors have paid special attention to the study of mathematical models of infectious diseases. Compartmental models and differential equations are the main tools for the mathematical study of natural phenomena in certain populations[6, 7, 12, 13, 20, 25–29, 32, 34]. Ecological models are another type of models that we use to study population dynamics in the living environment. One of the most famous ecological models is the prey-predator model [5, 8, 10, 11, 17, 18, 30]. In these models, the population in the environment is divided into two species: prey and predator; and in these models, we usually consider factors such as natural growth rate and carrying capacity of the environment. Mathematical ecology involves the study of populations that interact and thus influence each other's growth. The simplest predator-prey model is the Lotka-Volterra model, which is represented by the following differential equations:

$$\begin{cases} \frac{dx}{dt} = ax - \beta xy, \\ \frac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{1.1}$$

where x and y are number of prey and predators, respectively. a, β, δ and γ are positive real constants. The eco-epidemiological models have been applied to study disease either in predators [4, 9, 15] or in prey [14, 19, 33] or in both populations [16, 23]. In [22], Shuai et al. studied an epidemiological predator-prey model as follows:

Received: 01 February 2025 ; Accepted: 16 August 2025.
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$$\begin{cases} U_t = \delta_1 U_{xx} - \vartheta U_x + \gamma U(1 - \frac{U}{K}) - g(U)S, \quad x \in (0,l), \quad t > 0, \\ S_t = \delta_2 S_{xx} - \vartheta S_x + \epsilon g(U)S - h(I)S - \sigma S, \quad x \in (0,l), \quad t > 0, \\ I_t = \delta_3 I_{xx} - \vartheta I_x + h(I)S - (\sigma + \xi)I, \quad x \in (0,l), \quad t > 0, \\ \vartheta U(0;t) - \delta_1 U_x(0;t) = 0; \quad U_x(l;t) = 0; \quad t > 0; \\ \vartheta S(0;t) - \delta_1 S_x(0;t) = 0; \quad S_x(l;t) = 0; \quad t > 0; \\ \vartheta I(0;t) - \delta_1 I_x(0;t) = 0; \quad I_x(l;t) = 0; \quad t > 0; \end{cases} \tag{1.2}$$

The authors assumed that the interactive populations inhabit open advective environments with length l. The boundary conditions are Danckwert's boundary conditions, which mean that the stream source satisfies zero-flux conditions but the stream outflow meets free-flow conditions at the same rate as the advection rate [1]. Let $U(x;t)$, $S(x;t)$ and $I(x;t)$ be respectively the population densities of prey, susceptible predator and infected predator species at location x and moment t. The ODE form of this model has the following form:

$$\begin{cases} \frac{dU}{dt} = \gamma U(1 - \frac{U}{K}) - g(U)S, \\ \frac{dS}{dt} = \epsilon g(U)S - h(I)S - \sigma S, \\ \frac{dI}{dt} = h(I)S - (\sigma + \xi)I, \end{cases} \tag{1.3}$$

where $U(t)$, $S(t)$ and $I(t)$ are the population densities of prey, susceptible predator and infected predator species at moment t, respectively. In this eco-epidemiological model, we extend the typical structure of predator-prey models by considering the disease factor. In fact, disease will play a regulatory role in the population. When disease spreads in predators, the prey hunting rate decreases, as a result, prey can reproduce more and their population will increase, and the population balance will be disturbed. We further assume that the prey population is regulated by logistic growth when the predator is absent. γ and K denote respectively the per-capita intrinsic growth rate and carrying capacity of the prey; $\epsilon(< 1)$ is the conversion rate; σ and ξ stand respectively for the natural death rate and disease-induced death rate of predators. In this paper, we consider model (1.3) with the Holling functional response type II defined by $\frac{B\omega_0 U}{1+B\omega_1 U} = \frac{\beta_1 U}{1+\alpha_1 U}$ and the saturated incidence rate $\frac{\beta_2 SI}{1+\alpha_2 I}$ as follows:

$$\begin{cases} \frac{dU}{dt} = \gamma U(1 - \frac{U}{K}) - \frac{\beta_1 US}{1+\alpha_1 U}, \\ \frac{dS}{dt} = \frac{\epsilon \beta_1 US}{1+\alpha_1 U} - \frac{\beta_2 SI}{1+\alpha_2 I} - \sigma S, \\ \frac{dI}{dt} = \frac{\beta_2 SI}{1+\alpha_2 I} - (\sigma + \xi)I, \end{cases} \tag{1.4}$$

where ω_0 and ω_1 denote respectively the time taking by a Predator to search and capture Preys, B is the predation rate per unit of time, $\beta_1 = B\omega_0$ denotes the Prey searching rate and $\alpha_1 = B\omega_1$ denotes the satiety rate of Predators. Considering that predators are divided into two groups of susceptible and infected predator, so only susceptible predators can put pressure on prey and control the prey population with the Holling functional response type II. This relationship between prey and predator provides the basis for the stability of this ecological environment. The parameter ϵ indicates how prey contributes to the growth of a susceptible predator population. Changing this parameter can strongly affect the population balance and have important implications for wildlife management. The transmission rate of epidemic disease and saturation rate are denoted by β_2 and α_2, respectively. Choosing the Holling functional response type II and the specific form of the saturation incidence rate makes the model more logical from a biological perspective. Due to in our model there are infected predators, the only lever to control and prevent the excessive growth of prey is susceptible predators. In Holling functional response type II, when the prey population decreases, the hunting rate becomes linear. Since only susceptible predators can participate in the hunting process effectively and are limited in terms of physical strength and time and cannot hunt more than a certain amount, when the prey population is large, predator's hunting ability reaches saturation. By choosing the Holling functional response type II, we know that the prey population will be protected from extinction.

The reason for choosing the saturation Incidence function is that with an excessive increase in infection in predators, the predator population will not become extinct and the population balance in predators will remain. The saturation transmission rate $\frac{\beta_2 SI}{1+\alpha_2 I}$ shows that when the population of infected predators I increases, the disease transmission no longer increases linearly, but gradually saturates. This saturation is because the population of healthy predators may show preventive behaviors and not approach the infected, or only a limited number of them may become infected due to the limited capacity of the infection to spread, or the immune system in some predators may prevent the disease from spreading uncontrollably for biological reasons. This transmission rate allows us to predict the dynamics of the biological system in a more stable and balanced way, such that when the disease prevalence is very high, fewer people will become infected. Preventing peaks of high intensity in disease transmission. In fact, choosing these two rates helps us to examine the behavior of the population in three prey species, susceptible predator and infected predator, in a more realistic way.

This paper is organized as follows. In section 2, we discuss about positivity and boundedness of the model. Also, we obtain the trivial equilibrium, the axial equilibrium, the disease-free equilibrium, the positive equilibrium and the basic reproduction number of the model (1.4). In section 3, we study the stability of the Equilibria. We analyze the transcritical bifurcation in section 4. We investigate sensitivity index of parameters for R_0 in section 5. In section 6, numerical simulations are carried out to illustrate the main results.

