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# On the non-local $\varphi$-Hilfer hybrid fractional differential equations: an existence study

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**Abstract**

In this paper, we examine the existence of solutions for a hybrid fractional differential equation involving the $\varphi$-Hilfer derivative with a non-local condition. First, we establish the equivalence between our problem and an integral equation. Then, we utilize Dhage's renowned fixed point theorem to prove the existence of solutions. Finally, we present an illustrative example to validate our results.

**Keywords.** Hybrid fractional differential equation, $\varphi$-Hilfer derivative, Non-local condition, Fixed point theorem.

**2010 Mathematics Subject Classification.** 26A33, 34A08, 34A60.



## 1. Introduction

Fractional calculus is currently gaining great popularity among researchers in both fundamental and applied sciences. The concept of fractional order operators was defined in the 19th century by Riemann and Liouville, their aim was to extend the derivation or integration of fractional order by employing not only an integer order but also non-integer orders. Initially, it was almost a mental exercise for some renowned mathematicians who wanted to generalize the notion of differentiation from integer orders to fractional orders, allowing for the calculation of the real or complex order derivative of a differentiable function $f$, such as:

$$(D^{\alpha}f)(t) = \frac{d^{\alpha}f(t)}{dt^{\alpha}}.$$

Fractional differential equations (FDEs) naturally arise in various scientific fields such as physics, engineering, medicine, electrochemistry, control theory, etc. See [4, 11, 12, 15**?** , 16]. The effectiveness of these equations in modeling several real-world phenomena has motivated many researchers to study their quantitative and qualitative aspects.

Quadratically perturbed equations are interesting equations that form another step for solving problems in the modeling field. The problem perturbed in this way are called hybrid differential equations [10, 14, 19, 21]. The study of hybrid systems has caught the attention of the automation community, as well as that of the computing community. The objectives that can be assigned to the study of hybrid dynamical systems consist in providing a solution in terms of model, method, of performance and global quality to problems poorly treated by homogeneous approaches. For more details on the theory of Quadratically perturbed systems. See [1, 6, 9, 18–20].

Dhage and Lakshmikantham [6] discussed the following first order hybrid differential equation

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t), \quad t\in[0,1], x(t_0)=x_0, \tag{1.1}$$

where $f \in C^1([0,1]\times\mathbb{R},\mathbb{R}^*)$ and $g \in Car([0,1]\times\mathbb{R},\mathbb{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.

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Sun et al. [7], discuss the existence of solutions for the boundary value problem of fractional Hybrid differential equations

$$\begin{cases} D^{\beta}_{0_+}\left[\frac{x(t)}{f(t,x(t))}\right] + g(t,x(t) = 0, \quad t \in [0,1] \\ x(0) = x(1), \end{cases} \tag{1.2}$$

where $0 \leq \beta \leq 2$ is a real number,$D^{\beta}_{0_+}(.)$ is the Riemann-Liouville fractional derivative.

Hilal and Kajouni [9] studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 \leq \alpha \leq 1$

$$\begin{cases} {}^{c}D^{\alpha}_{0_+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t), \quad t \in [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{1.3}$$

where $f \in C^1([0,T] \times \mathbb{R}, \mathbb{R}^*)$ and $g \in Car([0,T] \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Caratheodory conditions.

Inspired by the aforementioned works, in this paper, we investigate the existence of solutions for the following nonlinear $\varphi$-Hilfer hybrid fractional differential equation:

$$\begin{cases} {}^{H}D^{\alpha,\sigma,\varphi}_{0_+}\left(\frac{u(t)}{\mathcal{V}(t,u(t)} + \mathcal{U}(t,u(t)\right) = \mathcal{H}(t,u(t)), \quad t \in \mathcal{J} = [0,b] \\ I^{1-\zeta;\varphi}_{0_+}\left(\frac{u(t)}{\mathcal{V}(t,u(t))}\right)_{t=0} + \chi(u) = \lambda \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $0 < \alpha < 1$, $0 < \sigma < 1$, $\zeta = \alpha + \sigma(1-\alpha)$, ${}^{H}D^{\alpha,\sigma,\varphi}_{0_+}(.)$ is the $\varphi$-Hilfer fractional derivative of order $\alpha$ and type $\sigma$, $\mathcal{V} \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R}^*)$, and $\mathcal{U}, \mathcal{H} \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R})$.

