

Multiple positive symmetric solutions for the fourth-order iterative differential equations with the IHPH operator

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Abstract

The purpose of this paper is to investigate the existence of positive symmetric solutions for fourth-order iterative systems with two-point integral boundary conditions involving increasing homeomorphism and positive homomorphism operators. Firstly, we establish the existence of at least two positive symmetric solutions by using the Avery-Henderson fixed-point theorem. Later, we derive conditions for the existence of at least three positive symmetric solutions by using the Avery-Peterson fixed-point theorem.

Keywords. Cone, Fixed-point theorems, Green's function, Iterative system, Positive symmetric solutions.

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1. INTRODUCTION

Fourth-order ordinary differential equations have gained significant attention due to their frequent occurrence in numerous academic and engineering disciplines, including chemistry, physics, biotechnology, and control theory for dynamical systems. Researchers have focused extensively on studying these equations, particularly with regard to the existence and multiplicity of positive solutions under various boundary conditions. To address these problems, a variety of mathematical approaches have been employed, such as fixed point theory[2, 3, 7, 8, 10, 15], the monotone iterative method[16, 24], upper and lower solution techniques[1, 30], and shooting methods[9, 28], resulting in valuable insights and advancements in this area. To explore turbulent flow through porous media, researchers frequently rely on mathematical models to capture the complexity of these systems. These models are essential in diverse applications, including petroleum engineering, groundwater hydrology, chemical engineering, environmental science and engineering, heat transfer applications, biomechanical and biomedical engineering; for example, see [12–14, 18, 21, 22]. One important mathematical framework for studying such phenomena is the p -Laplacian equation, introduced by Leibenson [17]. The equation is given by:

$$(\varphi_p(y'(s)))' = f(s, y(s), y'(s)),$$

where the operator $\varphi_p(y) = |y|^{p-2}y$ represents the p -Laplacian for some $p > 1$. This nonlinear operator generalizes the standard Laplace operator and plays a crucial role in modeling systems that involve nonlinear diffusion, such as turbulent flow in porous media. The inverse function of the p -Laplacian operator is denoted by $\varphi_q(y)$, where $\varphi_q(y) = |y|^{q-2}y$ and the exponents p and q are related by $\frac{1}{p} + \frac{1}{q} = 1$. The p -Laplacian operator appears in many applied fields, including turbulent filtration in porous media, blood flow dynamics, rheology, viscoplastic material modeling, and material science. The study of differential equations involving this operator is therefore of great interest in both theory and practical applications.

The generalization and improvement of the p -Laplacian operator is an operator φ and is called an increasing homeomorphism and positive homomorphism operator (IHPHO). Some of the works on IHPHO are mentioned here.

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In [19], Liu and Zhang studied the following boundary value problem with increasing homeomorphism and positive homomorphism,

$$\begin{aligned}(\varphi(y'(s)))' + a(s)f(y(s)) &= 0, \quad 0 < s < 1, \\ y(0) - \beta y(0) &= y(1) + \delta y'(1) = 0,\end{aligned}$$

and established the existence and uniqueness of positive solutions by using the fixed-point index theorem in cones.

In [20], Miao, Zhou, and Song studied the following boundary value problem with increasing homeomorphism and positive homomorphism

$$\begin{aligned}(\varphi(y'(s)))' + a(s)f(y(s)) &= 0, \quad 0 < s < 1, \\ y(0) &= \sum_{i=1}^{m-2} \alpha_i y(\xi_i), \quad y'(1) = 0,\end{aligned}$$

and established the existence and uniqueness of positive solutions by using the fixed-point theorem on partially ordered sets.

In [31] Zhang, Feng, and Ge considered the p -Laplacian fourth order boundary value problem

$$\begin{aligned}(\varphi_p(y''(s)))'' &= w(s)f(s, y(s)) = 0, \quad 0 < s < 1, \\ y(0) = y(1) &= \int_0^1 v(q)y(q)dq, \\ \varphi_p(y''(0)) &= \varphi_p(y''(1)) = \int_0^1 w(q)\varphi_p(y''(q))dq,\end{aligned}$$

they obtained the existence and multiplicity of symmetric positive solutions by utilizing fixed point theory.

Recently, Prasad and Bhushanam[25] considered the iterative boundary value problem

$$\begin{aligned}y_n''(s) + \Psi(s)f_n(y_{n+1}(s)) &= 0, \quad 1 \leq n \leq \ell, \quad s \in [0, \kappa], \\ y_{\ell+1}(s) &= y_1(s), \quad s \in [0, \kappa], \\ y_n(0) &= \int_0^\kappa w(q)y_n(q)dq, \quad 1 \leq n \leq \ell, \quad y_n(\kappa) = \int_0^\kappa w(q)y_n(q)dq, \quad 1 \leq n \leq \ell,\end{aligned}$$

and established the denumerable many positive symmetric solutions by using Krasnoselskii's fixed point theorem. Following that, the researchers have explored the study of symmetric positive solutions, see [6, 11, 23, 26, 27, 29].

Inspired by the works mentioned above, we investigate the existence of multiple positive symmetric solutions for the fourth-order iterative system involving IHPHO and satisfying integral boundary conditions.

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)), & 1 \leq n \leq \ell, \quad 0 \leq s \leq \kappa, \\ y_{\ell+1}(s) = y_1(s), & 0 \leq s \leq \kappa, \end{cases} \quad (1.1)$$

satisfying boundary conditions

$$\begin{cases} \xi_1 y_n(0) - \xi_2 y_n'(0) = \int_0^\kappa v(q)y_n(q)dq, \\ \xi_1 y_n(\kappa) + \xi_2 y_n'(\kappa) = \int_0^\kappa v(q)y_n(q)dq, & 1 \leq n \leq \ell, \end{cases} \quad (1.2)$$

$$\begin{cases} \varphi(g(0)y_n''(0)) = \int_0^\kappa w(q)\varphi(g(q)y_n''(q))dq, \\ \varphi(g(\kappa)y_n''(\kappa)) = \int_0^\kappa w(q)\varphi(g(q)y_n''(q))dq, & 1 \leq n \leq \ell, \end{cases} \quad (1.3)$$

where $\ell \in \mathbb{N}$, ξ_1, ξ_2 are positive constants and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an IHPHO with $\varphi(0) = 0$. A projection $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is called a IHPHO, if the following conditions are satisfied:



- (a) If $x \leq y$, then $\varphi(x) \leq \varphi(y)$ where x, y are real-valued functions.
- (b) $\varphi(x)$ is a continuous bijection and its inverse mapping is also continuous.
- (c) $\varphi(xy) = \varphi(x)\varphi(y)$, where x, y are real-valued functions.

The following conditions are presumed to be valid in the entire paper:

- (I1) $f_n : [0, \kappa] \times [0, \infty) \rightarrow [0, \infty)$ is continuous, $f_n(\kappa - s, y) = f_n(s, y)$, $1 \leq n \leq \ell$ for all $(s, y) \in [0, \kappa] \times [0, \infty)$.
- (I2) $g(s), h(s) \in L^1[0, \kappa]$ are positive, symmetric on $[0, \kappa]$ (i.e., $g(\kappa - s) = g(s)$ for $s \in [0, \kappa]$).
- (I3) $v(s), w(s) \in L^1[0, \kappa]$ are non-negative, symmetric on $[0, \kappa]$, and $\mu_1, \mu_2 \in (0, 1)$, $\mu_1 = \int_0^\kappa v(q) dq$, $\mu_2 = \int_0^\kappa w(q) dq$, $\xi_1 - \mu_1 > 0$.

The organization of the remaining part of the paper is as follows. In Section 2, we construct Green's function and estimate the bounds for Green's function. In Section 3, we establish the existence of at least two positive symmetric solutions by using the Avery-Henderson fixed point theorem. Using the Avery-Peterson fixed point theorem, we establish the existence of at least three positive symmetric solutions. In Section 4, we provide examples to check the validity of the results.