2. PRELIMINARY RESULTS

2.1. **Positivity and boundedness of the model.**

Theorem 2.1. $A = \{(U(t), S(t), I(t)) \in \mathbb{R}^3{}_+ \cup \{0\} : U(t) + S(t) + I(t) \leq \frac{\Omega}{\eta}, t \geq 0\}$ *is a positively invariant region of the system (1.4).*

Proof. Let $N = U + S + I$, then $\dot{N} = \dot{U} + \dot{S} + \dot{I} = \gamma U(1 - \frac{U}{K}) - \frac{\beta_1 U S}{1+\alpha_1 U}(1 - \epsilon) - \sigma S - (\sigma + \xi)I$. It is clear that

$$\dot{N} \leq \gamma U(1 - \frac{U}{K}) - \sigma S - (\sigma + \xi)I. \tag{2.1}$$

Now, we choose a positive constant $\eta > 0$, such that

$$\dot{N} + \eta N \leq U(\gamma(1 - \frac{U}{K}) + \eta) - (\sigma - \eta)S - (\sigma + \xi - \eta)I, \tag{2.2}$$

if we choose $\eta < \sigma$, we get

$$\dot{N} + \eta N \leq U(\gamma(1 - \frac{U}{K}) + \eta), \tag{2.3}$$

The supremum of $U(\gamma(1 - \frac{U}{K}) + \eta)$ is $\frac{K(\gamma+\eta)^2}{4\gamma}$. Therefore $\dot{N} + \eta N \leq \frac{K(\gamma+\eta)^2}{4\gamma} = \Omega > 0$. Now by the differential inequality introduced in [**?**], we obtain

$$N(t) \leq \frac{\Omega}{\eta}(1 - e^{-\eta t}) + N(0)e^{-\eta t}, \tag{2.4}$$

consequently

$$\limsup_{t \longrightarrow \infty} N(t) \leq \frac{\Omega}{\eta}. \tag{2.5}$$

Hence, the solutions of system (2.1) in $\mathbb{R}^3{}_+$ are confined in the following region

$$A = \{(U(t), S(t), I(t)) \in \mathbb{R}^3{}_+ \cup \{0\} : U(t) + S(t) + I(t) \leq \frac{\Omega}{\eta}, \Omega = \frac{K(\gamma+\eta)^2}{4\gamma}, t \geq 0\}. \tag{2.6}$$

□

2.2. **Existence of equilibria.** The system (1.4) has the trivial equilibrium $E_0 = (0, 0, 0)$, the axial equilibrium $E_1 = (K, 0, 0)$, the disease-free equilibrium $E_2 = (\frac{\sigma}{\epsilon\beta_1 - \sigma\alpha_1}, \frac{\gamma\epsilon\left(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma\right)}{K(\epsilon\beta_1 - \sigma\alpha_1)^2}, 0)$ and the positive equilibrium $E_3 = (U_3, S_3, I_3)$, where

$$S_3 = \frac{\gamma(K - U_3)(1 + \alpha_1 U_3)}{\beta_1 K}, \tag{2.7}$$

and

$$I_3 = \frac{\sigma(1 + \alpha_1 U_3) - \epsilon\beta_1 U_3}{\epsilon\beta_1\alpha_2 U_3 - \beta_2(1 + \alpha_1 U_3) - \sigma\alpha_2(1 + \alpha_1 U_3)}. \tag{2.8}$$

By substituting S_3 and I_3 in the third equation, we have

$$P_2 {U_3}^2 + P_1 U_3 + P_0 = 0, \tag{2.9}$$

where

$$\begin{aligned}
P_2 &= -\gamma\epsilon\beta_1\alpha_2 + \gamma\beta_2\alpha_1 + \gamma\sigma\alpha_1\alpha_2 = \gamma\big(\alpha_2(\sigma\alpha_1 - \epsilon\beta_1) + \beta_2\alpha_1\big),\\
P_1 &= \gamma K\epsilon\beta_1\alpha_2 - \gamma K\beta_2\alpha_1 - \gamma K\sigma\alpha_1\alpha_2 + \gamma\beta_2 + \gamma\sigma\alpha_2 = \gamma\big((\beta_2 + \sigma\alpha_2)(1 - K\alpha_1) + K\epsilon\beta_1\alpha_2\big),\\
P_0 &= -\gamma K\beta_2 - \gamma K\sigma\alpha_2 + \beta_1 K(\sigma + \xi) = K\big(\beta_1(\sigma + \xi) - \gamma(\beta_2 + \sigma\alpha_2)\big).
\end{aligned}$$

We describe all possible parametric conditions for the appearance of roots of Equation (2.9) in the following theorem.

Theorem 2.2. **(A)** *If* $\beta_1(\sigma + \xi) > \gamma(\beta_2 + \sigma\alpha_2)$,
  (i) *Choose* $K\alpha_1 < 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 > 0$, *then Equation (2.9) can exhibit unique positive root.*
  (ii) *Choose* $K\alpha_1 < 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 < 0$, *then Equation (2.9) has no positive root.*
  (iii) *Choose* $K\alpha_1 > 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 < 0$, *then Equation (2.9) can exhibit two positive roots.*
  (iv) *Choose* $K\alpha_1 > 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 > 0$, *then Equation (2.9) can exhibit unique positive root.*

(B) *If* $\beta_1(\sigma + \xi) < \gamma(\beta_2 + \sigma\alpha_2)$,
  (i) *Choose* $K\alpha_1 < 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 > 0$, *then Equation (2.9) can exhibit two positive roots.*
  (ii) *Choose* $K\alpha_1 < 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 < 0$, *then Equation (2.9) can exhibit unique positive root.*
  (iii) *Choose* $K\alpha_1 > 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 < 0$, *then Equation (2.9) can exhibit unique positive root.*
  (iv) *Choose* $K\alpha_1 > 1$ *and* $\alpha_2(\epsilon\beta_1 - \sigma\alpha_1) - \beta_2\alpha_1 > 0$, *then Equation (2.9) has no positive root.*

Proof. We examine the possible number of positive roots by applying Descartes' rule of signs [21] in Equation (2.9). In Table 1, we have shown the sign of the coefficients of Equation (2.9) under the different parametric conditions. □

According to [31], the basic reproduction number R_0 corresponding to model (1.4) is derived as follows:
Set

$$\bar{F} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\beta_2 SI}{1+\alpha_2 I} \end{bmatrix}, \quad \textit{and} \quad \bar{V} = \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\epsilon\beta_1 US}{1+\alpha_1 U} + \frac{\beta_2 SI}{1+\alpha_2 I} + \sigma S \\ (\sigma + \xi)I \end{bmatrix}. \tag{2.10}$$

with some calculation, we have

$$F = \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 S \end{bmatrix}, \qquad \textit{and} \qquad V = \begin{bmatrix} \sigma - \frac{\epsilon\beta_1 U}{1+\alpha_1 U} & \beta_2 S \\ 0 & \sigma + \xi \end{bmatrix},$$

with

$$V^{-1} = \begin{bmatrix} \frac{1}{\sigma - \frac{\epsilon\beta_1 U}{1+\alpha_1 U}} & \frac{-\beta_2 S}{(\sigma+\xi)(\sigma - \frac{\epsilon\beta_1 U}{1+\alpha_1 U})} \\ 0 & \frac{1}{\sigma+\xi} \end{bmatrix}.$$

Therefore, spectral radius of the matrix FV^{-1} can be found as,

$$R_0 = \frac{\beta_2 S_2}{\sigma + \xi} = \frac{\beta_2\gamma\epsilon(K[\epsilon\beta_1 - \sigma\alpha_1] - \sigma)}{K(\sigma + \xi)(\epsilon\beta_1 - \sigma\alpha_1)^2},$$

where $K[\epsilon\beta_1 - \sigma\alpha_1] - \sigma > 0$.

Table 1. Table indicates the sign of coefficients of Equation (2.9) under the parametric conditions.