The key contributions of this study can be outlined as follows:

1. In this work, we employ the fractional derivative in place of the classical derivative, leveraging its advantages and the enhanced analytical outcomes it offers. Specifically, we employ the $\varphi$-Hilfer fractional derivative which offers several advantages that enhance its applicability in mathematical modeling and analysis. As a generalization of various classical and fractional derivatives, including the Riemann–Liouville, Caputo, and Hilfer types, it provides a unified framework that can capture a wide range of dynamic behaviors. The presence of the function $\varphi$ introduces a high degree of flexibility, allowing the derivative to adapt to non-uniform time scales or heterogeneous media, which is particularly useful in modeling systems with time-dependent or nonlinear structures.
2. As far as we are aware, this study represents the first effort to analyze the structure of the nonlinear $\varphi$-Hilfer hybrid fractional differential equation with a non-local condition, within the framework of the system defined in (1.4) .
3. Utilizing the $\varphi$-Hilfer fractional derivative type and its properties we provide the integral solution to the given problem (1.4); see Lemma 3.1.
4. In Section 3, we establish a framework based on the hypotheses $(H_1)$, $(H_2)$, and $(H_3)$ concerning the continuous functions $\mathcal{U}$, $\mathcal{V}$, $\mathcal{H}$, and $\chi$. This framework supports the existence results through the application of fixed-point techniques.
5. The main existence theorem is derived using alternative fixed-point theorems developed by Dhage. Additionally, Section 4 provides a concrete example to illustrate the applicability of the obtained results.
6. Moreover, this study builds upon and advances earlier research found in the literature, particularly the works cited in [3, 8, 13].

## 2. Preliminaries

In this section, we provide definitions and lemmas related to the $\varphi$-Hilfer fractional derivative, which will be used throughout the following sections of this study.

• Let $\mathcal{J} = [0,b]$ with $b < \infty$ and $C(\mathcal{J},\mathbb{R})$ the space continuous functions, endowed with the norm $\|g\| = \sup_{t\in J}\{|g(t)|\}$.

• Throughout this paper, we consider that $\varphi \in C^1(\mathcal{J},\mathbb{R})$ is an increasing differentiable function, with $\varphi^{'}(t) \neq 0$, $\forall t \in \mathcal{J}$.

• We consider the following space

$$C_{1-\zeta,\varphi}([0,b]) = \left\{\Phi : (0,b] \to \mathbb{R}, (\varphi(t)-\varphi(0))^{1-\zeta}\Phi(t) \in C^1[0,b]\right\},$$

such that

$$\|\Phi(t)\|_{C_{1-\zeta,\varphi}([0,b])} = sup_{t\in[0,b]}|(\varphi(t)-\varphi(0))^{1-\zeta}\Phi(t)|.$$

**Definition 2.1.** [? ] The left-sided $\varphi$-Riemann-Liouville fractional integral and fractional derivative of order $\alpha$, $(n-1<\alpha<n, \quad n=[\alpha]+1)$ of a function $\Phi : [0,b] \to \mathbb{R}$, are respectively defined as follows

$$I_{0_+}^{\alpha;\varphi}\Phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \varphi^{'}(s)(\varphi(t)-\varphi(s))^{\alpha-1}\Phi(s)ds,$$

and

$$D_{0_+}^{\alpha;\varphi}\Phi(t) = \left(\frac{1}{\varphi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{n-\alpha;\varphi}\Phi(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\varphi^{'}(t)}\frac{d}{dt}\right)^n \int_0^t \varphi^{'}(s)(\varphi(t)-\varphi(s))^{n-\alpha-1}\Phi(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt, z>0.$$

**Definition 2.2.** [17] The left-sided $\varphi$-Hilfer fractional derivative of order $\alpha$ and type $\sigma \in [0,1]$ of a function $\Phi \in C^n(\mathcal{J},\mathbb{R})$ is given by

$${}^{H}D_{0_+}^{\alpha,\sigma,\varphi}\Phi(t) = I_{0_+}^{\sigma(n-\alpha);\varphi}\left(\frac{1}{\varphi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-\alpha);\varphi}\Phi(t),$$

in other way

$${}^{H}D_{0_+}^{\alpha,\sigma,\varphi}\Phi(t) = I_{0_+}^{\sigma(n-\alpha);\varphi}D_{0_+}^{\zeta,\varphi}\Phi(t),$$

where

$$D_{0_+}^{\zeta,\varphi}\Phi(t) = \left(\frac{1}{\varphi^{'}(t)}\frac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-\alpha);\varphi}\Phi(t),$$

with $\zeta = \alpha+\sigma(n-\alpha)$, $(n-1<\alpha<n)$, and $n=[\alpha]+1$.