2. GREEN'S FUNCTION AND ITS CHARACTERISTICS

Here, we derive the solution of (1.1)-(1.3) as a solution of an integral equation that includes Green's function. After that, we derive a few characteristics of the Green's function which are useful in establishing our main results.

Lemma 2.1. Assume that (I2) – (I3) hold. Then for any $u_1(s) \in C([0, \kappa], \mathbb{R})$, the BVP

$$\varphi(g(s)y_1''(s)) = u_1(s), \quad 0 \leq s \leq \kappa, \quad (2.1)$$

$$\xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q) dq, \quad \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q) dq, \quad (2.2)$$

has one and only one solution,

$$y_1(s) = - \int_0^\kappa H_1(s, t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt,$$

where $H_1(s, t)$ is the Green's function and is given by

$$H_1(s, t) = G_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G_1(q, t) v(q) dq, \quad (2.3)$$

in which

$$G_1(s, t) = \frac{1}{\xi_1(\xi_1\kappa + 2\xi_2)} \begin{cases} (\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2), & 0 \leq t \leq s, \\ (\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2), & s \leq t \leq \kappa. \end{cases} \quad (2.4)$$

Proof. An equivalent integral equation for (2.1) is

$$y_1(s) = \int_0^s (s - t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \alpha_1 s + \alpha_2.$$

By using boundary conditions (2.2), we get

$$\begin{aligned} \alpha_1 &= \frac{-1}{(\xi_1\kappa + 2\xi_2)} \int_0^\kappa (\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt, \\ \alpha_2 &= \frac{-\xi_2}{\xi_1(\xi_1\kappa + 2\xi_2)} \int_0^\kappa (\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \frac{1}{\xi_1} \int_0^\kappa v(q)y_1(q) dq. \end{aligned}$$

So, we have

$$\begin{aligned} y_1(s) &= \int_0^s (s - t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \frac{-s}{(\xi_1\kappa + 2\xi_2)} \int_0^\kappa (\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt \\ &\quad + \frac{-\xi_2}{\xi_1(\xi_1\kappa + 2\xi_2)} \int_0^\kappa (\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \frac{1}{\xi_1} \int_0^\kappa v(q)y_1(q) dq \end{aligned}$$



$$\begin{aligned}
&= \frac{-1}{\xi_1(\xi_1\kappa + 2\xi_2)} \left[\int_0^s (\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt \right. \\
&\quad \left. + \int_s^\kappa (\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt \right] + \frac{1}{\xi_1} \int_0^\kappa v(q) y_1(q) dq \\
&= - \int_0^\kappa G_1(s, t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \frac{1}{\xi_1} \int_0^\kappa v(q) y_1(q) dq.
\end{aligned}$$

After certain computations, we obtain

$$\int_0^\kappa v(q) y_1(q) dq = \frac{-\xi_1}{\xi_1 - \mu_1} \int_0^\kappa \int_0^\kappa G_1(q, t) v(q) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt dq.$$

Therefore,

$$\begin{aligned}
y_1(s) &= - \int_0^\kappa G_1(s, t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt + \frac{-1}{\xi_1 - \mu_1} \int_0^\kappa \int_0^\kappa G_1(q, t) v(q) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt dq \\
&= - \int_0^\kappa \left[G_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G_1(q, t) v(q) dq \right] \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt \\
&= - \int_0^\kappa H_1(s, t) \frac{1}{g(t)} \varphi^{-1}(u_1(t)) dt.
\end{aligned}$$

□

In establishing the unique solution of the problem, the following Lemma is required.

Lemma 2.2. Assume that (I2) – (I3) hold. Then for any $u_2(s) \in C([0, \kappa], \mathbb{R})$, the BVP

$$x_1''(s) = u_2(s), \quad 0 \leq s \leq \kappa, \quad (2.5)$$

$$x_1(0) = \int_0^\kappa w(q) x_1(q) dq, \quad x_1(\kappa) = \int_0^\kappa w(q) x_1(q) dq, \quad (2.6)$$

has one and only one solution

$$x_1(s) = - \int_0^\kappa H_2(s, t) u_2(t) dt,$$

where $H_2(s, t)$ is the Green's function and is given by

$$H_2(s, t) = G_2(s, t) + \frac{1}{1 - \mu_1} \int_0^\kappa G_2(q, t) w(q) dq, \quad (2.7)$$

in which

$$G_2(s, t) = \frac{1}{\kappa} \begin{cases} t(\kappa - s), & 0 \leq t \leq s, \\ s(\kappa - t), & s \leq t \leq \kappa. \end{cases} \quad (2.8)$$

Proof. An equivalent integral equation for (2.5) is

$$x_1(s) = \int_0^s (s - t) u_2(t) dt + \mathfrak{a}_3 s + \mathfrak{a}_4.$$

By using boundary conditions (2.6), we get

$$\mathfrak{a}_3 = \frac{-1}{\kappa} \int_0^\kappa (\kappa - t) u_2(t) dt, \quad \text{and} \quad \mathfrak{a}_4 = \int_0^\kappa w(q) x_1(q) dq.$$



So, we have

$$\begin{aligned} x_1(s) &= \int_0^s (s-t)u_2(t)dt + \frac{-1}{\kappa} \int_0^\kappa (s)(\kappa-t)u_2(t)dt + \int_0^\kappa w(q)x_1(q)dq \\ &= - \int_0^\kappa G_2(s,t)u_2(t)dt + \int_0^\kappa w(q)x_1(q)dq. \end{aligned}$$

After certain computations, we obtain

$$\int_0^\kappa w(q)x_1(q)dq = \frac{-1}{1-\mu_2} \int_0^\kappa \int_0^\kappa G_2(q,t)w(q)u_2(t)dtdq.$$

Therefore,

$$\begin{aligned} x_1(s) &= - \int_0^\kappa G_2(s,t)u_2(t)dt + \frac{-1}{1-\mu_2} \int_0^\kappa \int_0^\kappa G_2(q,t)w(q)u_2(t)dtdq \\ &= - \int_0^\kappa \left[G_2(s,t) + \frac{1}{1-\mu_2} \int_0^\kappa G_2(q,t)w(q)dq \right] u_2(t)dt \\ &= - \int_0^\kappa H_2(s,t)u_2(t)dt. \end{aligned}$$

□

Lemma 2.3. Assume that (I2) – (I3) hold. Then for any $u_3(s) \in C([0, \kappa], \mathbb{R})$, the BVP

$$(\varphi(g(s)y_1''(s)))'' = u_3(s), \quad 0 \leq s \leq \kappa, \quad (2.9)$$

satisfying boundary conditions

$$\begin{cases} \xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q)dq, & \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q)dq, \\ \varphi(g(0)y_1''(0)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq, & \varphi(g(\kappa)y_1''(\kappa)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq, \end{cases} \quad (2.10)$$

has a unique solution

$$y_1(s) = \int_0^\kappa H_1(s,t) \frac{1}{g(t)} \varphi^{-1} \left[\int_0^\kappa H_2(t,q)u_3(q)dq \right] dt,$$

where $H_1(s,t)$, $H_2(s,t)$ are defined in Lemma 2.1, Lemma 2.2 respectively.

