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Application of tanh–coth method for combined and double combined sinh–cosh–Gordon equations arising from chemical reactions to water surface gravity waves

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Abstract

An application of the generalized tanh–coth method to search for exact solutions of nonlinear partial differential equations is analyzed. This method is used for the combined and the double combined sinh-cosh-Gordon equations. The generalized tanh–coth method was used to construct periodic wave and solitary wave solutions of nonlinear evolution equations. This method is developed for searching for exact travelling wave solutions of nonlinear partial differential equations. It is shown that the generalized tanh–coth method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear problems.

Keywords. The generalized tanh–coth method; The combined and the double combined sinh–cosh–Gordon equations; Solitary wave and periodic wave solutions.

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1. Introduction

In the recent decade, the study of nonlinear partial differential equations in modelling physical phenomena have become an important tool. Nonlinear phenomena play a fundamental role in applied mathematics and physics. Also, the investigation of travelling-wave solutions plays an important role in nonlinear science. A variety of powerful methods have been presented, such as the inverse scattering transform [1], Hirota's bilinear method [22], the sine-cosine method [47], the homotopy perturbation method [10], the homotopy analysis method [11, 12], variational iteration method [13, 21], tanh-function method [16], Bäcklund transformation [35], Exp-function method [14, 15, 26, 27, 30, 31], tanh-coth method [8, 29, 40], $(\frac{G'}{G})$-expansion method [5, 7, 17], Laplace Adomian decomposition method [28], Differential transform method [6] and so on. Although mathematically challenging, fluid dynamics is a fascinating topic with numerous unresolved issues that can be addressed through numerical simulations and experimental methods such as computational fluid dynamics and particle image velocimetry [18, 20, 32, 33, 42]. So the study of NLPDEs, especially the study of the exact solution of NLPDEs, shows very important theoretical and application value [3, 9, 34, 38, 39, 44]. Here, we use an effective method to construct a range of exact solutions for the following nonlinear partial differential equations, for which we also developed solutions in this article. The standard tanh method is well-known analytical method, which was first presented by Malfliet's [23] and developed in [23, 24]. In this article, we explain a method called

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the generalized tanh–coth method is presented to look for travelling wave solutions of nonlinear evolution equations. The sinh-Gordon equation

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{1.1}$$

appears in integrable quantum field theory, kink dynamics, and fluid dynamics [19, 37, 41, 48–50]. The sinh-Gordon equation is completely integrable because it possesses similarity reductions to the third Painlevé equation [48]. The sine–Gordon and the double sine–Gordon, the sinh–Gordon, and the double sinh–Gordon equations given by

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + 2\alpha \sin \mathrm{u} = 0, \tag{1.2}$$

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + 2\alpha \sin \mathrm{u} + 2\beta \sin 2\mathrm{u} = 0, \tag{1.3}$$

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + 2\alpha \sinh \mathrm{u} = 0, \tag{1.4}$$

and

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + 2\alpha \sinh \mathrm{u} + 2\beta \sinh 2\mathrm{u} = 0, \tag{1.5}$$

respectively, were investigated by using the standard tanh method [23–25, 45, 46]. The (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa (DJKM) equation was investigated using the unified method, the modified Kudryashov scheme and the extended modified auxiliary equation mapping technique [43]. A mixed problem with a time derivative in the boundary conditions for the second-order inhomogeneous parabolic equation with complex coefficients was considered [2]. Authors of [36] presented the analytical and numerical solutions to the nonlinear fractional biological population equation with the fractional derivative Atangana Baleanu using the Kamal Adomian decomposition method. The principles of optimal design of mechanical drives of lifting units were investigated in a significant impact on the optimization criterion [4]. In this article, we used the generalized tanh–coth method to investigate the combined sinh–cosh-Gordon equation and the double combined sinh–cosh-Gordon equation ([48]) given by

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + \alpha \sinh \mathrm{u} + \beta \cosh \mathrm{u} = 0 \tag{1.6}$$

and

$$\mathrm{u_{tt}} - \mathrm{ku_{xx}} + \alpha \sinh \mathrm{u} + \alpha \cosh \mathrm{u} + \beta \sinh 2\mathrm{u} + \beta \cosh 2\mathrm{u} = 0, \tag{1.7}$$

respectively. The aim of this article is to obtain analytical solutions of the nonlinear combined and the double combined sinh-cosh-Gordon equations, and to determine the accuracy of the generalized tanh-coth method in solving this kind of problems. The article is organized as follows: In section 2, first, we briefly give the steps of the method and apply the method to solve the nonlinear partial differential equations. In sections 3 and 4, we examine the combined and the double combined sinh-cosh-Gordon equations, respectively. Also, a conclusion is given in section 5. Finally, some references are given at the end of this article.