**Lemma 2.3.** *[? ] Let $\alpha,\beta>0$. Then we have*

(i) $I_{0_+}^{\alpha;\varphi}(\varphi(t)-\varphi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\varphi(t)-\varphi(0))^{\alpha+\beta-1}$.

(ii) ${}^{H}D_{0_+}^{\alpha;\sigma;\varphi}(\varphi(t)-\varphi(0))^{\zeta-1} = 0$.

**Lemma 2.4.** *[2] Let $\alpha,\beta>0$. Then we have*

$$I_{0_+}^{\alpha;\varphi}I_{0_+}^{\beta;\varphi}\Phi(t) = I_{0_+}^{\alpha+\beta;\varphi}\Phi(t).$$

**Lemma 2.5.** *[? ] Let $\alpha>0$, then $I_{0_+}^{\alpha;\varphi}\Phi(t) \in C(J,\mathbb{R})$ and*

$$\lim_{t\to 0^+} I_{0_+}^{\alpha;\varphi}\Phi(t) = I_{0_+}^{\alpha;\varphi}\Phi(0) = 0.$$

**Lemma 2.6.** *[17] Let $\Phi \in C^n(J,\mathbb{R})$, $n-1<\alpha<n$, $0\le\sigma\le1$, and $\zeta = \alpha+\sigma(n-\alpha)$. Then for all $t\in(0,b]$*

$${}^{H}D_{0_+}^{\alpha;\sigma;\varphi}I_{0_+}^{\alpha;\varphi}\Phi(t) = \Phi(t).$$

*and*

$$I_{0^+}^{\alpha;\varphi}({}^{H}D_{0_+}^{\alpha;\sigma;\varphi}\Phi(t)) = \Phi(t) - \sum_{k=1}^{n}\frac{(\varphi(t)-\varphi(0))^{(\zeta-k)}}{\Gamma(\varsigma-k+1)}\Phi_{\varphi}^{(n-k)}I_{0_+}^{(n-\sigma)(n-\alpha);\varphi}\Phi(0),$$

*where* $\Phi_{\varphi}^{(n-k)}\Phi(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^{n-k}\Phi(t)$.

*In particular, if* $0 < \alpha < 1$, *we have*

$$I_{0^+}^{\alpha;\varphi}({}^{H}D_{0_+}^{\alpha;\sigma;\varphi}\Phi(t)) = \Phi(t) - \frac{(\varphi(t)-\varphi(0))^{\zeta-1}}{\Gamma(\varsigma)} I_{0_+}^{(1-\sigma)(1-\alpha);\varphi}\Phi(0).$$

**Theorem 2.7.** *[5] Let* $\mathcal{S}$ *be a bounded, closed, and convex subset of the Banach space* $X$*. Suppose that the operators* $\mathcal{P} : X \to X$ *and* $\mathcal{F} : \mathcal{S} \to X$ *satisfy the following conditions:*

a) $\mathcal{P}$ *is a Lipschitz operator with a Lipschitz constant* $\delta$*.*
b) $\mathcal{F}$ *is completely continuous.*
c) $u = \mathcal{P}u\mathcal{F}v \Rightarrow u \in \mathcal{S}$*, for all* $v \in \mathcal{S}$*.*
d) $\delta\mathcal{M} < 1$*, where* $\mathcal{M} = \|\mathcal{F}(\mathcal{S})\|$*.*

*Then, the operator* $\mathcal{T}u = \mathcal{P}u\mathcal{F}u$ *has at least a fixed point in* $\mathcal{S}$*.*

## 3. Results

In this section, we demonstrate the equivalence between the nonlinear $\varphi$-Hilfer hybrid fractional differential Equation (1.4) and an integral equation, while introducing several key hypotheses. These hypotheses will play a crucial role in proving the existence theorem for the solution of the nonlinear $\varphi$-Hilfer hybrid fractional differential Equation (1.4) .