Proof. Let $u_2(s) = \varphi(g(s)y_1''(s))$ for $0 \leq s \leq \kappa$. Then the BVP

$$(\varphi(g(s)y_1''(s)))'' = u_3(s), \quad 0 \leq s \leq \kappa,$$

$$\varphi(g(0)y_1''(0)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq, \quad \varphi(g(\kappa)y_1''(\kappa)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq$$

is equivalent to the problem

$$u_2''(s) = u_3(s), \quad 0 \leq s \leq \kappa, \quad (2.11)$$

$$u_2(0) = \int_0^\kappa w(q)u_2(q)dq, \quad u_2(\kappa) = \int_0^\kappa w(q)u_2(q)dq. \quad (2.12)$$

By Lemma 2.2, the BVP (2.11)-(2.12) has unique solution $u_2(s) = - \int_0^\kappa H_2(s,t)u_3(t)dt$. That is

$$\varphi(g(s)y_1''(s)) = - \int_0^\kappa H_2(s,t)u_3(t)dt. \quad (2.13)$$

Again by Lemma 2.1, the differential Equation (2.13) with boundary conditions

$$\xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q)dq, \quad \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q)dq,$$



has a unique solution

$$y_1(s) = \int_0^\kappa H_1(s, t) \frac{1}{g(t)} \varphi^{-1} \left[\int_0^\kappa H_2(t, q) u_3(q) dq \right] dt.$$

This completes the proof. $\square$

Lemma 2.4. Suppose (I3) holds. For $\lambda_1 \in (0, \frac{\kappa}{2})$, let $\sigma_1(\lambda_1) = \frac{\xi_1 \lambda_1 + \xi_2}{\xi_1 \kappa + \xi_2}$, $\sigma_2(\lambda_1) = \frac{\lambda_1}{\kappa}$, $\beta_1 = \frac{\xi_1}{\xi_1 - \mu_1}$, $\beta_2 = \frac{1}{1 - \mu_2}$. Then $G_i(s, t)$, $H_i(s, t)$, $i = 1, 2$ have the following properties:

- (A1) $0 \leq G_i(s, t) \leq G_i(t, t)$, $\forall s, t \in [0, \kappa]$,
- (A2) $0 \leq H_i(s, t) \leq \beta_i G_i(t, t)$, $\forall s, t \in [0, \kappa]$,
- (A3) $G_i(s, t) \geq \sigma_i(\lambda_1) G_i(t, t)$, $\forall s \in [\lambda_1, \kappa - \lambda_1]$ and $t \in [0, \kappa]$,
- (A4) $H_i(s, t) \geq \sigma_i(\lambda_1) \beta_i G_i(t, t)$, $\forall s \in [\lambda_1, \kappa - \lambda_1]$ and $t \in [0, \kappa]$,
- (A5) $G_i(\kappa - s, \kappa - t) = G_i(s, t)$, $H_i(\kappa - s, \kappa - t) = H_i(s, t)$, $\forall s, t \in [0, \kappa]$.

Proof. From (2.3)-(2.4) and (2.7)-(2.8), it is clear that (A1) and (A2) hold.

For inequality (A3), let $s \in [\lambda_1, \kappa - \lambda_1]$ and $t \leq s$, then

$$\frac{G_1(s, t)}{G_1(t, t)} = \frac{(\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2)}{(\xi_1 t + \xi_2)(\xi_1(\kappa - t) + \xi_2)} \geq \sigma_1(\lambda_1),$$

and for $s \leq t$,

$$\frac{G_1(s, t)}{G_1(t, t)} = \frac{(\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2)}{(\xi_1 t + \xi_2)(\xi_1(\kappa - t) + \xi_2)} \geq \sigma_1(\lambda_1).$$

Similarly, $G_2(s, t) \geq \sigma_2(\lambda_1) G_2(t, t)$, $\forall s \in [\lambda_1, \kappa - \lambda_1]$ and $t \in [0, \kappa]$. Hence, the inequality (A3). For the inequality (A4), consider

$$\begin{aligned} H_1(s, t) &= G_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G_1(q, t) v(q) dq \\ &\geq \sigma_1(\lambda_1) G_1(t, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa \sigma_1(\lambda_1) G_1(t, t) v(q) dq \\ &\geq \sigma_1(\lambda_1) \beta_1 G_1(t, t). \end{aligned}$$

Similarly, $H_2(s, t) \geq \sigma_2(\lambda_1) \beta_2 G_2(t, t)$. For inequality (A5), consider

$$\begin{aligned} G_1(\kappa - s, \kappa - t) &= \frac{1}{\xi_1(\xi_1 \kappa + 2\xi_2)} \begin{cases} (\xi_1(\kappa - t) + \xi_2)(\xi_1(\kappa - (\kappa - s)) + \xi_2), & 0 \leq \kappa - t \leq \kappa - s, \\ (\xi_1(\kappa - s) + \xi_2)(\xi_1(\kappa - (\kappa - t)) + \xi_2), & \kappa - s \leq \kappa - t \leq \kappa, \end{cases} \\ &= \frac{1}{\xi_1(\xi_1 \kappa + 2\xi_2)} \begin{cases} (\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2), & s \leq t \leq \kappa, \\ (\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2), & 0 \leq t \leq s, \end{cases} \\ &= G_1(s, t). \end{aligned}$$

Similarly, $G_2(\kappa - s, \kappa - t) = G_2(s, t)$.

Consider

$$\begin{aligned} H_1(\kappa - s, \kappa - t) &= G(\kappa - s, \kappa - t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G(q, \kappa - t) v(q) dq \\ &= G(s, t) + \frac{1}{\xi_1 - \mu_1} \int_\kappa^0 G(\kappa - q, \kappa - t) v(\kappa - q) d(\kappa - q) \\ &= G(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G(q, t) v(q) dq \\ &= H_1(s, t). \end{aligned}$$

Similarly, $H_2(\kappa - s, \kappa - t) = H_2(s, t)$, $\forall s, t \in [0, \kappa]$. $\square$



Note that an ℓ -tuple $(y_1(s), y_2(s), \dots, y_\ell(s))$ is a solution of (1.1)-(1.3) if and only if

$$y_n(s) = \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_n(t_2, y_{n+1}(t_2)) dt_2 \right] dt_1, \quad n = 1, 2, \dots, \ell,$$

$$y_{\ell+1}(s) = y_1(s), \quad s \in [0, \kappa], \quad 1 \leq n \leq \ell.$$

Therefore,

$$y_1(s) = \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \right. \right. \right. \\ \left. \left. \left. \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] \right. \right. \right. \\ \left. \left. \left. dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1.$$

Now we state the following Avery-Henderson and Avery and Peterson fixed point theorems to establish the existence of at least two and at least three positive symmetric solutions respectively.

Theorem 2.5. [4] Let K be a cone in a real Banach space B . Suppose α_1 and γ_1 are increasing, nonnegative continuous functionals on K and θ_1 is a nonnegative continuous functional on K with $\theta_1(0) = 0$ such that, for some positive numbers c_1 and k , $\gamma_1(y) \leq \theta_1(y) \leq \alpha_1(y)$ and $\|y\| \leq k\gamma_1(y)$, for all $y \in \overline{P(\gamma_1, c_1)}$. Suppose that there exist positive numbers a_1 and b_1 with $a_1 < b_1 < c_1$ such that $\theta_1(\lambda y) \leq \lambda\theta_1(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial K(\theta_1, b_1)$. Further, let $T : y \in \overline{P(\gamma_1, c_1)} \rightarrow K$ be a completely continuous operator such that

- (B1) $\gamma_1(Ty) > c_1$, for all $y \in \partial K(\gamma_1, c_1)$,
- (B2) $\theta_1(Ty) < b_1$, for all $y \in \partial K(\theta_1, b_1)$,
- (B3) $K(\alpha_1, a_1) \neq \emptyset$ and $\alpha_1(Ty) > a_1$, for all $y \in \partial K(\alpha_1, c_1)$.

Then, T has atleast two fixed points $y^*, y^{**} \in \overline{P(\gamma_1, c_1)}$ such that $a_1 < \alpha_1(y^*)$ with $\theta_1(y^*) < b_1$ and $b_1 < \theta_1(y^{**})$ with $\gamma_1(y^{**}) < c_1$.