Parametric conditions	P_2	P_1	P_0	Number of positive roots of Equation (2.9)
A(i)	-	+	+	1
A(ii)	+	+	+	0
A(iii)	+	-	+	2
A(iv)	-	-	+	1
B(i)	-	+	-	2
B(ii)	+	+	-	1
B(iii)	+	-	-	1
B(iv)	-	-	-	0

Theorem 2.3. *Model (1.4) has no interior equilibrium point if* $R_0 < 1$. *For* $R_0 > 1$, *if* $\beta_2\epsilon\gamma < 1$ *and if the following conditions are met for the parameters:*

$$\begin{cases} (H1) \quad (\epsilon\beta_1 - \sigma\alpha_1)(1+\alpha_2 I) > \beta_2\alpha_1 I, \\ (H2) \quad \alpha_2\big(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma\big) < \beta_2(K\alpha_1 + 1), \\ (H3) \quad \big(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma\big)\big(1+\alpha_2 I\big) > \beta_2 I(K\alpha_1 + 1), \\ (H4) \quad \alpha_2(\epsilon\beta_1 - \sigma\alpha_1) > \beta_2\alpha_1 > 0, \end{cases}$$

then Model (1.4) generates unique interior equilibrium point $E_3 = (U_3, S_3, I_3)$ *with* $U_3, S_3 > 0$ *and*

$$I_3 \in \Big(0, \frac{K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma}{\beta_2(K\alpha_1+1) - \alpha_2(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)}\Big).$$

Proof. The equilibria are found by solving the following system:

$$\begin{cases} \gamma U(1 - \frac{U}{K}) - \frac{\beta_1 US}{1+\alpha_1 U} = 0, \\ \frac{\epsilon\beta_1 US}{1+\alpha_1 U} - \frac{\beta_2 SI}{1+\alpha_2 I} - \sigma S = 0, \\ \frac{\beta_2 SI}{1+\alpha_2 I} - (\sigma+\xi)I = 0. \end{cases} \tag{2.11}$$

which leads to

$$U = \frac{\beta_2 I + \sigma(1+\alpha_2 I)}{(\epsilon\beta_1 - \sigma\alpha_1)(1+\alpha_2 I) - \beta_2\alpha_1 I},$$

and

$$S = \frac{\gamma(K-U)(1+\alpha_1 U)}{\beta_1 K}.$$

Considering the above two expressions, we get

$$S = \frac{\epsilon\gamma(1+\alpha_2 I)\left((K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(1+\alpha_2 I) - \beta_2 I(K\alpha_1+1)\right)}{K\left((\epsilon\beta_1 - \sigma\alpha_1)(1+\alpha_2 I) - \beta_2\alpha_1 I\right)^2},$$

Note that $S \geq 0$ implies that $I \leq \frac{K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma}{\beta_2(K\alpha_1+1) - \alpha_2(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)}$. Therefore, if $I > \frac{K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma}{\beta_2(K\alpha_1+1) - \alpha_2(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)}$, then there is no equilibrium point. With second equation of (2.11), we obtain

$$\frac{\beta_2\epsilon\gamma\left((K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(1+\alpha_2 I) - \beta_2 I(K\alpha_1+1)\right)}{K\left((\epsilon\beta_1 - \sigma\alpha_1)(1+\alpha_2 I) - \beta_2\alpha_1 I\right)^2} = (\sigma+\xi).$$

Define

$$Q(I) = \frac{\beta_2\epsilon\gamma\left((K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(1+\alpha_2 I) - \beta_2 I(K\alpha_1+1)\right)}{K\left((\epsilon\beta_1 - \sigma\alpha_1)(1+\alpha_2 I) - \beta_2\alpha_1 I\right)^2} - (\sigma+\xi).$$

Clearly,

$$Q(0) = \frac{\beta_2\epsilon\gamma\left((K(\epsilon\beta_1-\sigma\alpha_1)-\sigma)\right)}{K(\epsilon\beta_1-\sigma\alpha_1)^2} - (\sigma+\xi) = (\sigma+\xi)(R_0-1) > 0,$$

and

$$\begin{aligned} Q\Big(\frac{K(\epsilon\beta_1-\sigma\alpha_1)-\sigma}{\beta_2(K\alpha_1+1)-\alpha_2(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma)}\Big) &= \frac{(\beta_2\epsilon\gamma-1)(K\alpha_1+1)(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma))}{K\beta_2((\epsilon\beta_1-\sigma\alpha_1)(K\alpha_1+1)-\alpha_1(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma))^2} \\ &\times \frac{(\beta_2(K\alpha_1+1)-\alpha_2(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma))}{K\beta_2((\epsilon\beta_1-\sigma\alpha_1)(K\alpha_1+1)-\alpha_1(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma))^2} \\ &-(\sigma+\xi), \end{aligned}$$

$$\begin{aligned} Q^{'}(I) &= \frac{\beta_2\epsilon\gamma\Big(\big(\alpha_2(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma)-\beta_2(K\alpha_1+1)\big)\big((\epsilon\beta_1-\sigma\alpha_1)(1+\alpha_2 I)-\beta_2\alpha_1 I\big)\Big)}{K((\epsilon\beta_1-\sigma\alpha_1)(1+\alpha_2 I)-\beta_2\alpha_1 I)^3} \\ &- \frac{2\beta_2\epsilon\gamma\Big(\big((\epsilon\beta_1-\sigma\alpha_1)\alpha_2-\beta_2\alpha_1\big)\big((k(\epsilon\beta_1-\sigma\alpha_1)-\sigma)(1+\alpha_2 I)-\beta_2 I(K\alpha_1+1)\big)\Big)}{K((\epsilon\beta_1-\sigma\alpha_1)(1+\alpha_2 I)-\beta_2\alpha_1 I)^3}. \end{aligned}$$

Using assumptions (H1-H4), we have $Q\left(\frac{K(\epsilon\beta_1-\sigma\alpha_1)-\sigma}{\beta_2(K\alpha_1+1)-\alpha_2(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma)}\right) < 0$ and $Q^{'}(I) < 0$. Thus, a unique endemic equilibrium $E_3 = (U_3, S_3, I_3)$ exists with $I_3 \in (0, \frac{K(\epsilon\beta_1-\sigma\alpha_1)-\sigma}{\beta_2(K\alpha_1+1)-\alpha_2(K(\epsilon\beta_1-\sigma\alpha_1)-\sigma)})$ and $S_3, U_3 > 0$. □

3. Stability

The Jacobian matrix of the system (1.4) is as follows:

$$J = \begin{pmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S}{(1+\alpha_1 U)^2} & \frac{-\beta_1 U}{(1+\alpha_1 U)} & 0 \\ \frac{\epsilon\beta_1 S}{(1+\alpha_1 U)^2} & \frac{\epsilon\beta_1 U}{(1+\alpha_1 U)} - \frac{\beta_2 I}{(1+\alpha_2 I)} - \sigma & -\frac{\beta_2 S}{(1+\alpha_2 I)^2} \\ 0 & \frac{\beta_2 I}{(1+\alpha_2 I)} & \frac{\beta_2 S}{(1+\alpha_2 I)^2} - (\sigma+\xi) \end{pmatrix}.$$

Theorem 3.1. *The trivial equilibrium point* $E_0 = (0,0,0)$ *is a saddle.*

Proof. The Jacobian matrix of the system (1.4) at E_0, denoted by J_0, is given by

$$J_0 = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & -(\sigma+\xi) \end{pmatrix}.$$

So the eigenvalues of J_0 are $\lambda_1 = \gamma$, $\lambda_2 = -\sigma$ and $\lambda_3 = -(\sigma+\xi)$. Clearly λ_1 is positive and λ_2 and λ_3 are negative. Therefore, E_0 is a saddle. □

Theorem 3.2. *(i) The axial equilibrium* $E_1 = (K,0,0)$ *is locally asymptotically stable provided that* $\epsilon\beta_1 K < \sigma(1+\alpha_1 K)$.
(ii) The axial equilibrium $E_1 = (K,0,0)$ *is unstable provided that* $\epsilon\beta_1 K > \sigma(1+\alpha_1 K)$.

Proof. At E_1, the Jacobian matrix J_1, is

$$J_1 = \begin{pmatrix} -\gamma & -\frac{\beta_1 K}{1+\alpha_1 K} & 0 \\ 0 & \frac{\epsilon\beta_1 K}{1+\alpha_1 K} - \sigma & 0 \\ 0 & 0 & -(\sigma+\xi) \end{pmatrix}.$$

The eigenvalues of matrix J_1 are $\lambda_1 = -\gamma$, $\lambda_2 = \frac{\epsilon\beta_1 K}{1+\alpha_1 K} - \sigma$ and $\lambda_3 = -(\sigma+\xi)$. We see λ_1 and λ_3 are negative. So, when $\epsilon\beta_1 K < \sigma(1+\alpha_1 K)$, E_1 is LAS[1]. On the other hand, if $\epsilon\beta_1 K > \sigma(1+\alpha_1 K)$; then $\lambda_2 > 0$, which means that E_1 is unstable.