**Lemma 3.1.** *Let* $\mathcal{V} \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R}^*)$ *and* $\mathcal{U}, \mathcal{H} \in C(J \times \mathbb{R}, \mathbb{R})$*, for all* $u \in C_{1-\zeta,\varphi}(J, \mathbb{R})$*. Then, the problem (1.4) has a solution given by*

$$u(t) = \mathcal{V}(t, u(t))\left\{-\mathcal{V}(t, u(t)) + \Omega_{\varphi}^{\zeta}(t, 0)(\lambda - \chi(u)) + I_{0+}^{\alpha;\varphi}\mathcal{H}(t, u(t))\right\}, \tag{3.1}$$

*where* $\Omega_{\varphi}^{\zeta}(t, 0) = \frac{(\varphi(t)-\varphi(0))^{\zeta-1}}{\Gamma(\zeta)}$.

*Proof.* Applying the operator $I_{0_+}^{\alpha;\varphi}$ on both sides of the problem (1.4) and using Lemma 2.6 , we obtain

$$\frac{u(t)}{\mathcal{V}(t, u(t))} + \mathcal{U}(t, u(t)) - \Omega_{\varphi}^{\zeta}(t, 0) I_{0_+}^{(1-\zeta);\varphi}\left(\frac{u(t)}{\mathcal{V}(t, u(t))} + \mathcal{U}(t, u(t))\right)_{t=0} = I_{0_+}^{\alpha;\varphi}\mathcal{H}(t, u(t)), \quad t \in \mathcal{J},$$

using Lemma (2.5)(ii) and the initial condition $I_{0_+}^{1-\zeta;\varphi}\left(\frac{u(t)}{\mathcal{V}(t,u(t))}\right)_{t=0} + \chi(u) = \lambda$, we get

$$u(t) = \mathcal{V}(t, u(t))\left\{-\mathcal{U}(t, u(t)) + \Omega_{\varphi}^{\zeta}(t, 0)(\lambda - \chi(u)) + I_{0_+}^{\alpha;\varphi}\mathcal{H}(t, u(t))\right\}, \quad t \in \mathcal{J}. \tag{3.2}$$

On the other hand, suppose that $u(t)$ satisfies Equation (3.1) for all $t \in \mathcal{J}$. Then, we have

$$\left(\frac{u(t)}{\mathcal{V}(t, u(t))} + \mathcal{U}(t, u(t))\right) = \Omega_{\varphi}^{\zeta}(t, 0)(\lambda - \chi(u)) + I_{0_+}^{\alpha;\varphi}\mathcal{H}(t, u(t)), \quad t \in \mathcal{J}, \tag{3.3}$$

Applying the operator ${}^{H}D_{0_+}^{\alpha,\sigma,\varphi}(.)$ on both sides of Equation (3.3) , then thanks to Lemma 2.6 , and Lemma 2.3(ii) , we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\varphi}\left(\frac{u(t)}{\mathcal{V}(t, u(t))} + \mathcal{U}(t, u(t))\right) = \mathcal{H}(t, u(t)), \quad t \in \mathcal{J}.$$

To verify the initial condition, we apply the operator $I_{0+}^{1-\zeta;\varphi}(.)$ to both sides of Equation (3.3) , putting $t = 0$ and using Lemmas 2.3(i) and 2.5(ii), we get

$$I_{0_+}^{1-\zeta;\varphi}\left(\frac{u(t)}{\mathcal{V}(t, u(t))}\right)_{t=0} + \chi(u) = \lambda \in \mathbb{R}.$$

The proof is complated. $\square$

Next, we introduce the following hypotheses:
$(H_1)$ The functions $\mathcal{V}$ and $\mathcal{U}$ are bounded, and there exists a constant $\delta > 0$ such that

$$|\mathcal{V}(t,p) - \mathcal{V}(t,q)| \leq \delta |p-q|, \quad \forall p,q \in \mathbb{R}, \quad t \in \mathcal{J}.$$