Theorem 2.6. [5] Let K be a cone in a real Banach space B . Let γ_2 and θ_2 be a nonnegative continuous convex functionals on K , α_2 be a nonnegative continuous concave functional on K , and ψ_2 be a nonnegative continuous functional on K satisfying $\psi_2(\lambda u) \leq \lambda\psi_2(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $\mathcal{M}$ and d_2 ,

$$\alpha_2(u) \leq \psi_2(u) \text{ and } \|u\| \leq \mathcal{M}\gamma_2(u),$$

for all $u \in \overline{K(\gamma_2, d_2)}$. Suppose $T : \overline{K(\gamma_2, d_2)} \rightarrow \overline{K(\gamma_2, d_2)}$ is completely continuous and there exist positive numbers a_2, b_2 and c_2 with $a_2 < b_2$ such that

- (B4) $\{u \in K(\gamma_2, \theta_2, \alpha_2, b_2, c_2, d_2) : \alpha_2(u) > b_2\} \neq \emptyset$ and $\alpha_2(Tu) > b_2$ for $u \in K(\gamma_2, \theta_2, \alpha_2, b_2, c_2, d_2)$;
- (B5) $\alpha_2(Tu) > b_2$ for $u \in K(\gamma_2, \alpha_2, b_2, d_2)$ with $\theta_2(Tu) > c_2$;
- (B6) $0 \notin R(\gamma_2, \psi_2, a_2, d_2)$ and $\psi_2(Tu) < a_2$ for $u \in R(\gamma_2, \psi_2, a_2, d_2)$ with $\psi_2(u) = a_2$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{K(\gamma_2, d_2)}$ such that

$$\gamma_2(u_i) \leq d_2, \text{ for } i = 1, 2, 3;$$

$$b_2 < \alpha_2(u_1);$$

$$a_2 < \psi_2(u_2) \text{ with } \alpha_2(u_2) < b_2;$$

and

$$\psi_2(u_3) < a_2.$$



3. EXISTENCE OF POSITIVE SYMMETRIC SOLUTIONS

Let $B = \{y : y \in C^{(4)}([0, \kappa], \mathbb{R})\}$ be a Banach space with norm $\|y\| = \max_{s \in [0, \kappa]} |y(s)|$. For $\lambda_1 \in (0, \frac{\kappa}{2})$, we define the cone $K \subset B$ as

$$K = \{y \in B : y(s) \geq 0, y(s) \text{ is concave, symmetric on } [0, \kappa] \text{ and } \min_{s \in [\lambda_1, \kappa - \lambda_1]} y(s) \geq \sigma_1(\lambda_1) \|y\|\}.$$

Define operator $T : K \rightarrow B$ by

$$\begin{aligned} Ty_1(s) = & \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \right. \right. \right. \\ & \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] \right. \\ & \left. \left. \left. dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1. \end{aligned}$$

Let,

$$\begin{aligned} m = & \max \left\{ \beta_1 \int_0^\kappa G_1(t_j, t_j) \frac{1}{g(t_j)} \varphi^{-1} \left[\beta_2 \int_0^\kappa G_2(t_{j+1}, t_{j+1}) h(t_{j+1}) dt_{j+1} \right] dt_j, j = 1, 2, \dots, 2\ell - 1 \right\}, \\ M = & \min \left\{ \sigma_1(\lambda_1) \beta_1 \int_\lambda^{\kappa-\lambda} G_1(t_j, t_j) \frac{1}{g(t_j)} \varphi^{-1} \left[\sigma_2(\lambda_1) \beta_2 \int_\lambda^{\kappa-\lambda} G_2(t_{j+1}, t_{j+1}) h(t_{j+1}) dt_{j+1} \right] dt_j \right\}, \\ & j = 1, 2, \dots, 2\ell - 1. \end{aligned}$$

Lemma 3.1. For each $\lambda_1 \in (0, \frac{\kappa}{2})$, $T(K) \subset K$ and $T : K \rightarrow K$ is completely continuous.

Proof. Since $H_1(s, t) \geq 0$, $H_2(s, t) \geq 0$, $\forall s, t \in [0, \kappa]$, $(Ty_1)(s) \geq 0$. Let $y_1 \in K$, then consider

$$\begin{aligned} (Ty_1)(\kappa - s) = & \int_0^\kappa H_1(\kappa - s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \\ & \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\ & \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \left. \right) \cdots dt_4 \left. \right] dt_3 \left. \right] dt_2 \left. \right] dt_1 \\ = & \int_\kappa^0 H_1(\kappa - s, \kappa - t_1) \frac{1}{g(\kappa - t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(\kappa - t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \cdots \right. \right. \\ & \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \left. \right) \cdots dt_4 \left. \right] dt_3 \left. \right] dt_2 \left. \right] d(\kappa - t_1) \\ = & \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_\kappa^0 H_2(\kappa - t_1, \kappa - t_2) h(\kappa - t_2) f_1 \left(\kappa - t_2, \int_0^\kappa H_1(\kappa - t_2, t_3) \cdots \right. \right. \\ & \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \left. \right) \cdots dt_4 \left. \right] dt_3 \left. \right] d(\kappa - t_2) \left. \right] dt_1 \\ & \vdots \\ = & \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \right. \right. \right. \\ & \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] \right. \\ & \left. \left. \left. dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \end{aligned}$$



$$=(Ty_1)(s).$$

Hence Ty_1 is symmetric on $[0, \kappa]$. From Lemma 2.4, we get

$$\begin{aligned} (Ty_1)(s) &= \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \right. \right. \right. \\ &\quad \left. \left. \left. \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] \right. \right. \right. \\ &\quad \left. \left. \left. dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\ &\leq \beta_1 \int_0^\kappa G(t_1, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) \right. \right. \right. \\ &\quad \left. \left. \left. f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) \right. \right. \right. \right. \\ &\quad \left. \left. \left. dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1. \end{aligned}$$

So,

$$\begin{aligned} \|Ty_1\| &\leq \beta_1 \int_0^\kappa G(t_1, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \right. \right. \right. \\ &\quad \left. \left. \left. \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] \right. \right. \right. \\ &\quad \left. \left. \left. dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1. \end{aligned}$$

Again from Lemma 2.4, we get

$$\begin{aligned} \min_{s \in [\lambda_1, \kappa - \lambda_1]} \{(Ty_1)(s)\} &\geq \beta_1 \sigma_1(\lambda_1) \int_0^\kappa G(t_1, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \\ &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1. \end{aligned}$$

By using the above two inequalities, one can write

$$\min_{s \in [\lambda_1, \kappa - \lambda_1]} \{(Ty_1)(s)\} \geq \sigma_1(\lambda_1) \|Ty_1\|.$$

So, $Ty_1 \in K$ and thus $T(K) \subset K$. By the Arzela-Ascoli theorem and standard methods, we can prove that T is completely continuous. $\square$

3.1. Existence of at least two positive solutions. To prove the existence of at least two positive symmetric solutions of the BVP (1.1)-(1.3), the Avery-Henderson fixed point theorem will be the main tool. Let ψ_1 be a nonnegative continuous functional on a cone K of the real Banach space B . Then for a positive real number c_1 , we define the sets

$$K(\psi_1, c_1) = \{y \in K : \psi_1(y) < c_1\}$$

and

$$K_{c_1} = \{y \in K : \|y\| < c_1\}.$$



Also, we define the nonnegative, increasing, continuous functionals γ_1, θ_1 and α_1 on the cone K

$$\begin{aligned}\gamma_1(y) &= \min_{s \in I_1} |y(s)|, \\ \theta_1(y) &= \max_{s \in I_1} |y(s)|, \text{ and} \\ \alpha_1(y) &= \max_{[0, \kappa]} |y(s)|.\end{aligned}$$

For any $y_1 \in K$, we have

$$\gamma_1(y_1) \leq \theta_1(y_1) \leq \alpha_1(y_1) \quad (3.1)$$

and

$$\|y_1\| \leq \frac{1}{\sigma_1(\lambda_1)} \min_{s \in I_1} |y_1(s)| = \frac{1}{\sigma_1(\lambda_1)} \gamma_1(y_1) \leq \frac{1}{\sigma_1(\lambda_1)} \theta_1(y_1) \leq \frac{1}{\sigma_1(\lambda_1)} \alpha_1(y_1). \quad (3.2)$$