2. Basic idea of generalized tanh–coth method

We now describe the generalized tanh–coth method for the given partial differential equations. We give a detailed description of the method to use this method, we take the following steps:

Step 1. For a given NLPDE with independent variables $\mathrm{X} = (\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ and dependent variable u, we consider a general form of nonlinear equation:

$$\mathcal{P}(\mathrm{u}, \mathrm{u_t}, \mathrm{u_x}, \mathrm{u_y}, \mathrm{u_z}, \mathrm{u_{xx}}, \mathrm{u_{yy}}, \mathrm{u_{zz}}, \mathrm{u_{xy}}, \mathrm{u_{tt}}, \mathrm{u_{tx}}, \mathrm{u_{ty}}, \mathrm{u_{tz}} ...) = 0, \tag{2.1}$$

which can be converted to an ODE

$$\mathcal{Q}(\mathrm{u}, -\mathrm{cu}', \mathrm{u}', \mathrm{u}', \mathrm{u}', \mathrm{u}'', ...) = 0, \tag{2.2}$$

which transformation $\xi = \mathrm{x} + \mathrm{y} - \mathrm{ct}$ is wave variable. Also, c is a constant to be determined later.

Step 2. We introduce the Riccati equation as following

$$\Phi' = \mathrm{r} + \mathrm{p}\Phi + \mathrm{q}\Phi^2, \qquad \Phi = \Phi(\xi), \qquad \xi = \mathrm{x} + \mathrm{y} - \mathrm{ct}, \tag{2.3}$$

leads to the change of derivatives

$$\frac{d}{d\xi} = \left(r + p\Phi + q\Phi^2\right)\frac{d}{d\Phi}, \tag{2.4}$$

$$\frac{d^2}{d\xi^2} = \left(r + p\Phi + q\Phi^2\right)\left[(p + 2q\Phi)\frac{d}{d\Phi} + (r + p\Phi + q\Phi^2)\frac{d^2}{d\Phi^2}\right], \tag{2.5}$$

$$\begin{aligned}\frac{d^3}{d\xi^3} = \left(r + p\Phi + q\Phi^2\right)\Bigg[&\left(6q^2\Phi^2 + 6pq\Phi + 2rq + p^2\right)\frac{d}{d\Phi} \\ &+ \left(6q^2\Phi^3 + 9pq\Phi^2 + 3(p^2+2rq)\Phi + 3rp\right)\frac{d^2}{d\Phi^2} + \left(r + p\Phi + q\Phi^2\right)^2\frac{d^3}{d\Phi^3}\Bigg],\end{aligned} \tag{2.6}$$

which admits the use of a finite series of functions of the form:

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} a_k\Phi^k + \sum_{k=1}^{m} b_k\Phi^{-k}, \tag{2.7}$$

where $a_k(k = 0, 2, ..., m), b_k(k = 1, 2, ..., m), p, r$ and q are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2). If m is not an integer, then a transformation formula should be used to overcome this difficulty. For the aforementioned method, expansion (2.7) reduces to the standard tanh method [23] for $b_k = 0, 1 \leq k \leq m$.

Step 3. Substituting Eqs. (2.3)–(2.6) into Eq. (2.2) with the value of m obtained in Step 2. Collecting the coefficients of $\Phi^k(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $a_0, a_i(i = 1, 2, ..., m), b_i(i = 1, 2, ..., m)$ p, q and r with the aid of symbolic computation Maple 13.

Step 4. Solving the algebraic equations in Step 3, then substituting $a_0, a_1, b_1, ..., a_m, b_m, c$ in Eq. (2.7).

We will consider the following special solutions of the Riccati equation (2.3):

Case 1: **For each** p, r **and** $q \neq 0$**, Eq. (2.3) has the following solutions**

$$\Phi(\xi) = \frac{-p}{2q} + \frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right), \qquad \Delta = p^2 - 4qr, \qquad \xi = x + y - ct, \tag{2.8}$$

or

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2} + C\right), \qquad \Delta = p^2 - 4qr, \qquad \xi = x + y - ct, \tag{2.9}$$

or

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right), \qquad \Delta = p^2 - 4qr, \qquad \xi = x + y - ct, \tag{2.10}$$

or

$$\Phi(\xi) = \frac{-p}{2q} - \frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2} + C\right), \qquad \Delta = p^2 - 4qr, \qquad \xi = x + y - ct, \tag{2.11}$$

where C is constant.