$(H_2)$ There exists a function $K \in C_{1-\zeta,\varphi}(J,\mathbb{R})$ such that

$$|\mathcal{H}(t,p)| \leq (\varphi(t)-\varphi(0))^{1-\zeta} K(t) \quad t \in \mathcal{J}, p \in \mathbb{R}.$$

$(H_3)$ There exists a constant $\varepsilon > 0$ such that the function $\chi : C_{1-\zeta,\varphi} \to \mathbb{R}$ is continuous and satisfies:

$$|\chi(u)| \leq \varepsilon.$$

Let $X := (C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R}), \|.\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})})$, then $X$ is a Banach space. Define

$$\mathcal{S} = \left\{ v \in X, \|v\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} \leq R \right\},$$

where

$$\mathcal{R} = L_1 \left( (\varphi(b)-\varphi(0))^{1-\zeta} L_2 + \frac{\varepsilon + |\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)} \|K\|_{C_{1-\zeta,\varphi}(J,\mathbb{R})} \right),$$

with $L_1, L_2 > 0$ are constants such that

$$|\mathcal{V}(t,p)| \leq L_1, \quad |\mathcal{U}(t,p)| \leq L_2, \qquad p,q \in \mathbb{R},\, t \in \mathcal{J}.$$

It is clear that $\mathcal{S}$ is a bounded, closed, and convex subset of $X$.

Let $\mathcal{P} : X \to X$ and $\mathcal{F} : S \to X$ such that

$$\mathcal{P}u(t) = \mathcal{V}(t,u(t)).$$

$$\mathcal{F}u(t) = -\mathcal{U}(t,u(t)) + \Omega_{\varphi}^{\zeta}(t,0)(\lambda - \chi(u)) + I_{0^+}^{\alpha;\varphi} \mathcal{H}(t,u(t)),$$

and let $\mathcal{T} : S \to X$ defined by:

$$\mathcal{T}u(t) = \mathcal{P}u(t)\mathcal{F}u(t).$$

**Theorem 3.2.** *Consider that the hypotheses $(H_1)$, $(H_2)$, and $(H_1)$ hold. Then, the nonlinear $\varphi$-Hilfer hybrid fractional differential Equation (1.4) has a solution $u \in C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})$ provided*

$$\delta \left\{ L_2(\varphi(b)-\varphi(0))^{1-\zeta} + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)} \|K\|_{C_{1-\zeta,\varphi}(J,\mathbb{R})} \right\} < 1. \tag{3.4}$$

*Proof.* According to Theorem 2.7, the proof is given in the following steps:

**Step 1:** $\mathcal{P}$ is a Lipschitz operator.

Let $t \in [0,b]$ and $u, v \in \mathcal{S}$. Thanks to hypothesis $(H_1)$, we have

$$\begin{aligned}
&\left|(\varphi(t)-\varphi(0))^{1-\zeta}(\mathcal{P}u(t) - \mathcal{P}v(t))\right| \\
&= \left|(\varphi(t)-\varphi(0))^{1-\zeta}(\mathcal{V}(t,u(t)) - \mathcal{V}(t,v(t)))\right| \\
&\leq \delta \left|(\varphi(t)-\varphi(0))^{1-\zeta}(u(t)-v(t))\right| \\
&\leq \delta \|u-v\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})},
\end{aligned}$$

then, $\mathcal{P}$ is a Lipschitz operator.

**Step 2:** The operator $\mathcal{F}$ is completely continuous.

$(i)$ $\mathcal{F} : \mathcal{S} \to X$ is continuous.

Let $t \in [0,b]$, we consider the sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$ such that $u_n \to n$ as $n \to \infty$ in $\mathcal{S}$. Then we have

$$\begin{aligned}
\|\mathcal{F}u_n - \mathcal{F}u\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} &= \max_{t\in\mathcal{J}} \Big| (\varphi(t)-\varphi(0))^{1-\zeta} \Big( -\mathcal{U}(t,u_n(t)) + \Omega_{\varphi}^{\zeta}(t,0)\chi(u_n) + I_{0^+}^{\alpha;\varphi}\mathcal{H}(t,u_n(t)) + \mathcal{U}(t,u(t)) \\
&\quad - \Omega_{\varphi}^{\zeta}(t,0)\chi(u) - I_{0^+}^{\alpha;\varphi}\mathcal{H}(t,u(t)) \Big) \Big| \\
&\leq \max_{t\in\mathcal{J}} (\varphi(t)-\varphi(0))^{1-\zeta} \left\{ |\mathcal{U}(t,u_n(t)) - \mathcal{U}(t,u(t))| + \Omega_{\varphi}^{\zeta}(t,0)|\chi(u_n) - \chi(u)| \right.
\end{aligned}$$

$$+ \left| I_{0^+}^{\alpha;\varphi}\mathcal{H}(t,u_n(t)) - I_{0^+}^{\alpha;\varphi}\mathcal{H}(t,u(t)) \right| \Big\}.$$