□

Theorem 3.3. *If $R_0 < 1$, then $E_2 = \left(\frac{\sigma}{\epsilon\beta_1 - \sigma\alpha_1}, \frac{\gamma\epsilon[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{K(\epsilon\beta_1-\sigma\alpha_1)^2}, 0\right)$ is locally asymptotically stable provided $a_{11} < 0$.*

Proof. At E_2, the Jacobian matrix J_2, is given by

$$J_2 = \begin{pmatrix} \gamma - \frac{2\gamma U_2}{K} - \frac{\beta_1 S_2}{(1+\alpha_1 U_2)^2} & -\frac{\beta_1 U_2}{1+\alpha_1 U_2} & 0 \\ \frac{\epsilon\beta_1 S_2}{(1+\alpha_1 U_2)^2} & \frac{\epsilon\beta_1 U_2}{1+\alpha_1 U_2} - \sigma & -\beta_2 S_2 \\ 0 & 0 & \beta_2 S_2 - (\sigma+\xi) \end{pmatrix}.$$

We calculate the $\det(J_2 - \lambda I) = 0$ as follows:

$$\begin{vmatrix} a_{11} - \lambda & -\frac{\sigma}{\epsilon} & 0 \\ \frac{\gamma[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{\beta_1 K} & -\lambda & -(\sigma+\xi)R_0 \\ 0 & 0 & (\sigma+\xi)(R_0-1) - \lambda \end{vmatrix} = 0.$$

where $a_{11} = \gamma - \frac{2\gamma\sigma}{K(\epsilon\beta_1-\sigma\alpha_1)} - \frac{\gamma[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{\epsilon\beta_1 K}$. We obtain the characteristic equation as follows:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{3.1}$$

where

$$\begin{cases} A = -a_{11} - (\sigma+\xi)(R_0-1), \\ B = a_{11}(\sigma+\xi)(R_0-1) + \frac{\sigma}{\epsilon}\left(\frac{\gamma[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{\beta_1 K}\right), \\ C = -\frac{\sigma}{\epsilon}\left(\frac{\gamma[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{\beta_1 K}(\sigma+\xi)(R_0-1)\right). \end{cases}$$

We know that $A > 0, B > 0$ and $C > 0$. Now, we check the sign of $AB - C$. With simple calculations, we have

$$AB - C = -{a_{11}}^2(\sigma+\xi)(R_0-1) - a_{11}\frac{\sigma}{\epsilon}\left(\frac{\gamma[K(\epsilon\beta_1-\sigma\alpha_1)-\sigma]}{\beta_1 K}\right) - a_{11}(\sigma+\xi)^2(R_0-1)^2.$$

We can easily see $AB - C > 0$, therefore by Routh-Hurwitz stability criterion, we find that E_2 is LAS. □

Theorem 3.4. *The positive equilibrium E_3 is locally asymptotically stable if the following inequalities are satisfied:*

$$(i)\quad \gamma + \frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3} + \frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} < \eta_1, \tag{3.2}$$

$$(ii)\begin{cases} \gamma\left(\frac{\beta_2 I_3}{1+\alpha_2 I_3} + \sigma\right) + \left(\frac{2\gamma U_3}{K} + \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)}\right) + \gamma(\sigma+\xi) + \left(\frac{2\gamma U_3}{K} + \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2}\right) \\ +\left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)}\right)(\sigma+\xi) + \left(\frac{\beta_2 I_3}{1+\alpha_2 I_3} + \sigma\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2}\right) < \eta_2, \end{cases} \tag{3.3}$$

$$(iii)\begin{cases} \left(\frac{\gamma\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_1 U_3)(1+\alpha_2 I_3)^2}\right) + \gamma(\sigma+\xi)\left(\frac{\beta_2 I_3}{1+\alpha_2 I_3} + \sigma\right) + \left(\frac{2\gamma U_3}{K} + \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)}\right)(\sigma+\xi) \\ +\left(\frac{2\gamma U_3}{K} + \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\beta_2 I_3}{1+\alpha_2 I_3} + \sigma\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2}\right) + \gamma\left(\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2 I_3)^3}\right) + \frac{\epsilon{\beta_1}^2\beta_2{S_3}^2 U_3}{(1+\alpha_1 U_3)^3(1+\alpha_2 I_3)^2} < \eta_3, \end{cases} \tag{3.4}$$

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[1] Locally Asymptotically Stable

$$(iv)\begin{cases}\Big((\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+(\sigma+\xi)\Big)\Big(\frac{\gamma\epsilon\beta_1 U_3}{1+\alpha_1 U_3}+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma) \\ +\frac{\gamma\beta_2 S_3}{(1+\alpha_2 I_3)^2}+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\sigma+\xi)+\frac{\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_1 U_3)(1+\alpha_1 I_3)^2}+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\sigma+\xi)+\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_1 I_3)^3} \\ +\frac{\epsilon{\beta_1}^2 S_3 U_3}{(1+\alpha_1 U_3)^3}\Big)+\Big(\gamma+\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3}+\frac{\beta_2 S}{(1+\alpha_2 I_3)^2}][\gamma(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3})+\gamma(\sigma+\xi) \\ +(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})(\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3})(\sigma+\xi)+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})\Big) \\ +\Big(\frac{\gamma\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_1 U_3)(1+\alpha_1 I_3)^2}+\gamma(\sigma+\xi)(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3})(\sigma+\xi) \\ +(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})+\gamma(\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2)^3})+\frac{\epsilon{\beta_1}^2\beta_2 {S_3}^2 U_3}{(1+\alpha_1 U_3)^3(1+\alpha_1 I_3)^2}\Big)>\eta_4.\end{cases} \tag{3.5}$$

where

$$\begin{aligned}\eta_1 &= \frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}+\frac{\beta_2 I_3}{(1+\alpha_2 I_3)}+\sigma+(\sigma+\xi),\\ \eta_2 &= \frac{\gamma\epsilon\beta_1 U_3}{1+\alpha_1 U_3}+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+\frac{\gamma\beta_2 S_3}{(1+\alpha_2 I_3)^2}+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\sigma+\xi)\\ &+\frac{\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_1 U_3)(1+\alpha_1 I_3)^2}+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\sigma+\xi)+\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2 I_3)^3}+\frac{\epsilon{\beta_1}^2 S_3 U_3}{(1+\alpha_1 U_3)^3},\\ \eta_3 &= \frac{\gamma\epsilon\beta_1 U_3}{1+\alpha_1 U_3}(\sigma+\xi)+\gamma(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_2 I_3)^2(1+\alpha_1 U_3)})\\ &+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\sigma+\xi)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2 I_3)^3})+\frac{\epsilon{\beta_1}^2 S_3 U_3}{(1+\alpha_1 U_3)^3}(\sigma+\xi),\end{aligned}$$

and

$$\begin{aligned}\eta_4 = &\Big((\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+(\sigma+\xi)\Big)\Big(\gamma(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+(\frac{2\gamma U_3}{K}\\ &+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3})+\gamma(\sigma+\xi)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})\\ &+(\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3})(\sigma+\xi)+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})\Big)+\Big(\gamma+\frac{\epsilon\beta_1 U_3}{1+\alpha_1 U_3}\\ &+\frac{\beta_2 S}{(1+\alpha_2 I_3)^2}\Big)\Big(\frac{\gamma\epsilon\beta_1 U_3}{1+\alpha_1 U_3}+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)+\frac{\gamma\beta_2 S_3}{(1+\alpha_2 I_3)^2}\\ &+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\sigma+\xi)+\frac{\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_1 U_3)(1+\alpha_1 I_3)^2}+(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\sigma+\xi)\\ &+\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2 I_3)^3}+\frac{\epsilon{\beta_1}^2 S_3 U_3}{(1+\alpha_1 U_3)^3}\Big)+\Big((\frac{\gamma\epsilon\beta_1 U_3}{1+\alpha_1 U_3})(\sigma+\xi)+\gamma(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\frac{\beta_2 S}{(1+\alpha_2 I_3)^2})\\ &+\frac{\epsilon{\beta_1}^2 S_3 U_3}{(1+\alpha_1 U_3)^3}(\sigma+\xi)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\epsilon\beta_1\beta_2 S_3 U_3}{(1+\alpha_2 I_3)^2(1+\alpha_1 U_3)})+(\frac{2\gamma U_3}{K}\\ &+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{\beta_2 I_3}{1+\alpha_2 I_3}+\sigma)(\sigma+\xi)+(\frac{2\gamma U_3}{K}+\frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2})(\frac{{\beta_2}^2 S_3 I_3}{(1+\alpha_2 I_3)^3})\Big).\end{aligned}$$