Theorem 3.2. Suppose there exist $0 < a'_1 < b'_1 < c'_1$ such that f_n , for $n = 1, 2, \dots, \ell$ satisfies the following conditions:

$$\begin{aligned}(\text{I4}) \quad & f_n(s, y) > \varphi\left(\frac{c'_1}{M}\right), \quad s \in I_1 \text{ and } y \in \left[c'_1, \frac{c'_1}{\sigma_1(\lambda_1)}\right], \\ (\text{I5}) \quad & f_n(s, y) < \varphi\left(\frac{b'_1}{m}\right), \quad s \in [0, \kappa] \text{ and } y \in \left[0, \frac{b'_1}{\sigma_1(\lambda_1)}\right], \\ (\text{I6}) \quad & f_n(s, y) > \varphi\left(\frac{a'_1}{M}\right), \quad s \in I_1 \text{ and } y \in \left[c'_1, \frac{c'_1}{\sigma_1(\lambda_1)}\right].\end{aligned}$$

Then the iterative boundary value problem (1.1)-(1.3) has at least two positive solutions $(y_1^*, y_2^*, \dots, y_\ell^*)$ and $(y_1^{**}, y_2^{**}, \dots, y_\ell^{**})$ such that

$$\begin{aligned}a'_1 &< \max_{s \in [0, \kappa]} \{|y_n^*(s)|\} \text{ with } \max_{s \in I_1} \{|y_n^*(s)|\} < b'_1, \quad n = 1, 2, \dots, \ell, \\ b'_1 &< \max_{s \in I_1} \{|y_n^{**}(s)|\} \text{ with } \min_{s \in I_1} \{|y_n^{**}(s)|\} < c'_1, \quad n = 1, 2, \dots, \ell.\end{aligned}$$

Proof. From Lemma 3.1, we know $T : K \rightarrow K$ is a completely continuous operator. From (3.1)-(3.2), for each $y_1 \in K$, $\gamma_1(y_1) \leq \theta_1(y_1) \leq \alpha_1(y_1)$ and $\|y_1\| \leq \frac{1}{\sigma_1(\lambda_1)} \gamma_1(y_1)$. Also for any $\lambda \in [0, 1]$ and $y_1 \in K$, $\theta_1(\lambda y_1) = \max_{s \in I_1} \{\lambda |y_1(s)|\} = \lambda \max_{s \in I_1} \{|y_1(s)|\} = \lambda \theta_1(y_1)$. It is clear that $\theta_1(0) = 0$. Since $y_1 \in \partial K(\gamma_1, c'_1)$, from (3.2) we have that

$$c'_1 = \min_{s \in I_1} |y_1(s)| \leq \|y_1\| \leq \frac{c'_1}{\sigma_1(\lambda_1)}.$$

By (I4), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned}& \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\ & \geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \varphi\left(\frac{c'_1}{M}\right) dt_{2\ell} \right] dt_{2\ell-1} \\ & \geq \frac{c'_1}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\sigma_2(\lambda_1) \beta_2 \int_{\lambda_1}^{\kappa-\lambda_1} G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\ & \geq c'_1.\end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned}& \int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\ & \quad \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) dt_{2\ell-3} \right] dt_{2\ell-3}\end{aligned}$$



$$\begin{aligned}
&\geq \int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, c'_1 \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \varphi \left(\frac{c'_1}{M} \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\geq \frac{c'_1}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} G_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\geq c'_1.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\gamma_1(Ty_1(s)) &= \min_{s \in I_1} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\
&\quad \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\
&\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\
&\geq c'_1.
\end{aligned}$$

Since $y_1 \in \partial K(\theta_1, b'_1)$, from (3.2) we have that

$$0 \leq y_1(s) \leq \|y_1\| \leq \frac{b'_1}{\sigma_1(\lambda_1)}, \text{ for } s \in [0, \kappa].$$

By (I5), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned}
&\int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\
&\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \varphi \left(\frac{b'_1}{m} \right) dt_{2\ell} \right] dt_{2\ell-1} \\
&\leq \frac{b'_1}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\beta_2 \int_0^\kappa G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\
&\leq b'_1.
\end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned}
&\int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\
&\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\leq \int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, b'_1 \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \varphi \left(\frac{b'_1}{m} \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\leq \frac{b'_1}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\leq b'_1.
\end{aligned}$$



Continuing in this fashion, we get

$$\begin{aligned}
 \theta_1(\mathbb{T}y_1(s)) &= \max_{s \in I_1} \left[\int_0^\kappa \mathbb{H}_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa \mathbb{H}_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\
 &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbb{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\
 &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\
 &\leq \int_0^\kappa \mathbb{H}_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa \mathbb{H}_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \\
 &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbb{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\
 &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\
 &\leq c'_1.
 \end{aligned}$$

Since $0 \in K$ and $\mathbf{a}'_1 > 0$, we get $K(\alpha_1, \mathbf{a}'_1) \neq \emptyset$. Since $y_1 \in \partial K(\alpha_1, \mathbf{a}'_1)$,

$$\mathbf{a}'_1 = \max_{s \in [0, \kappa]} |y_1(s)| \leq \|y_1\| \leq \frac{\mathbf{a}'_1}{\sigma_1(\lambda_1)}.$$

By (I6), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned}
 &\int_0^\kappa \mathbb{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\
 &\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \varphi\left(\frac{\mathbf{a}'_1}{M}\right) dt_{2\ell} \right] dt_{2\ell-1} \\
 &\geq \frac{\mathbf{a}'_1}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\sigma_2(\lambda_1) \beta_2 \int_{\lambda_1}^{\kappa-\lambda_1} G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\
 &\geq \mathbf{a}'_1.
 \end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned}
 &\int_0^\kappa \mathbb{H}_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbb{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\
 &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) dt_{2\ell-3} \right] dt_{2\ell-4} \\
 &\geq \int_0^\kappa \mathbb{H}_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \mathbb{H}_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \mathbf{a}'_1 \right) dt_{2\ell-1} \right] dt_{2\ell-4} \\
 &\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \varphi\left(\frac{\mathbf{a}'_1}{M}\right) dt_{2\ell-1} \right] dt_{2\ell-4} \\
 &\geq \frac{\mathbf{a}'_1}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-4} \\
 &\geq \mathbf{a}'_1.
 \end{aligned}$$



Continuing in this fashion, we get

$$\begin{aligned} \gamma_1(Ty_1(s)) &= \min_{s \in I_1} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\ &\geq a'_1. \end{aligned}$$

So, proved all the conditions of Avery-Henderson fixed point Theorem 2.5. Hence T has at least two fixed points $y_1^*, y_1^{**} \in \overline{P}(\gamma_1, c'_1)$ such that $a'_1 < \alpha_1(y^*)$ with $\theta_1(y^*) < b'_1$ and $b'_1 < \theta_1(y^{**})$ with $\gamma_1(y^{**}) < c'_1$. Setting $y_{\ell+1}(s) = y_1(s)$, we obtain a two positive symmetric solutions $(y_1^*, y_2^*, \dots, y_\ell^*)$ and $(y_1^{**}, y_2^{**}, \dots, y_\ell^{**})$ of the boundary value problem (1.1)-(1.3) given iteratively by

$$y_n(s) = \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_n(t_2, y_{n+1}(t_2)) dt_2 \right] dt_1, \quad n = 1, 2, \dots, \ell,$$

such that

$$\begin{aligned} a'_1 &< \max_{s \in [0, \kappa]} \{|y_n^*(s)|\} \text{ with } \max_{s \in I_1} \{|y_n^*(s)|\} < b'_1, \quad n = 1, 2, \dots, \ell, \\ b'_1 &< \max_{s \in I_1} \{|y_n^{**}(s)|\} \text{ with } \min_{s \in I_1} \{|y_n^{**}(s)|\} < c'_1, \quad n = 1, 2, \dots, \ell. \end{aligned}$$

□

3.2. Existence of at least three positive solutions. Let γ_2 and θ_2 be nonnegative continuous convex functionals on K , α_2 be a nonnegative continuous concave functional on a K , and ψ_2 be a nonnegative continuous functional K . Then for positive real numbers a_2 , b_2 , c_2 , and d_2 we define the following convex sets:

$$\begin{aligned} K(\gamma_2, d_2) &= \{u \in K : \gamma_2(u) < d_2\}, \\ K(\gamma_2, \alpha_2, b_2, d_2) &= \{u \in K : b_2 \leq \alpha_2(u), \gamma_2(u) \leq d_2\}, \\ K(\gamma_2, \theta_2, \alpha_2, b_2, c_2, d_2) &= \{u \in K : b_2 \leq \alpha_2(u), \theta_2(u) \leq c_2, \gamma_2(u) \leq d_2\}, \end{aligned}$$

and a closed set

$$R(\gamma_2, \psi_2, a_2, d_2) = \{u \in K : a_2 \leq \psi_2(u), \gamma_2(u) \leq d_2\}.$$