By the continuity of the functions $\mathcal{H}$, $\mathcal{U}$, $\chi$ and Lebesgue dominated convergence theorem, we get

$$\|\mathcal{F}u_n - \mathcal{F}u\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} \to 0, \ as\, n \to \infty.$$

This implies that $\mathcal{F} : \mathcal{S} \to X$ is continuous.

$(ii)$ $\mathcal{F}(\mathcal{S}) = \{\mathcal{F}u : u \in \mathcal{S}\}$ is uniformly bounded.

Let $u \in \mathcal{S}$ and $t \in \mathcal{J}$. Using hypotheses $(H_1)$ and $(H_2)$, then we have

$$\begin{aligned}
|(\varphi(t)-\varphi(0))^{1-\zeta}\mathcal{F}u(t)| &= \left|(\varphi(t)-\varphi(0))^{1-\zeta}\left(-\mathcal{U}(t,u(t)) + \Omega_\varphi^\zeta(t,0)(\lambda-\chi(u)) + I_{0^+}^{\alpha;\varphi}\mathcal{H}(t,u(t))\right)\right| \\
&\leq (\varphi(t)-\varphi(0))^{1-\zeta}\,|\mathcal{U}(t,u(t))| + \left|\frac{\lambda-\chi(u)}{\Gamma(\zeta)}\right| \\
&+ (\varphi(t)-\varphi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^t \varphi^{'}(s)(\varphi(t)-\varphi(s))^{\alpha-1}\,|\mathcal{H}(s,u(s))|\,ds \\
&\leq (\varphi(b)-\varphi(0))^{1-\zeta}L_2 + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} \\
&+ (\varphi(b)-\varphi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^t \varphi^{'}(s)(\varphi(t)-\varphi(s))^{\alpha-1}\left|(\varphi(s)-\varphi(0))^{1-\zeta}K(s)\right|ds \\
&\leq (\varphi(b)-\varphi(0))^{1-\zeta}L_2 + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} \\
&+ (\varphi(b)-\varphi(0))^{1-\zeta}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}\frac{1}{\Gamma(\alpha)}\int_0^t \varphi^{'}(s)(\varphi(t)-\varphi(s))^{\alpha-1}ds \\
&\leq (\varphi(b)-\varphi(0))^{1-\zeta}L_2 + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})},
\end{aligned}$$

therfore

$$\|\mathcal{F}u\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} \leq (\varphi(b)-\varphi(0))^{1-\zeta}L_2 + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}. \tag{3.5}$$

$(iii)$ $\mathcal{F}(\mathcal{S})$ is equicontinuous.

Let $u \in \mathcal{S}$ and $\tau_1, \tau_2 \in \mathcal{J}$ with $\tau_1 < \tau_2$. By using hypothesis $(H_2)$, we get