Case 2: **For** $p = r = 1$ **and** $q = 0$ **Eq. (2.3) has the following solution**

$$\Phi(\xi) = e^{\xi} - 1, \qquad \xi = x + y - ct. \tag{2.12}$$

Case 3: For $r=\frac{1}{2}, p=0$ and $q=-\frac{1}{2}$ Eq. (2.3) has the following solutions

$$\Phi(\xi)=-\mathrm{i}\tan\left(\frac{\xi \mathrm{i}}{2}\right), \quad \text{or} \quad \Phi(\xi)=\tanh\left(\frac{\xi}{2}\right), \quad \xi=x+y-ct, \tag{2.13}$$

$$\Phi(\xi)=\mathrm{i}\cot\left(\frac{\xi \mathrm{i}}{2}\right), \quad \text{or} \quad \Phi(\xi)=\coth\left(\frac{\xi}{2}\right), \quad \xi=x+y-ct, \tag{2.14}$$

$$\Phi(\xi)=-\mathrm{i}\tan(\mathrm{i}\xi)\pm \mathrm{i}\sec(\mathrm{i}\xi), \quad \text{or} \quad \Phi(\xi)=\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi), \quad \xi=x+y-ct, \tag{2.15}$$

but, we know

$$\tanh\left(\frac{\xi}{2}\right)=\coth(\xi)-\mathrm{csch}(\xi), \quad \coth\left(\frac{\xi}{2}\right)=\coth(\xi)+\mathrm{csch}(\xi). \tag{2.16}$$

Case 4: For $r=1, p=1$ and $q=-1$ Eq. (2.3) has the following solutions

$$\Phi(\xi)=\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh\left(\frac{\sqrt{5}}{2}\xi\right), \quad \text{or} \quad \Phi(\xi)=\frac{1}{2}-\frac{\sqrt{5}}{2}\mathrm{i}\tan\left(\frac{\sqrt{5}}{2}\mathrm{i}\xi\right), \tag{2.17}$$

$$\Phi(\xi)=\frac{1}{2}+\frac{\sqrt{5}}{2}\coth\left(\frac{\sqrt{5}}{2}\xi\right), \quad \text{or} \quad \Phi(\xi)=\frac{1}{2}+\frac{\sqrt{5}}{2}\mathrm{i}\cot\left(\frac{\sqrt{5}}{2}\mathrm{i}\xi\right). \tag{2.18}$$

3. The combined sinh–cosh-Gordon equation

In this section, we employ the combined sine–cosine–Gordon equation as follows

$$u_{tt}-ku_{xx}+\alpha\sinh u+\beta\cosh u=0. \tag{3.1}$$

Using the wave variable as follow $\xi=x-ct$, is carried to an ODE

$$(c^2-k)u''+\alpha\sinh u+\beta\cosh u=0. \tag{3.2}$$

We use the Painlevé property

$$v=e^u, \tag{3.3}$$

or equivalently

$$u=\ln v, \tag{3.4}$$

where we have

$$u'=\frac{v'}{v}, \tag{3.5}$$

$$u''=\left(\frac{v''}{v}-\frac{(v')^2}{v^2}\right).$$

The transformation (3.3) also gives

$$\sinh u=\frac{v-v^{-1}}{2}, \qquad \cosh u=\frac{v+v^{-1}}{2}, \tag{3.6}$$

that also gives

$$u=\mathrm{arccosh}\left[\frac{v+v^{-1}}{2}\right]. \tag{3.7}$$

Substituting these transformations namely (3.5) and (3.6) into Eq. (3.2) we obtain

$$2(c^2-k)(vv''-(v')^2)+(\alpha+\beta)v^3-(\alpha-\beta)v=0. \tag{3.8}$$

In order to determine value of m, we balance vv'' with v^3 in Eq. (3.8), and by using Eq. (2.7) we obtain $m = 2$. We can suppose that the solution of Eq. (3.1) is of the form

$$v(\xi) = a_0 + a_1\Phi + a_2\Phi^2 + \frac{b_1}{\Phi} + \frac{b_2}{\Phi^2}. \tag{3.9}$$