In obtaining the existence of at least three positive symmetric solutions of the BVP (1.1)-(1.3), the Avery and Peterson fixed point theorem will be the fundamental tool. For that, we define the nonnegative, continuous concave functional α_2 , the nonnegative continuous convex functionals θ_2 , γ_2 and the nonnegative continuous functional ψ_2 on the cone K by

$$\begin{aligned} \gamma_2(y) &= \max_{s \in I_1} |y(s)|, & \theta_2(y) &= \max_{s \in I_1} |y(s)|, \\ \psi_2(y) &= \min_{s \in I_2} |y(s)|, & \alpha_2(y) &= \min_{s \in I_1} |y(s)|, \end{aligned}$$

where $I_1 = [\lambda_1, \kappa - \lambda_1]$, $I_2 = [\lambda_2, \kappa - \lambda_2]$, $\lambda_1 < \lambda_2$, $\lambda_1, \lambda_2 \in (0, \frac{\kappa}{2})$. For any $y_1 \in K$, we have

$$\alpha_2(y_1) \leq \psi_2(y_1) \leq \theta_2(y_1) \leq \gamma_2(y_1), \quad (3.3)$$

and

$$\|y_1\| \leq \frac{1}{\sigma_1(\lambda_1)} \min_{s \in I_1} |y_1(s)| = \frac{1}{\sigma_1(\lambda_1)} \alpha_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)} \psi_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)} \theta_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)} \gamma_2(y_1). \quad (3.4)$$

Theorem 3.3. Suppose there exist $0 < a'_2 < b'_2 < d'_2$, $c'_2 = \frac{b'_2}{\sigma_1(\lambda_1)}$ such that f_n , for $n = 1, 2, \dots, \ell$ satisfies the following conditions:



$$(I7) \quad f_n(s, y) \leq \varphi(\frac{d'_2}{m}), \quad s \in I \text{ and } y_n \in \left[0, \sigma_1(\lambda_1)d'_2\right],$$

$$(I8) \quad f_n(s, y) > \varphi(\frac{b'_2}{M}), \quad s \in I_1 \text{ and } y_n \in \left[b'_2, c'_2\right],$$

$$(I9) \quad f_n(s, y) < \varphi(\frac{a'_2}{m}), \quad s \in I \text{ and } y_n \in \left[0, a'_2\right].$$

Then the iterative boundary value problem (1.1)-(1.3) has at least three positive solutions $(y_1^*, y_2^*, \dots, y_\ell^*), (y_1^{**}, y_2^{**}, \dots, y_\ell^{**})$ and $(y_1^{***}, y_2^{***}, \dots, y_\ell^{***})$ such that

$$\begin{aligned} \gamma_2(y_n^i) &\leq d'_2, \text{ for } i = *, **, *** , \quad n = 1, 2, \dots, \ell; \\ b'_2 &< \alpha_2(y_n^*), \quad n = 1, 2, \dots, \ell; \\ a'_2 &< \psi_2(y_n^{**}) \text{ with } \alpha_2(y_n^{**}) < b'_2, \quad n = 1, 2, \dots, \ell; \end{aligned}$$

and

$$\psi_2(y_n^{***}) < a'_2, \quad n = 1, 2, \dots, \ell.$$

Proof. From Lemma 3.1, $T : K \rightarrow K$ is a completely continuous operator. From (3.3)-(3.4), for each $y_1 \in K$, $\alpha_2(y_1) \leq \psi_2(y_1) \leq \theta_2(y_1) \leq \gamma_2(y_1)$, and $\|y_1\| \leq \frac{1}{\sigma_1(\lambda)}\gamma_2(y_1)$. Also for any $\lambda \in [0, 1]$ and $y_1 \in K$, $\psi_2(\lambda y_1) = \min_{s \in I_2} \{\lambda(|y_1(s)|)\} = \lambda \min_{s \in I_2} \{|y_1(s)|\} = \lambda \psi_2(y_1)$. It is clear that $\psi_2(0) = 0 < a'_2$, and thus $0 \notin R(\gamma_2, \psi_2, a'_2, d'_2)$. If $y_1 \in \overline{K(\gamma_2, d'_2)}$, then $\|y_1\| \leq \frac{1}{\sigma_1(\lambda)}\gamma_2(y_1) \leq \frac{1}{\sigma_1(\lambda)}d'_2$. By (I7), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned} &\int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \phi_p\left(\frac{d'_2}{m}\right) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq \frac{d'_2}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\beta_2 \int_0^\kappa G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq d'_2. \end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned} &\int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\ &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, d'_2 \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \phi_p\left(\frac{d'_2}{m}\right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \frac{d'_2}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\beta_2 \int_0^\kappa G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq d'_2. \end{aligned}$$



Continuing in this fashion, we get

$$\begin{aligned} \gamma_2(Ty_1) &= \max_{s \in I} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2 h(t_2)) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \Big] \\ &\leq d'_2. \end{aligned}$$

Therefore, $T : \overline{K(\gamma_2, d'_2)} \rightarrow \overline{K(\gamma_2, d'_2)}$, since we have $T : K \rightarrow K$ by Lemma 3.1. It is immediate that $\{y_1 \in K(\gamma_2, \theta_2, \alpha_2, b'_2, c'_2, d'_2) : \alpha_2(y_1) > b'_2\} \neq \emptyset$

Now, we prove that, if $y_1 \in R(\gamma_2, \psi_2, a'_2, d'_2)$ with $\psi_2(y_1) = a'_2$ then $\psi_2(Ty_1) < a'_2$.

By (I9), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned} &\int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \phi_p\left(\frac{a'_2}{m}\right) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq \frac{a'_2}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\beta_2 \int_0^\kappa G_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\ &\leq a'_2. \end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned} &\int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\ &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \int_0^\kappa H_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, a'_2 \right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \int_0^\kappa \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \phi_p\left(\frac{a'_2}{m}\right) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq \frac{a'_2}{m} \beta_1 \int_0^\kappa G_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \beta_2 G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-3} \\ &\leq a'_2. \end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned} \psi_2(Ty_1) &= \min_{s \in I_2} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2 h(t_2)) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \Big] \\ &\leq \max_{s \in I} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2 h(t_2)) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \end{aligned}$$



$$\varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \cdots dt_4 \left[dt_3 \right] dt_2 \left[dt_1 \right] \leq \mathbf{a}'_2.$$

Now, we prove that, if $y_1 \in K(\gamma_2, \alpha_2, \mathbf{b}'_2, \mathbf{d}'_2)$ with $\theta_2(\mathbf{T}y_1) > \mathbf{c}'_2$, then $\alpha_2(\mathbf{T}y_1) > \mathbf{b}'_2$.

Consider

$$\begin{aligned} \alpha_2(\mathbf{T}y_1(s)) &= \min_{s \in I_1} \left[\int_0^\kappa \mathbf{H}_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa \mathbf{H}_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbf{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\ &\geq \beta_1 \sigma_1(\lambda_1) \int_0^\kappa \mathbf{G}(t_1, t_1) \frac{1}{g(t_1)} \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa \mathbf{H}_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \\ &\quad \left. \left. \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbf{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\ &\geq \sigma_1(\lambda_1) \int_0^\kappa \mathbf{H}_1(s, t_1) \frac{1}{g(t_1)} \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa \mathbf{H}_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \\ &\quad \left. \left. \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbf{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \phi_q \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right) dt_2 \right] dt_1 \\ &= \sigma_1(\lambda_1) \theta_2(\mathbf{T}y_1) \\ &\geq \sigma_1(\lambda_1) \mathbf{c}'_2 = \mathbf{b}'_2. \end{aligned}$$

Now, we prove that, if $y_1 \in K(\gamma_2, \theta_2, \alpha_2, \mathbf{b}'_2, \mathbf{c}'_2, \mathbf{d}'_2)$, then $\alpha_2(\mathbf{T}y_1) > \mathbf{b}'_2$.