$$\begin{aligned}
&\left|(\varphi(\tau_2)-\varphi(0))^{1-\zeta}\mathcal{F}u(\tau_2) - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\mathcal{F}u(\tau_1)\right| \\
&= \Big|-(\varphi(\tau_2)-\varphi(0))^{1-\zeta}\mathcal{U}(\tau_2,u(\tau_2)) + \left((\varphi(\tau_2)-\varphi(0))^{1-\zeta}I_{0^+}^{\alpha;\varphi}\mathcal{H}(\tau_2,u(\tau_2))\right) \\
&+ (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\mathcal{U}(\tau_1,u(\tau_1)) - \left((\varphi(\tau_1)-\tau(0))^{1-\zeta}I_{0^+}^{\alpha;\varphi}\mathcal{H}(\tau_1,u(\tau_1))\right)\Big| \\
&\leq \left|(\varphi(\tau_2)-\tau(0))^{1-\zeta}\mathcal{U}(\tau_2,u(\tau_2)) - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\mathcal{U}(\tau_1,u(\tau_1))\right| \\
&+ \left|(\varphi(\tau_2)-\varphi(0))^{1-\zeta}I_{0^+}^{\alpha;\varphi}\mathcal{H}(\tau_2,u(\tau_2)) - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}I_{0^+}^{\alpha;\varphi}\mathcal{H}(\tau_1,u(\tau_1))\right| \\
&\leq \left|(\varphi(\tau_2)-\varphi(0))^{1-\zeta}\mathcal{U}(\tau_2,u(\tau_2)) - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\mathcal{U}(\tau_1,u(\tau_1))\right| \\
&+ \Big|(\varphi(\tau_2)-\varphi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}\varphi^{'}(s)(\varphi(\tau_2)-\varphi(s))^{\alpha-1}|\mathcal{H}(s,u(s))|ds \\
&- (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\varphi^{'}(s)(\varphi(\tau_1)-\varphi(s))^{\alpha-1}|\mathcal{H}(s,u(s))|ds\Big| \\
&\leq L_2\left|(\varphi(\tau_2)-\varphi(0))^{1-\zeta} - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\right| \\
&+ \left|\frac{(\varphi(\tau_2)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} - \frac{(\varphi(\tau_1)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}\right|
\end{aligned}$$

$$\leq L_2 \left|(\varphi(\tau_2)-\varphi(0))^{1-\zeta} - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\right|$$
$$+ \frac{\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}}{\Gamma(\alpha+1)}|(\varphi(\tau_2)-\varphi(0))^{\alpha+1-\zeta} - (\varphi(\tau_1)-\varphi(0))^{\alpha+1-\zeta}|.$$

Thanks to the continuity of the function $\varphi$, we get

$$|(\varphi(\tau_2)-\varphi(0))^{1-\zeta}\mathcal{F}u(\tau_2) - (\varphi(\tau_1)-\varphi(0))^{1-\zeta}\mathcal{F}u(\tau_1)| \to 0, \quad as \quad |\tau_1-\tau_2| \to 0.$$

From $(ii)$, $(iii)$, and by Arzela-Ascoli theorem it follows that $\mathcal{F}(\mathcal{S})$ is relatively compact. Since $\mathcal{F}$ is continuous, then it is completely continuous.

**Step 3:**

Let $t \in \mathcal{J}$, $u \in C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})$, and $v \in \mathcal{S}$ such that $u(t) = \mathcal{P}u(t)\mathcal{F}v(t)$ then, we have

$$\begin{aligned}
|(\varphi(t)-\psi(0))^{1-\zeta}u(t)| &= |\mathcal{V}(t,u(t))| \left((\varphi(t)-\varphi(0))^{1-\zeta}\left|-\mathcal{U}(t,v(t)) + \Omega_{\varphi}^{\zeta}(t,0)(\lambda-\chi(u)) + I_{0_+}^{\alpha;\varphi}\mathcal{H}(t,v(t))\right|\right)\\
&\leq |\mathcal{V}(t,u(t))| \left((\varphi(t)-\varphi(0))^{1-\zeta}|U(t,v(t))| + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \left|(\varphi(t)-\varphi(0))^{1-\zeta}I_{0_+}^{\alpha;\varphi}\mathcal{H}(t,v(t))\right|\right)\\
&\leq L_1\left((\varphi(b)-\varphi(0))^{1-\zeta}L_2 + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}\right),
\end{aligned}$$

this implies that $\|u\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} \leq \mathcal{R}$, then $u \in \mathcal{S}$.

**Step 4:**

Let $\mathcal{M} = \|\mathcal{F}(\mathcal{S})\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} = \sup\{\|Fu\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})} : u \in \mathcal{S}\}$.