Substituting Eq. (3.9) into Eq. (3.8) and by using the well–known Maple software, we obtain the system of the following results

$$\begin{aligned} &a_0 = -\frac{4(c^2-k)qr}{\alpha+\beta}, \quad a_1 = -\frac{4(c^2-k)qp}{\alpha+\beta}, \quad a_2 = -\frac{4(c^2-k)q^2}{\alpha+\beta}, \\ &b_1 = 0, \quad b_2 = 0, \quad \alpha = \beta, \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} &a_0 = -\frac{4(c^2-k)qr}{\alpha+\beta}, \quad b_1 = -\frac{4(c^2-k)pr}{\alpha+\beta}, \quad b_2 = -\frac{4(c^2-k)r^2}{\alpha+\beta}, \\ &a_1 = 0, \quad a_2 = 0, \quad \alpha = \beta, \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} &a_0 = -\frac{(c^2-k)p^2}{\alpha+\beta}, \quad a_1 = -\frac{4(c^2-k)qp}{\alpha+\beta}, \quad a_2 = -\frac{4(c^2-k)q^2}{\alpha+\beta}, \\ &b_1 = 0, \quad b_2 = 0, \quad c = \sqrt{k + \frac{\sqrt{\beta^2-\alpha^2}}{\Delta}}, \end{aligned} \tag{3.12}$$

or

$$\begin{aligned} &a_0 = -\frac{(c^2-k)p^2}{\alpha+\beta}, \quad b_1 = -\frac{4(c^2-k)pr}{\alpha+\beta}, \quad b_2 = -\frac{4(c^2-k)r^2}{\alpha+\beta}, \\ &a_1 = 0, \quad a_2 = 0, \quad c = \sqrt{k + \frac{\sqrt{\beta^2-\alpha^2}}{\Delta}}, \end{aligned} \tag{3.13}$$

where p, q, r, and c are arbitrary constants. Substituting Eqs. (3.10)–(3.13) into expression Eq. (3.9) along with using Eq. (3.7) and using before section we obtain

$$\begin{aligned} u(\xi) &= \operatorname{arccosh}\left[-\frac{2(c^2-k)q}{\alpha+\beta}\left(r+p\Phi+q\Phi^2\right) - \frac{\alpha+\beta}{8(c^2-k)qr}\frac{1}{r+p\Phi+q\Phi^2}\right] \\ &= \operatorname{arccosh}\left[-\frac{1+\frac{16(c^2-k)^2q^2}{(\alpha+\beta)^2}\ \Phi'^2}{\frac{8(c^2-k)q}{\alpha+\beta}\ \Phi'}\right], \end{aligned} \tag{3.14}$$

$$\begin{aligned} u(\xi) &= \operatorname{arccosh}\left[-\frac{2(c^2-k)r}{\alpha+\beta}\left(\frac{r+p\Phi+q\Phi^2}{\Phi^2}\right) - \frac{\alpha+\beta}{8(c^2-k)r}\left(\frac{\Phi^2}{r+p\Phi+q\Phi^2}\right)\right] \\ &= \operatorname{arccosh}\left[-\frac{\alpha+\beta}{8(c^2-k)r}\ \frac{\frac{16(c^2-k)^2r^2}{(\alpha+\beta)^2}\Phi'^2+\Phi^4}{\Phi'\Phi^2}\right], \end{aligned} \tag{3.15}$$

$$\begin{aligned} u(\xi) &= \operatorname{arccosh}\left[-\frac{(c^2-k)}{2(\alpha+\beta)}\left(p+2q\Phi\right)^2 - \frac{\alpha+\beta}{2(c^2-k)}\ \frac{1}{\left(p+2q\Phi\right)^2}\right] \\ &= \operatorname{arccosh}\left[-\frac{\alpha+\beta}{2(c^2-k)}\frac{1+\frac{(c^2-k)^2}{(\alpha+\beta)^2}\ \left(p+2q\Phi\right)^4}{\left(p+2q\Phi\right)^2}\right], \end{aligned} \tag{3.16}$$

and

$$u(\xi) = \operatorname{arccosh}\left[-\frac{(c^2-k)}{2(\alpha+\beta)}\left(\frac{2r+p\Phi}{\Phi}\right)^2 - \frac{\alpha+\beta}{2(c^2-k)}\left(\frac{\Phi}{2r+p\Phi}\right)^2\right]$$
$$= \operatorname{arccosh}\left[-\frac{\alpha+\beta}{2(c^2-k)}\frac{\Phi^4+\frac{(c^2-k)^2}{(\alpha+\beta)^2}\left(2r+p\Phi\right)^2}{\left(2r\Phi+p\Phi^2\right)^2}\right]. \tag{3.17}$$