By (I8), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned} &\int_0^\kappa \mathbf{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \\ &\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 \mathbf{G}_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 \mathbf{G}_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) \phi_p \left(\frac{\mathbf{b}'_2}{M} \right) dt_{2\ell} \right] dt_{2\ell-1} \\ &\geq \frac{\mathbf{b}'_2}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} \mathbf{G}_1(t_{2\ell-1}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\sigma_2(\lambda_1) \beta_2 \int_{\lambda_1}^{\kappa-\lambda_1} \mathbf{G}_2(t_{2\ell}, t_{2\ell}) h(t_{2\ell}) dt_{2\ell} \right] dt_{2\ell-1} \\ &\geq \mathbf{b}'_2. \end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned} &\int_0^\kappa \mathbf{H}_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa \mathbf{H}_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \right. \right. \\ &\quad \left. \left. \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-1}, t_{2\ell}) h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right] dt_{2\ell-1} \right] dt_{2\ell-3} \right] dt_{2\ell-4} \\ &\geq \int_0^\kappa \mathbf{H}_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_0^\kappa \mathbf{H}_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) f_{\ell-1} \left(t_{2\ell-2}, \mathbf{b}'_2 \right) dt_{2\ell-1} \right] dt_{2\ell-4} \end{aligned}$$



$$\begin{aligned}
&\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1) \beta_1 G_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \phi_p\left(\frac{b'_2}{M}\right) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\geq \frac{b'_2}{M} \sigma_1(\lambda_1) \beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} G_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1} \left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1) \beta_2 G_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1} \right] dt_{2\ell-3} \\
&\geq b'_2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\alpha_2(Ty_1(s)) &= \min_{s \in I_1} \left[\int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \varphi^{-1} \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_1 \left(t_2, \int_0^\kappa H_1(t_2, t_3) \frac{1}{g(t_3)} \right. \right. \right. \\
&\quad \left. \left. \left. \varphi^{-1} \left[\int_0^\kappa H_2(t_3, t_4) h(t_4) f_2 \cdots f_{\ell-1} \left(t_{2\ell-2}, \int_0^\kappa H_1(t_{2\ell-2}, t_{2\ell-1}) \frac{1}{g(t_{2\ell-1})} \varphi^{-1} \left[\int_0^\kappa H_2(t_{2\ell-1}, t_{2\ell}) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. h(t_{2\ell}) f_\ell(t_{2\ell}, y_1(t_{2\ell})) dt_{2\ell} \right) dt_{2\ell-1} \right) \cdots dt_4 \right] dt_3 \right] dt_2 \right] dt_1 \\
&\geq b'_2.
\end{aligned}$$

So, proved all the conditions of Avery and Peterson fixed point theorem 2.6. Hence, there exist three positive fixed points $y_1^*, y_1^{**}, y_1^{***} \in \overline{K(\gamma_2, d_2)}$ such that

$$\begin{aligned}
&\gamma_2(y_1^*) \leq d'_2; \\
&b'_2 < \alpha_2(y_1^*); \\
&a'_2 < \psi_2(y_1^{**}) \text{ with } \alpha_2(y_1^{**}) < b'_2;
\end{aligned}$$

and

$$\psi_2(y_1^{***}) < a'_2.$$

Setting $y_{\ell+1}(s) = y_1(s)$, we obtain a three positive symmetric solutions $(y_1^*, y_2^*, \dots, y_\ell^*)$, $(y_1^{**}, y_2^{**}, \dots, y_\ell^{**})$ and $(y_1^{***}, y_2^{***}, \dots, y_\ell^{***})$ of the boundary value problem (1.1)-(1.3) given iteratively by

$$y_n(s) = \int_0^\kappa H_1(s, t_1) \frac{1}{g(t_1)} \phi_q \left[\int_0^\kappa H_2(t_1, t_2) h(t_2) f_n(t_2, y_{n+1}(t_2)) dt_2 \right] dt_1, \quad n = 1, 2, \dots, \ell,$$

such that

$$\begin{aligned}
&\gamma_2(y_n^i) \leq d'_2, \text{ for } i = *, **, *** , \quad n = 1, 2, \dots, \ell; \\
&b'_2 < \alpha_2(y_n^*), \quad n = 1, 2, \dots, \ell; \\
&a'_2 < \psi_2(y_n^{**}) \text{ with } \alpha_2(y_n^{**}) < b'_2, \quad n = 1, 2, \dots, \ell;
\end{aligned}$$

and

$$\psi_2(y_n^{***}) < a'_2, \quad n = 1, 2, \dots, \ell.$$

□

4. EXAMPLES

Example 4.1. Consider the following problem

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)), & 1 \leq n \leq 2, 0 \leq s \leq 1, \\ y_3(s) = y_1(s), & 0 \leq s \leq 1, \end{cases} \quad (4.1)$$

satisfying boundary conditions

$$\begin{cases} 3y_n(0) - y_n'(0) = \int_0^1 v(q)y_n(q)dq, \quad 3y_n(1) + y_n'(1) = \int_0^1 v(q)y_n(q)dq, & 1 \leq n \leq 2, \\ \varphi(g(0)y_n''(0)) = \int_0^1 w(q)\varphi(g(q)y_n''(q))dq, \quad \varphi(g(1)y_n''(1)) = \int_0^1 w(q)\varphi(g(q)y_n''(q))dq, & 1 \leq n \leq 2, \end{cases} \quad (4.2)$$



where $g(s) = -5s^2 + 5s + 4$, $h(s) = 100$, $v(s) = \frac{1}{5}$, $w(q) = \frac{1+s-s^2}{13}$,

$$f_1(s, y) = f_2(s, y) = \begin{cases} y^3 s^4 - 2y^3 s^3 + y^3 s^2 + 15, & (s, y) \in [0, 1] \times (0, 6], \\ 6y^2 s^4 - 12y^2 s^3 + 6y^2 s^2 + 3y^3 e^y - 18y^2 e^y + 15, & (s, y) \in [0, 1] \times [6, \infty). \end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{1}{5}$, $\mu_2 = \frac{7}{78}$, $\beta_1 = \frac{15}{14}$, $\beta_2 = \frac{78}{71}$,

$$\begin{aligned} H_1(s, t) &= G_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G_1(q, t) v(q) dq, \\ H_2(s, t) &= G_2(s, t) + \frac{1}{1 - \mu_1} \int_0^\kappa G_2(q, t) w(q) dq, \end{aligned}$$

in which

$$\begin{aligned} G_1(s, t) &= \frac{1}{15} \begin{cases} (3t+1)(4-3s), & 0 \leq t \leq s, \\ (3s+1)(4-3t), & s \leq t \leq 1, \end{cases} \\ G_2(s, t) &= \begin{cases} t(1-s), & 0 \leq t \leq s, \\ s(1-t), & s \leq t \leq 1. \end{cases} \end{aligned}$$

Let $\lambda_1 = \frac{1}{7}$, then $\sigma_1(\lambda_1) = \frac{5}{14}$, $\sigma_2(\lambda_1) = \frac{1}{7}$, $m = 0.3465210722$, $M = 0.03234219173$. Choose $a'_1 = 0.1$, $b'_1 = 2$, $c'_1 =$

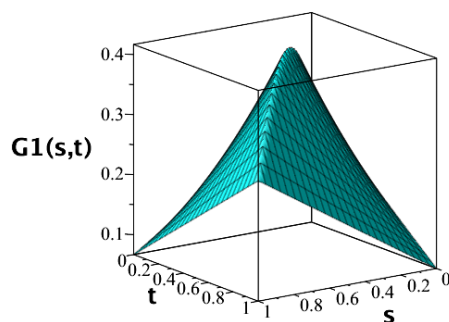


FIGURE 1. Pictorial representation of $G_1(s, t)$.