Proof. The Jacobian matrix of the system (1.4) at E_3, is as follows:

$$J_3 = \begin{pmatrix} \gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2} & \frac{-\beta_1 U_3}{(1+\alpha_1 U_3)} & 0 \\ \frac{\epsilon \beta_1 S_3}{(1+\alpha_1 U_3)^2} & \frac{\epsilon \beta_1 U_3}{(1+\alpha_1 U_3)} - \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} - \sigma & -\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} \\ 0 & \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} & \frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma + \xi) \end{pmatrix}.$$

The characteristics equation of J_3 can be given as

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{3.6}$$

where

$$A = -\Bigg((\gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}) + (\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)} - \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} - \sigma) + (\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma+\xi)) \Bigg),$$

$$\begin{aligned} B = &\left(\gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)} - \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} - \sigma\right) \\ &+ \left(\gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma+\xi)\right) \\ &+ \left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)} - \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} - \sigma\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma+\xi)\right) \\ &- \left(-\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2}\right)\Bigg(\frac{\beta_2 I_3}{(1+\alpha_2 I_3)}\Bigg] - \Bigg[\frac{-\beta_1 U_3}{(1+\alpha_1 U_3)}\Bigg)\left(\frac{\epsilon\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right), \end{aligned}$$

and

$$\begin{aligned} C = &-\left(\gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(\frac{\epsilon\beta_1 U_3}{(1+\alpha_1 U_3)} - \frac{\beta_2 I_3}{(1+\alpha_2 I_3)} - \sigma\right)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma+\xi)\right) \\ &+ \left(\gamma - \frac{2\gamma U_3}{K} - \frac{\beta_1 S_3}{(1+\alpha_1 U_3)^2}\right)\left(-\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2}\right)\left(\frac{\beta_2 I_3}{(1+\alpha_2 I_3)}\right) \\ &+ \Bigg(\frac{-\beta_1 U_3}{(1+\alpha_1 U_3)}\Bigg]\Bigg[\frac{\epsilon\beta_1 S_3}{(1+\alpha_1 U_3)^2}\Bigg)\left(\frac{\beta_2 S_3}{(1+\alpha_2 I_3)^2} - (\sigma+\xi)\right). \end{aligned}$$

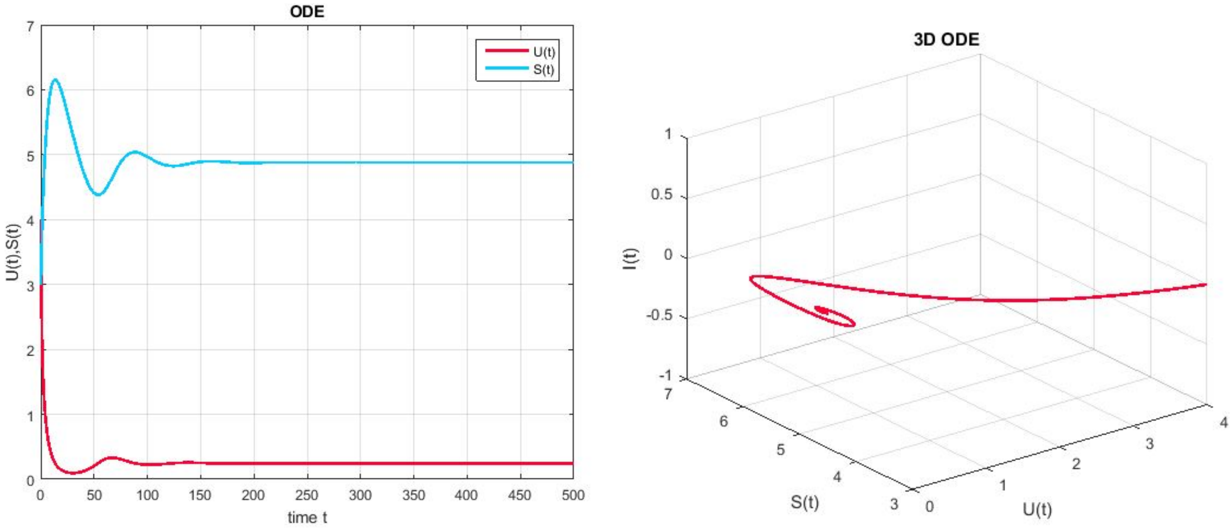
Using the Routh-Hurwitz criterion, we know that eigenvalues of J_3 have negative real parts if and only if $A > 0$, $B > 0$,$C > 0$ and $AB - C > 0$. This implies that E_3 is LAS if and only if inequalities (3.2)-(3.5) hold true.

□

Remark 3.5. Equilibrium E_0 represents a state in which none of the prey, susceptible predator and infected predator populations exist. In this paper, the equilibrium E_0 is unstable(Saddle), meaning that with the entry of even a small number of prey or predator species into the environment, the equilibrium moves away from state $(0, 0, 0)$ and we witness population growth and species survival in the environment. Equilibrium $E_1 = (K, 0, 0)$ indicates that there are no predators, either susceptible or Infected, in the system. In this case, prey will grow to capacity K and reach saturation level, and the equilibrium will be stable. In the event of the entry of any predator and subsequently with the reproduction of them, E_1 becomes unstable.

In the disease-free equilibrium E_2, prey and susceptible predator species are present in the system. With the help of the R_0, the threshold behavior of this equilibrium can be observed, because when $R_0 < 1$ and the infected predator is absent in the system, the infection transmission rate decreases, which means that the ecological environment of the prey and predator life remains stable(E_2 is stable). When E_2 is unstable, an infected predator enters the environment and the infection transmission rate increases, or equivalently $R_0 > 1$, then the environment clearly becomes unstable and the system loses its steady state and an endemic equilibrium appears.

In the case where E_3 is stable, the disease is endemic in the environment. We have the disease among predators, but in this case there is no peak of disease in the population and the three species coexist together. But when E_3 becomes unstable, the species cannot have a stable state together. The incidence of the disease may increase to such an extent that it causes the extinction of predators or oscillatory and chaotic behavior, or the disease is completely eliminated



(a) (b)

FIGURE 1. Numerical simulation of the three dimensional model (1.4) with parameter values $\beta_1 = 0.09$, $\beta_2 = 0.03$, $\gamma = 0.5$, $\epsilon = 0.9$, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$, $K = 2$, $\sigma = 0.02$, and $\xi = 0.2$; (a) Populations as functions of time t; (b) The trajectory in 3 dimensions for the disease-free equilibrium $E_2 = (0.2475, 4.88, 0)$ and initial conditions $(4, 3, 0)$.

from the population. If predators are generally eliminated, we will return to the axial equilibrium $E_1 = (K; 0; 0)$, if only infected predators are eliminated, we will return to a disease-free case. In general, the stability or instability of the endemic equilibrium determines the long-term health and stability of the ecosystem and plays a key role in disease management and species conservation.