By the manipulation as explained in the previous section, we have

(I) The first set for Eq. (3.14)

By using **Case 1** we have

$$u_1(x,t) = \operatorname{arccosh}\left[\frac{1+\frac{(c^2-k)^2\Delta^2}{(\alpha+\beta)^2}\ \sec^4\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+C\right]}{\frac{2(c^2-k)\Delta}{\alpha+\beta}\ \sec^2\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+C\right]}\right], \tag{3.18}$$

$$u_2(x,t) = \operatorname{arccosh}\left[\frac{1+\frac{(c^2-k)^2\Delta^2}{(\alpha+\beta)^2}\ \operatorname{sech}^4\left[\frac{\sqrt{\Delta}}{2}(x-ct)+C\right]}{\frac{2(c^2-k)\Delta}{\alpha+\beta}\ \operatorname{sech}^2\left[\frac{\sqrt{\Delta}}{2}(x-ct)+C\right]}\right], \tag{3.19}$$

$$u_3(x,t) = \operatorname{arccosh}\left[\frac{1+\frac{(c^2-k)^2\Delta^2}{\alpha^2}\ \csc^4\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+C\right]}{\frac{2(c^2-k)\Delta}{\alpha}\ \csc^2\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+C\right]}\right], \tag{3.20}$$

$$u_4(x,t) = \operatorname{arccosh}\left[-\frac{1+\frac{(c^2-k)^2\Delta^2}{\alpha^2}\ \operatorname{csch}^4\left[\frac{\sqrt{\Delta}}{2}(x-ct)+C\right]}{\frac{2(c^2-k)\Delta}{\alpha}\ \operatorname{csch}^2\left[\frac{\sqrt{\Delta}}{2}(x-ct)+C\right]}\right]. \tag{3.21}$$

By using **Case 3** and $\Phi = \coth(\xi) \pm \operatorname{csch}(\xi)$, before section we have

$$u_5(x,t) = \operatorname{arccosh}\left[-\frac{1+\frac{16(c^2-k)^2q^2}{(\alpha+\beta)^2}\ \operatorname{csch}^2(x-ct)[\coth(x-ct)\pm\operatorname{csch}(x-ct)]^2}{\frac{8(c^2-k)q}{\alpha+\beta}\ \operatorname{csch}(x-ct)[\coth(x-ct)\pm\operatorname{csch}(x-ct)]}\right]. \tag{3.22}$$

By using **Case 3** and $\Phi = \tanh(\xi) \pm i\operatorname{sech}(\xi)$, we get

$$u_6(x,t) = \operatorname{arccosh}\left[-\frac{1+\frac{16(c^2-k)^2q^2}{(\alpha+\beta)^2}\ \operatorname{sech}^2(x-ct)[\operatorname{sech}(x-ct)\pm i\tanh(x-ct)]^2}{\frac{8(c^2-k)q}{\alpha+\beta}\ \operatorname{sech}(x-ct)[\operatorname{sech}(x-ct)\pm i\tanh(x-ct)]}\right]. \tag{3.23}$$

By using **Case 3** and $\Phi = \tanh(\frac{\xi}{2})$ or $\Phi = \coth(\frac{\xi}{2})$, we obtain

$$u_7(x,t) = \operatorname{arccosh}\left[-\frac{1+\frac{4(c^2-k)^2q^2}{(\alpha+\beta)^2}\ \operatorname{sech}^4[\frac{(x-ct)}{2}]}{\frac{4(c^2-k)q}{\alpha+\beta}\ \operatorname{sech}^2[\frac{(x-ct)}{2}]}\right], \tag{3.24}$$

$$u_8(x,t) = \operatorname{arccosh}\left[-\frac{1+\frac{4(c^2-k)^2q^2}{(\alpha+\beta)^2}\ \operatorname{csch}^4[\frac{(x-ct)}{2}]}{\frac{4(c^2-k)q}{\alpha+\beta}\ \operatorname{csch}^2[\frac{(x-ct)}{2}]}\right], \tag{3.25}$$