From inequality (3.4) and (3.5) , we have

$$\delta\mathcal{M} \leq \delta\left\{L_2(\varphi(b)-\varphi(0))^{1-\zeta} + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \frac{(\varphi(b)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})}\right\} < 1.$$

We observe that all the conditions of Dhage's fixed point Theorem 2.7 hold. Then the operator $\mathcal{T}$ has at least a fixed point in $\mathcal{S}$, this implies that the nonlinear $\varphi$-Hilfer hybrid fractional differential Equation (1.4) has at least a solution in $C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})$. □

## 4. Example

In this section, we give an illustrative example to demonstrate our results. We consider the particular case when $\varphi(t) = t$ and $\sigma = 1$, in this case, the space $C_{1-\zeta,\varphi}(\mathcal{J},\mathbb{R})$ reduces to the space $C(\mathcal{J},\mathbb{R})$ and the $\varphi$-Hilfer derivative coincides with the Caputo fractional derivative.

Let $\mathcal{J} = [0,1]$, $\mathcal{V}(t,u(t)) = 1 + \frac{cos(t)}{4}u(t)$, $\mathcal{U}(t,u(t)) = \frac{sin(t)|u(t)|}{1+|u(t)|}$, $\mathcal{H}(t,u(t)) = \frac{|u(t)|}{2+|u(t)|}$, and $\chi(u) = \frac{1}{5(1+|u(\frac{1}{3})|)}$.

Then, consider the following nonlinear hybrid fractional differential equation involving Caputo fractional derivative:

$$\begin{cases}
{}^{C}D_{0_+}^{\frac{1}{2}}\left(\frac{u(t)}{1+\frac{cos(t)}{4}u(t)} + \frac{sin(t)|u(t)|}{1+|u(t)|}\right) = \frac{|u(t)|}{2+|u(t)|}, \quad t \in [0,1],\\
I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{1+\frac{cos(t)}{4}|u(t)|}\right)_{t=0} + \frac{1}{5(1+|u(\frac{1}{3})|)} = \frac{1}{2}.
\end{cases} \tag{4.1}$$

Comparing problem (4.1) with the nonlinear hybrid fractional differential Equation (1.4). Then $\alpha = \frac{1}{2}$, $\sigma = 1$, $\zeta = 1$, $\varphi(t) = t$, and $\lambda = \frac{1}{2}$. Let's check the hypotheses $(H_1)$, $(H_2)$, and $(H_3)$. Then, we have:

$$|\mathcal{V}(t,u(t)) - \mathcal{V}(t,v(t))| = \left|\frac{cos(t)}{4}u(t) - \frac{cos(t)}{4}v(t)\right| \leq \frac{1}{4}|u-v|.$$

$$|\mathcal{U}(t,u(t))| = \left|\frac{sin(t)|u(t)|}{1+|u(t)|}\right| \leq 1.$$

$$|\mathcal{H}(t,u(t))| = \left|\frac{|u(t)|}{2+|u(t)|}\right| \leq 1.$$

$$|\chi(u)| = \frac{1}{5(1+|u(\frac{1}{3})|)} \leq \frac{1}{5}.$$

Therefore, the hypotheses $(H_1)$, $(H_2)$, and $(H_3)$ hold, with $k(t) = 1$, $L_2 = 1$, and $\delta = \frac{1}{4}$.
Now we check for condition (3.4). Further, consider

$$\begin{aligned}\delta\left\{L_2(\varphi(1)-\varphi(0))^{1-\zeta} + \frac{\varepsilon+|\lambda|}{\Gamma(\zeta)} + \|K\|_{C_{1-\zeta,\varphi}(J,\mathbb{R})}\frac{(\varphi(1)-\varphi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\right\} &= \frac{1}{4}\left\{1+\frac{\frac{1}{5}+\frac{1}{2}}{\Gamma(1)}+\frac{1}{\Gamma(\frac{3}{2})}\right\} \\ &\approx 0,707 < 1.\end{aligned}$$

We observe that all the conditions of Theorem 3.2 hold. Therefore, the system of nonlinear hybrid fractional differential Equation (4.1) has a solution in $C(\mathcal{J}, \mathbb{R})$.

## 5. Conclusion

In this paper, we have developed an existence theory for solutions to a nonlinear hybrid fractional differential equation involving the $\varphi$-Hilfer derivative under non-local conditions. The method employed is based on Dhage's fixed point theorem, and an example is provided to illustrate the main results. As a perspective for future research, one may explore the development of numerical schemes for approximating these solutions or extend the analysis to more general classes of hybrid systems involving multi-term or distributed-order fractional operators.



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