4. TRANSCRITICAL BIFURCATION ANALYSIS

Let

$$g(X,\beta_2) = \begin{pmatrix} g_1(X,\beta_2) \\ g_2(X,\beta_2) \\ g_3(X,\beta_2) \end{pmatrix} = \begin{pmatrix} \gamma U(1-\frac{U}{K}) - \frac{\beta_1 US}{1+\alpha_1 U} \\ \frac{\epsilon\beta_1 US}{1+\alpha_1 U} - \frac{\beta_2 SI}{1+\alpha_2 I} - \sigma S \\ \frac{\beta_2 SI}{1+\alpha_2 I} - (\sigma+\xi)I \end{pmatrix}, \qquad \textit{and} \qquad X = \begin{pmatrix} U \\ S \\ I \end{pmatrix}.$$

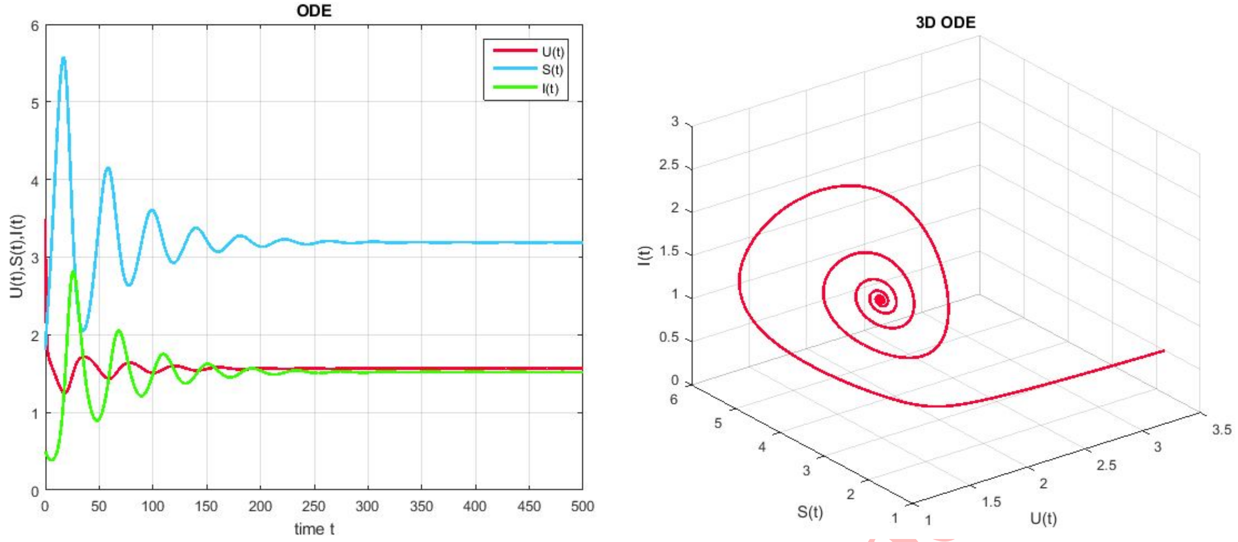
Theorem 4.1. *The system (1.4) undergoes a transcritical bifurcation at E_2 when bifurcation parameter $R_0 = 1$, provided that $\gamma \neq \frac{K(\sigma+\xi)\left[(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2\right]}{\sigma\alpha_2\epsilon[\epsilon\beta_1-\sigma\alpha_1]^4}$ and $(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2 > 0$.*

Proof. When $R_0 = 1$, the Jacobian matrix at E_2 is

$$J_2 = \begin{pmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S_2}{(1+\alpha_1 U_2)^2} & -\frac{\beta_1 U_2}{1+\alpha_1 U_2} & 0 \\ \frac{\epsilon\beta_1 S_2}{(1+\alpha_1 U_2)^2} & \frac{\epsilon\beta_1 U_2}{1+\alpha_1 U_2} - \sigma & (\sigma+\xi) \\ 0 & 0 & 0 \end{pmatrix},$$

which has a simple zero eigenvalue, with corresponding right and left eigenvectors given by:

$$V_R = \begin{bmatrix} 1 \\ 0 \\ \frac{\epsilon\beta_1 S_2}{(\sigma+\xi)(1+\alpha_1 U_2)^2} \end{bmatrix},$$



(a) (b)

FIGURE 2. Numerical simulation of the three dimensional model (1.4) with parameter values $\beta_1 = 0.09$, $\beta_2 = 0.07$, $\gamma = 1.3$, $\epsilon = 0.9$, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$, $K = 2$, $\sigma = 0.02$, and $\xi = 0.2$; (a) Populations as functions of time t; (b) The trajectory in 3 dimensions for the interior equilibrium $E_3 = (1.5650, 3.1908, 1.5201)$ and initial conditions $(3.5, 1.8, 0.5)$.

and

$$W_L = \begin{bmatrix} \frac{(\sigma+\xi)(1+\alpha_1 U_2)}{\beta_1 U} & 0 & 1 \end{bmatrix},$$

respectively. Then, we get

$$D_{\beta_2} g = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-SI}{(1+\alpha_2 I)} \\ \frac{SI}{(1+\alpha_2 I)} \end{bmatrix}.$$

By applying the Sotomayor theorem ([24], Theorem 1), we find that

$$(W_L D_{\beta_2} g)_{E_2} = 0 \tag{4.1}$$

and

$$(D_X D_{\beta_2} g)_{E_2} = \left(D_X \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \right)_{E_2} = \begin{bmatrix} \psi_{1U} & \psi_{1S} & \psi_{1I} \\ \psi_{2U} & \psi_{2S} & \psi_{2I} \\ \psi_{3U} & \psi_{3S} & \psi_{3I} \end{bmatrix}_{E_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -S \\ 0 & 0 & S \end{bmatrix}.$$

Then

$$(V_L(D_X D_{\beta_2} g)V_R)_{E_0} = \frac{\epsilon \beta_1 S^2}{(\sigma+\xi)(1+\alpha_1 U)^2} \neq 0. \tag{4.2}$$

Also, by simple computation, we get

$$D_X g = D_X \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} g_{1U} & g_{1S} & g_{1I} \\ g_{2U} & g_{2S} & g_{2I} \\ g_{3U} & g_{3S} & g_{3I} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S}{(1+\alpha_1 U)^2} & -\frac{\beta_1 U}{(1+\alpha_1 U)} & 0 \\ \frac{\epsilon\beta_1 S}{(1+\alpha_1 U)^2} & \frac{\epsilon\beta_1 U}{(1+\alpha_1 U)} - \frac{\beta_1 I}{(1+\alpha_2 I)} - \sigma & -\frac{\beta_2 S}{(1+\alpha_2 I)^2} \\ 0 & \frac{\beta_2 I}{(1+\alpha_2 I)} & \frac{\beta_2 S}{(1+\alpha_2 I)^2} - (\sigma+\xi) \end{bmatrix}.$$

Observe that

$$(D_X g_1)^T = \begin{bmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S}{(1+\alpha_1 U)^2} & -\frac{\beta_1 U}{(1+\alpha_1 U)} & 0 \end{bmatrix}^T = \begin{bmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S}{(1+\alpha_1 U)^2} \\ -\frac{\beta_1 U}{(1+\alpha_1 U)} \\ 0 \end{bmatrix},$$

$$(D_X g_2)^T = \begin{bmatrix} \frac{\epsilon\beta_1 S}{(1+\alpha_1 U)^2} & \frac{\epsilon\beta_1 U}{(1+\alpha_1 U)} - \frac{\beta_2 I}{(1+\alpha_1 I)} - \sigma & -\frac{\beta_2 S}{(1+\alpha_1 I)^2} \end{bmatrix}^T = \begin{bmatrix} \frac{\epsilon\beta_1 S}{(1+\alpha_1 U)^2} \\ \frac{\epsilon\beta_1 U}{(1+\alpha_1 U)} - \frac{\beta_2 I}{(1+\alpha_2 I)} - \sigma \\ -\frac{\beta_2 S}{(1+\alpha_2 I)^2} \end{bmatrix}$$

and

$$(D_X g_3)^T = \begin{bmatrix} 0 & \frac{\beta_2 I}{(1+\alpha_2 I)} & \frac{\beta_2 S}{(1+\alpha_2 I)^2} - (\sigma+\xi) \end{bmatrix}^T = \begin{bmatrix} 0 \\ \frac{\beta_2 I}{(1+\alpha_2 I)} \\ \frac{\beta_2 S}{(1+\alpha_2 I)^2} - (\sigma+\xi) \end{bmatrix}.$$