By using **Case 3** and $\Phi = -\mathrm{i}\tan(\frac{i\xi}{2})$ or $\Phi = i\coth(\frac{i\xi}{2})$, we have

$$\mathrm{u}_9(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{1+\frac{4(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \sec^4[\frac{\mathrm{i}(\mathrm{x}-\mathrm{ct})}{2}]}{\frac{4(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \sec^2[\frac{\mathrm{i}(\mathrm{x}-\mathrm{ct})}{2}]}\right], \tag{3.26}$$

$$\mathrm{u}_{10}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{1+\frac{(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \csc^4[\frac{\mathrm{i}(\mathrm{x}-\mathrm{ct})}{2}]}{\frac{2(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \csc^2[\frac{\mathrm{i}(\mathrm{x}-\mathrm{ct})}{2}]}\right]. \tag{3.27}$$

By using **Case 4** we have

$$\mathrm{u}_{11}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{1+\frac{25(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \mathrm{sech}^4[\frac{\sqrt{5}}{2}(\mathrm{x}-\mathrm{ct})]}{\frac{10(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \mathrm{sech}^2[\frac{\sqrt{5}}{2}(\mathrm{x}-\mathrm{ct})]}\right], \tag{3.28}$$

$$\mathrm{u}_{12}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{1+\frac{25(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \sec^4[\frac{\sqrt{5}}{2}\mathrm{i}(\mathrm{x}-\mathrm{ct})]}{\frac{10(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \sec^2[\frac{\sqrt{5}}{2}\mathrm{i}(\mathrm{x}-\mathrm{ct})]}\right], \tag{3.29}$$

$$\mathrm{u}_{13}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[\frac{1+\frac{25(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \mathrm{csch}^4[\frac{\sqrt{5}}{2}(\mathrm{x}-\mathrm{ct})]}{\frac{10(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \mathrm{csch}^2[\frac{\sqrt{5}}{2}(\mathrm{x}-\mathrm{ct})]}\right], \tag{3.30}$$

$$\mathrm{u}_{14}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{1+\frac{25(\mathrm{c}^2-\mathrm{k})^2\mathrm{q}^2}{(\alpha+\beta)^2}\ \csc^4[\frac{\sqrt{5}}{2}\mathrm{i}(\mathrm{x}-\mathrm{ct})]}{\frac{10(\mathrm{c}^2-\mathrm{k})\mathrm{q}}{\alpha+\beta}\ \csc^2[\frac{\sqrt{5}}{2}\mathrm{i}(\mathrm{x}-\mathrm{ct})]}\right]. \tag{3.31}$$

(II) The second set for Eq. (3.15)

By using **case 1** we have

$$\mathrm{u}_{15}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[\frac{\mathrm{q}(\alpha+\beta)}{2(\mathrm{c}^2-\mathrm{k})\mathrm{r}\Delta}\ \frac{\frac{(\mathrm{c}^2-\mathrm{k})^2\mathrm{r}^2\Delta^2}{\mathrm{q}^2(\alpha+\beta)^2}\sec^4\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)+\left[\frac{-\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{-\Delta}}{2\mathrm{q}}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\right]^4}{\sec^2\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\left[\frac{-\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{-\Delta}}{2\mathrm{q}}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\right]^2}\right], \tag{3.32}$$

$$\mathrm{u}_{16}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[\frac{\mathrm{q}(\alpha+\beta)}{2(\mathrm{c}^2-\mathrm{k})\mathrm{r}\Delta}\ \frac{\frac{(\mathrm{c}^2-\mathrm{k})^2\mathrm{r}^2\Delta^2}{\mathrm{q}^2(\alpha+\beta)^2}\mathrm{sech}^4\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)+\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{\Delta}}{2\mathrm{q}}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\right]^4}{\mathrm{sech}^2\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{\Delta}}{2\mathrm{q}}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\right]^2}\right], \tag{3.33}$$

$$\mathrm{u}_{17}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[\frac{\mathrm{q}(\alpha+\beta)}{2(\mathrm{c}^2-\mathrm{k})\mathrm{r}\Delta}\ \frac{\frac{(\mathrm{c}^2-\mathrm{k})^2\mathrm{r}^2\Delta^2}{\mathrm{q}^2(\alpha+\beta)^2}\csc^4\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)+\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{-\Delta}}{2\mathrm{q}}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\right]^4}{\csc^2\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{-\Delta}}{2\mathrm{q}}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+\mathrm{C}\right)\right]^2}\right], \tag{3.34}$$