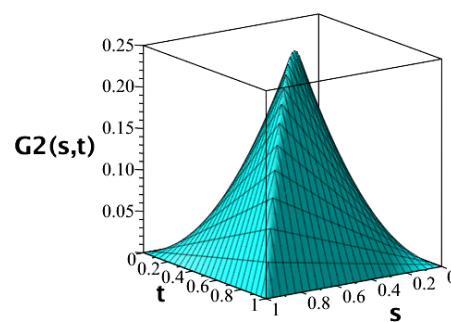


FIGURE 2. Pictorial representation of $G_2(s, t)$.

6.5, then

$$\begin{aligned} f_n(s, y) &> \varphi\left(\frac{c'_1}{M}\right) = 40391.29882, \text{ for } s \in [0.14, 0.86], y \in [6.50, 18.20], \\ f_n(s, y) &< \varphi\left(\frac{b'_1}{m}\right) = 33.31199881, \text{ for } s \in [0, 1], y \in [0, 5.60], \\ f_n(s, y) &> \varphi\left(\frac{a'_1}{M}\right) = 9.560070727, \text{ for } s \in [0.14, 0.86], y \in [0.1, 0.28]. \end{aligned}$$

Hence by Theorem 3.2, the BVP (4.1)-(4.2) has at least two positive symmetric solutions (y_1^*, y_2^*) and (y_1^{**}, y_2^{**}) such that

$$\begin{aligned} 0.1 &< \max_{s \in [0, 1]} \{|y_n^*(s)|\} \text{ with } \max_{s \in [0.14, 0.86]} \{|y_n^*(s)|\} < 2, \quad n = 1, 2, \\ 2 &< \max_{s \in [0.14, 0.86]} \{|y_n^{**}(s)|\} \text{ with } \min_{s \in [0.14, 0.86]} \{|y_n^{**}(s)|\} < 6.5, \quad n = 1, 2. \end{aligned}$$



Example 4.2. Consider the following problem

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)), & 1 \leq n \leq 2, \quad 0 \leq s \leq 4, \\ y_3(s) = y_1(s), & 0 \leq s \leq 4, \end{cases} \quad (4.3)$$

satisfying boundary conditions

$$\begin{cases} 5y_n(0) - 2y_n'(0) = \int_0^4 v(q)y_n(q)dq, \quad 5y_n(1) + 2y_n'(1) = \int_0^4 v(q)y_n(q)dq, & 1 \leq n \leq 2, \\ \varphi(g(0)y_n''(0)) = \int_0^4 w(q)\varphi(g(q)y_n''(q))dq, \quad \varphi(g(1)y_n''(1)) = \int_0^4 w(q)\varphi(g(q)y_n''(q))dq, & 1 \leq n \leq 2, \end{cases} \quad (4.4)$$

where $g(s) = \frac{137+300s-75s^2}{19}$, $h(s) = 78$, $v(s) = \frac{1+8s-2s^2}{17}$, $w(s) = \frac{4}{19}$,

$$f_1(s, y) = f_2(s, y) = \begin{cases} \frac{5ys^4-10ys^3+5ys^2+1}{24450}, & (s, y) \in [0, 4] \times (0, 2], \\ \frac{5ys^4-10ys^3+5ys^2+733500y-1466999}{24450}, & (s, y) \in [0, 4] \times (2, 3], \\ \frac{5ys^4-10ys^3+5ys^2+24450y+660151}{24450}, & (s, y) \in [0, 4] \times (3, \infty). \end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{76}{51}$, $\mu_2 = \frac{16}{19}$, $\beta_1 = \frac{255}{179}$, $\beta_2 = \frac{19}{3}$,

$$\begin{aligned} H_1(s, t) &= G_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa G_1(q, t)v(q)dq, \\ H_2(s, t) &= G_2(s, t) + \frac{1}{1 - \mu_1} \int_0^\kappa G_2(q, t)w(q)dq, \end{aligned}$$

in which

$$\begin{aligned} G_1(s, t) &= \frac{1}{120} \begin{cases} (5t+2)(22-5s), & 0 \leq t \leq s, \\ (5s+2)(22-5t), & s \leq t \leq 4, \end{cases} \\ G_2(s, t) &= \begin{cases} \frac{t}{4}(4-s), & 0 \leq t \leq s, \\ \frac{s}{4}(4-t), & s \leq t \leq 4. \end{cases} \end{aligned}$$

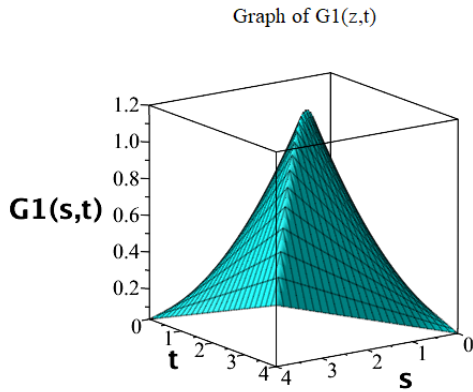


FIGURE 3. Pictorial representation of $G_1(s, t)$.

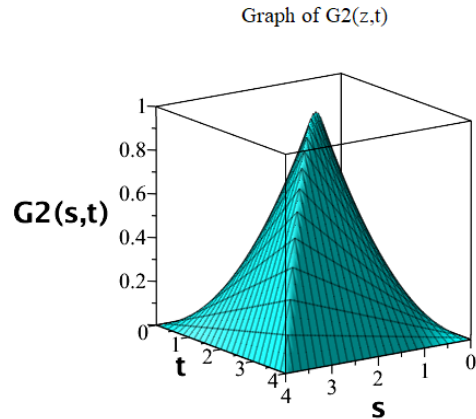


FIGURE 4. Pictorial representation of $G_2(s, t)$.

Let $\lambda_1 = 1$, $\lambda_2 = 1.5$ then $\sigma_1(\lambda_1) = \frac{7}{22}$, $\sigma_2(\lambda_1) = \frac{1}{4}$, $m = 10.73716627$, $M = 0.7111785106$. Choose $a'_2 = 1.5$, $b'_2 = 2.5$, $c'_2 = 7.857142857$, $d'_2 = 100$ then

$$f_n(s, y) < \varphi\left(\frac{d'_2}{m}\right) = 18.44215292, \text{ for } s \in [1, 3], y \in [0, 31.82],$$

$$f_n(s, y) > \varphi\left(\frac{b'_2}{M}\right) = 3.178111994, \text{ for } s \in [0, 4], y \in [2.5, 7.857142857],$$

$$f_n(s, y) < \varphi\left(\frac{a'_2}{m}\right) = 1.758390288, \text{ for } s \in [1, 3], y \in [0, 1.5].$$

Hence by Theorem 3.3, the BVP (4.1)-(4.2) has at least three positive symmetric solutions (y_1^*, y_2^*) , (y_1^{**}, y_2^{**}) and (y_1^{***}, y_2^{***}) such that

$$\begin{aligned} \max_{s \in [0, 4]} |y_n^i(s)| &\leq 100, \text{ for } i = *, **, *** , n = 1, 2; \\ 2.5 &< \min_{s \in [1, 3]} |y_n^*(s)|, n = 1, 2; \\ 1.5 &< \min_{s \in [1.5, 2.5]} |y_n^{**}(s)| \text{ with } \min_{s \in [1, 3]} |y_n^{**}(s)| < 2.5, n = 1, 2; \end{aligned}$$

and

$$\min_{s \in [1.5, 2.5]} |y_n^{***}(s)| < 1.5, n = 1, 2.$$

5. CONCLUSION

This paper presents the existence of at least two positive symmetric solutions and three positive symmetric solutions for the fourth-order iterative systems involving the IHPH operator satisfying two-point integral boundary conditions. The results are established by applying Avery-Henderson and Avery-Peterson fixed point theorems. These results may be extended to the higher order multi-point boundary value problems and fractional order boundary value problems with suitable fractional derivatives.

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