Thus, we conclude that

$$\left(D_X (D_X g_1)^T\right)_{E_2} = \left(D_X \begin{bmatrix} \gamma - \frac{2\gamma U}{K} - \frac{\beta_1 S}{(1+\alpha_1 U)^2} \\ -\frac{\beta_1 U}{(1+\alpha_1 U)} \\ 0 \end{bmatrix} \right)_{E_2} = \begin{bmatrix} -\frac{2\gamma}{K} + \frac{2\alpha_1\beta_1 S}{(1+\alpha_1 U)^3} & -\frac{\beta_1}{(1+\alpha_1 U)^2} & 0 \\ -\frac{\beta_1}{(1+\alpha_1 U)^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{E_2},$$

and

$$\left(D_X (D_X g_2)^T\right)_{E_2} = \left(D_X \begin{bmatrix} \frac{\epsilon\beta_1 S}{(1+\alpha_1 U)^2} \\ \frac{\epsilon\beta_1 U}{(1+\alpha_1 U)} - \frac{\beta_2 I}{(1+\alpha_2 I)} - \sigma \\ -\frac{\beta_2 S}{(1+\alpha_2 I)^2} \end{bmatrix} \right)_{E_2} = \begin{bmatrix} -\frac{2\epsilon\alpha_1\beta_1 S}{(1+\alpha_1 U)^3} & \frac{\epsilon\beta_1}{(1+\alpha_1 U)^2} & 0 \\ \frac{\epsilon\beta_1}{(1+\alpha_1 U)^2} & 0 & -\frac{\beta_2}{(1+\alpha_2 I)^2} \\ 0 & -\frac{\beta_2}{(1+\alpha_2 I)^2} & \frac{2\alpha_2\beta_2 S}{(1+\alpha_2 I)^3} \end{bmatrix}_{E_2},$$

and

$$\left(D_X (D_X g_3)^T\right)_{E_2} = \left(D_X \begin{bmatrix} 0 \\ \frac{\beta_2 I}{(1+\alpha_2 I)} \\ \frac{\beta_2 S}{(1+\alpha_2 I)^2} - (\sigma+\xi) \end{bmatrix} \right)_{E_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2}{(1+\alpha_2 I)^2} \\ 0 & \frac{\beta_2}{(1+\alpha_2 I)^2} & \frac{-2\alpha_2\beta_2 S}{(1+\alpha_2 I)^3} \end{bmatrix}_{E_2}.$$

Consequently, we have

$$\begin{aligned} W_L((D_{XX} g)(V_R, V_R)) &= (W_L \sum [e_i {V_R}^T D_X (D_X g_i)^T V_R])_{E_2} \\ &= (W_L [e_1 {V_R}^T D_X (D_X g_1)^T V_R])_{E_2} + (W_L [e_3 {V_R}^T D_X (D_X g_3)^T V_R])_{E_2} \\ &= \left(\begin{bmatrix} \frac{(\sigma+\xi)(1+\alpha_1 U)}{\beta_1 U} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{\epsilon\beta_1 S}{(\sigma+\xi)(1+\alpha_1 U)^2} \end{bmatrix} \begin{bmatrix} -\frac{2\gamma}{K} + \frac{2\alpha_1\beta_1 S}{(1+\alpha_1 U)^3} & -\frac{\beta_1}{(1+\alpha_1 U)^2} & 0 \\ -\frac{\beta_1}{(1+\alpha_1 U)^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right. \\ &\quad \left. \begin{bmatrix} 1 \\ 0 \\ \frac{\epsilon\beta_1 S}{(\sigma+\xi)(1+\alpha_1 U)^2} \end{bmatrix} \right)_{E_2} + \left(\begin{bmatrix} \frac{(\sigma+\xi)(1+\alpha_1 U)}{\beta_1 U} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{\epsilon\beta_1 S}{(\sigma+\xi)(1+\alpha_1 U)^2} \end{bmatrix} \right. \\ &\quad \left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2}{(1+\alpha_2 I)^2} \\ 0 & \frac{\beta_2}{(1+\alpha_2 I)^2} & \frac{-2\alpha_2\beta_2 S}{(1+\alpha_2 I)^3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{\epsilon\beta_1 S}{(\sigma+\xi)(1+\alpha_1 U)^2} \end{bmatrix} \right)_{E_2}, \end{aligned}$$

therefore

$$W_L((D_{XX}g)(V_R,V_R)) = \left(\frac{2\gamma(\sigma+\xi)[(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2]}{\sigma K\epsilon{\beta_1}^2}\right) - \left(\frac{2\alpha_2\beta_2\epsilon\gamma^3(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)^2}{{\beta_1}^2K^3(\sigma+\xi)}\right).$$

As $R_0 = 1$, so

$$\beta_2 = \frac{K(\sigma+\xi)(\epsilon\beta_1-\sigma\alpha_1)^2}{\gamma\epsilon(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)}.$$

Finally, we have

$$W_L((D_{XX}g)(V_R,V_R)) = \frac{2\gamma}{K{\beta_1}^2}\left(\frac{(\sigma+\xi)[(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2]}{\sigma\epsilon} - \frac{\alpha_2\gamma[\epsilon\beta_1-\sigma\alpha_1]^4}{K}\right) \neq 0, \tag{4.3}$$

provided that

$$\gamma \neq \frac{K(\sigma+\xi)\left[(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2\right]}{\sigma\alpha_2\epsilon[\epsilon\beta_1-\sigma\alpha_1]^4},$$

and

$$(K[\epsilon\beta_1-\sigma\alpha_1]-\sigma)(\epsilon\beta_1-\sigma\alpha_1)-(\epsilon\beta_1)^2 > 0.$$

From (4.1)- (4.3), the theorem is proved. ☐

Remark 4.2. The transcritical bifurcation occurs at a critical point where the stability of the disease-free equilibrium is replaced by the stability of the endemic equilibrium. In our model, which includes prey, susceptible predators, and infected predators, the transcritical bifurcation states that $R_0 = 1$ acts as a threshold in determining the stability or instability of the disease in the ecosystem. Thus, if the value of $R_0 < 1$, the disease cannot be stable in predators, and the disease-free equilibrium will be stable in this state, so the ecosystem is in a stable state. However, if the amount of disease transmission increases and as a result R_0 crosses 1 and goes higher, then the disease spreads to a larger number of susceptible predators, and therefore the population of susceptible predators will decrease significantly. In this case, predators will hunt less, and as a result, the number of prey will increase, causing instability of the ecosystem. Ultimately, it can be said that with changes in some of the parameters, we will witness sudden changes in the ecosystem. By analyzing the transcritical bifurcation, it can be predicted whether the introduction of infection will lead to the collapse of the population or whether the ecosystem will tolerate the disease. We can investigate how to change parameters such as β_2, σ, ξ, etc. to keep the basic reproduction number less than 1 and thus control the disease.

5. SENSITIVITY ANALYSIS OF BASIC REPRODUCTION (R_0)

This section focuses on analyzing the sensitivity of the basic reproduction number R_0 to changes in the model parameters defined in model (1.4). Our goal is to determine which parameters have the greatest impact on the spread of the disease and play a key role in the variation in R_0.

Definition 5.1. The normalized sensitivity index of R_0 with respect to parameter ς is defined as:

$$C_{\varsigma}^{R_0} = \frac{\partial R_0}{\partial \varsigma} \times \frac{\varsigma}{R_0}, \tag{5.1}$$

By referring to the formulation of Equation (5.1) and the parameter values in Table 2, we calculate the parameter sensitivity index as follows:

Take sensitivity for parameter β_2,

$$C_{\beta_2}^{R_0} = \frac{\partial R_0}{\partial \beta_2} \times \frac{\beta_2}{R_0} = 1 > 0.$$

TABLE 2. Parameters used to show the local stability of E_2.

Parameters	β_1	β_2	γ	ϵ	α_1	α_2	K	σ	ξ
Values	0.09	0.03	0.5	0.9	0.01	0.01	2	0.02	0.2

TABLE 3. Parameters used to show the local stability of E_3.

Parameters	β_1	β_2	γ	ϵ	α_1	α_2	K	σ	ξ
Values	0.09	0.07	1.3	0.9	0.01	0.01	2	0.02	0.2

TABLE 4. Sensitivity index.

Parameters	β_1	σ	ξ	K	α_1	β_2	ϵ	γ
Values	-0.8609	-0.230	-0.9091	0.1412	0.0021	1	0.1391	1

Take sensitivity for parameter σ,

$$\begin{aligned} C_{\sigma}^{R_0} &= \frac{\partial R_0}{\partial \sigma} \times \frac{\sigma}{R_0} \\ &= \frac{-\sigma({\alpha_1}^2\xi\sigma K + 2{\alpha_1}^2\sigma^2 K - \epsilon\alpha_1\xi\beta_1 K + \alpha_1\xi\sigma - 3\epsilon\alpha_1\beta_1\sigma K + 2\alpha_1\sigma^2 + \epsilon\xi\beta_1 + \epsilon^2{\beta_1}^2 K)}{(\sigma+\xi)(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(\epsilon\beta_1 - \alpha_1\sigma)} = -0.2300 < 0. \end{aligned}$$

Take sensitivity for parameter ξ,

$$C_{\xi}^{R_0} = \frac{\partial R_0}{\partial \xi} \times \frac{\xi}{R_0} = \frac{-\xi}{(\sigma+\xi)} = -0.9091 < 0.$$

Take sensitivity for parameter K,

$$C_{K}^{R_0} = \frac{\partial R_0}{\partial K} \times \frac{K}{R_0} = \frac{\sigma}{(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)} = 0.1412 > 0.$$

Take sensitivity for parameter α_1,

$$C_{\alpha_1}^{R_0} = \frac{\partial R_0}{\partial \alpha_1} \times \frac{\alpha_1}{R_0} = \frac{\sigma\alpha_1(K(\epsilon\beta_1 - \sigma\alpha_1) - 2\sigma)}{(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(\epsilon\beta_1 - \sigma\alpha_1)} = 0.0021 > 0,$$

and take sensitivity for parameter β_1,

$$C_{\beta_1}^{R_0} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0} = \frac{-\beta_1\epsilon(K(\epsilon\beta_1 - \sigma\alpha_1) - 2\sigma)}{(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)(\epsilon\beta_1 - \sigma\alpha_1)} = -0.8609 < 0.$$

and take sensitivity for parameter ϵ,

$$C_{\epsilon}^{R_0} = \frac{\partial R_0}{\partial \epsilon} \times \frac{\epsilon}{R_0} = \frac{\sigma(\sigma\alpha_1 + \beta_1\epsilon + \sigma{\alpha_1}^2 K - K\alpha_1\beta_1\epsilon)}{(K(\epsilon\beta_1 - \sigma\alpha_1) - \sigma)[\epsilon\beta_1 - \sigma\alpha_1]} = 0.1391 > 0,$$

and take sensitivity for parameter γ,

$$C_{\gamma}^{R_0} = \frac{\partial R_0}{\partial \gamma} \times \frac{\gamma}{R_0} = 1 > 0.$$

Using the calculations above, we derived Table 4. It is clear that the sensitivity indices of the K, α_1, β_2, ϵ, and γ is positive. This indicates that increasing the value of any of these parameters, while the other parameters remain unchanged, leads to an increase in the basic reproduction number R_0, while decreasing them leads to a decrease in R_0. In contrast, the β_1, σ, and ξ parameters have negative sensitivity indices, indicating that increasing their values while the other parameters are held constant, leads to a decrease in R_0, and decreasing them instead leads to an increase in it.

6. NUMERICAL SIMULATIONS

In this section, we present numerical simulations for the model (1.4). Using the parameter values in Table 2, we obtain $R_0 = 0.6655$ and $E_2 = (0.2475, 4.88, 0)$. The simulation in Figures 1(a) and 1(b) illustrates the stability of E_2. According to the values in Table 3, we obtain $R_0 = 4.3222$ and $E_3 = (1.5650, 3.1908, 1.5201)$. It is clear that inequalities (3.2)-(3.5) also hold. As a result, according to Theorem 3.4, the endemic (the positive) equilibrium E_3 is *LAS*. In Figures 2(a) and 2(b), we can see the stability of E_3. In Figure 3, we see the effect of each parameter on the basic reproduction number.

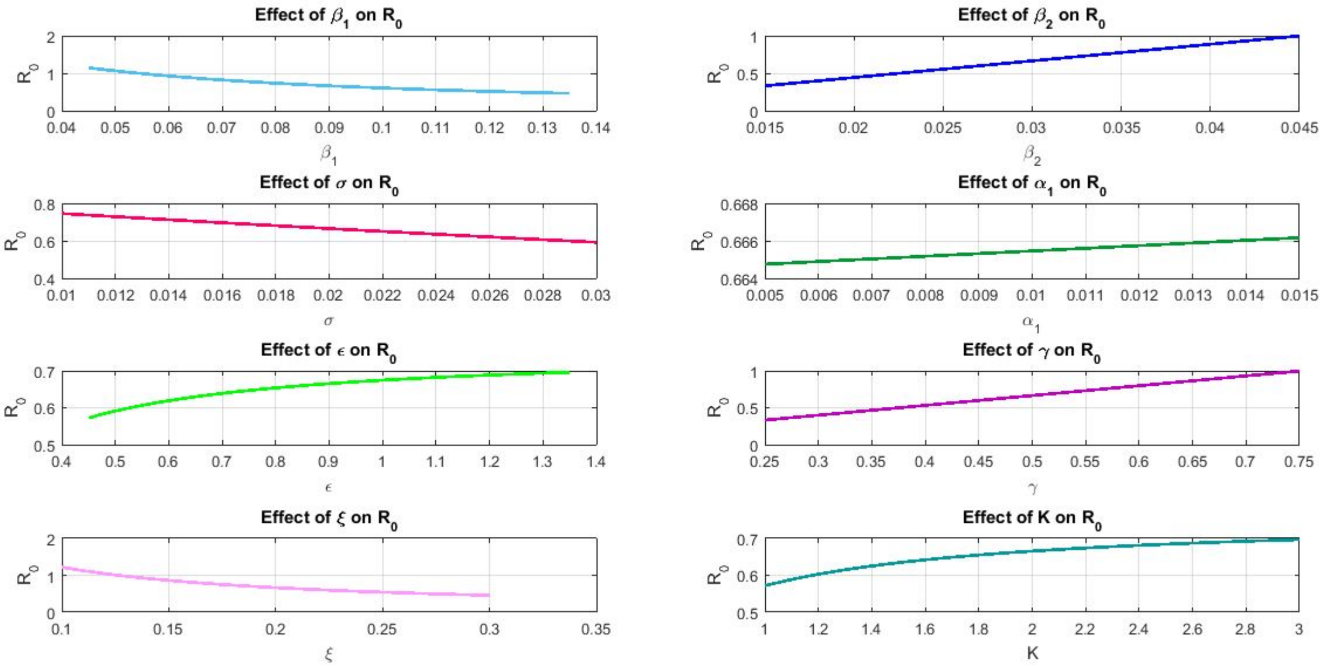


FIGURE 3. Effect on R_0 of the variation of the parameters when $\gamma = 0.5, \beta_2 = 0.03, \alpha_1 = 0.01, \beta_1 = 0.09, k = 2, \epsilon = 0.9, \sigma = 0.02, \xi = 0.9$.

7. CONCLUSION

In this paper, we studied a Prey-Predator model with an epidemic disease in predators, Holling functional response type II and the saturated incidence rate. The system's equilibria and the basic reproduction number R_0 were obtained. Also, the local stability of the equilibria was studied. By using the Sotomayor theorem, Transcritical bifurcation was investigated. Using MATLAB software, the numerical simulation of the behavior of the 3D model (1.4) is shown in Figures 1 and 2. Also, in Figure 3, we simulated the effect of parameter changes on R_0.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for his/her valuable comments on the first version of the manuscript which have led to an improvement in this revised version. There are no funders to report for this submission.

CONFLICT OF INTERESTS

This work does not have any conflicts of interest.

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