$$\mathrm{u}_{18}(\mathrm{x},\mathrm{t}) = \mathrm{arccosh}\left[-\frac{\mathrm{q}(\alpha+\beta)}{2(\mathrm{c}^2-\mathrm{k})\mathrm{r}\Delta}\ \frac{\frac{(\mathrm{c}^2-\mathrm{k})^2\mathrm{r}^2\Delta^2}{\mathrm{q}^2(\alpha+\beta)^2}\mathrm{csch}^4\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)+\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{\Delta}}{2\mathrm{q}}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\right]^4}{\mathrm{csch}^2\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\left[\frac{\mathrm{p}}{2\mathrm{q}}+\frac{\sqrt{\Delta}}{2\mathrm{q}}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+\mathrm{C}\right)\right]^2}\right]. \tag{3.35}$$

By using **Case 2** and $\Phi = e^{\xi} - 1$, before section we have

$$u_{19}(x,t) = \operatorname{arccosh}\left[-\frac{\alpha+\beta}{8(c^2-k)}\,\frac{\frac{16(c^2-k)^2}{(\alpha+\beta)^2}e^{2(x-ct)} + (e^{x-ct}-1)^4}{e^{x-ct}(e^{x-ct}-1)^2}\right]. \tag{3.36}$$

By using **Case 3** and $\Phi = \coth(\xi) \pm \operatorname{csch}(\xi)$, before section we have

$$u_{20}(x,t) = \operatorname{arccosh}\left[-\frac{\alpha+\beta}{8(c^2-k)r}\,\frac{\frac{16(c^2-k)^2r^2}{(\alpha+\beta)^2}\operatorname{csch}^2(x-ct) + [\coth(x-ct) \pm \operatorname{csch}(x-ct)]^2}{\operatorname{csch}(x-ct)[\coth(x-ct) \pm \operatorname{csch}(x-ct)]}\right]. \tag{3.37}$$

By using **Case 3** and $\Phi = \tanh(\xi) \pm i\operatorname{sech}(\xi)$, we obtain

$$u_{21}(x,t) = \operatorname{arccosh}\left[\frac{\alpha+\beta}{8(c^2-k)r}\,\frac{\frac{16(c^2-k)^2r^2}{(\alpha+\beta)^2}\operatorname{sech}^2(x-ct) + [\operatorname{sech}(x-ct) \pm i\tanh(x-ct)]^2}{\operatorname{sech}(x-ct)[\operatorname{sech}(x-ct) \pm i\tanh(x-ct)]^3}\right]. \tag{3.38}$$

By using **Case 3** and $\Phi = \tanh(\frac{\xi}{2})$ or $\Phi = \coth(\frac{\xi}{2})$, we get

$$u_{22}(x,t) = \operatorname{arccosh}\left[-\frac{\alpha+\beta}{4(c^2-k)r}\,\frac{\frac{4(c^2-k)^2r^2}{\alpha^2}\operatorname{sech}^4[\frac{x-ct}{2}] + \tanh^4[\frac{x-ct}{2}]}{\operatorname{sech}^2[\frac{x-ct}{2}]\tanh^2[\frac{x-ct}{2}]}\right], \tag{3.39}$$

$$u_{23}(x,t) = \operatorname{arccosh}\left[\frac{\alpha+\beta}{4(c^2-k)r}\,\frac{\frac{4(c^2-k)^2r^2}{\alpha^2}\operatorname{csch}^4[\frac{x-ct}{2}] + \coth^4[\frac{x-ct}{2}]}{\operatorname{csch}^2[\frac{x-ct}{2}]\coth^2[\frac{x-ct}{2}]}\right]. \tag{3.40}$$

By using **Case 4** before Section we have

$$u_{24}(x,t) = \operatorname{arccosh}\left[-\frac{\alpha+\beta}{10(c^2-k)r}\,\frac{\frac{25(c^2-k)^2r^2}{(\alpha+\beta)^2}\operatorname{sech}^4\left[\frac{\sqrt{5}}{2}(x-ct)\right] + \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}(x-ct)\right]\right)^4}{\operatorname{sech}^2\left[\frac{\sqrt{5}}{2}(x-ct)\right]\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}(x-ct)\right]\right)^2}\right], \tag{3.41}$$

$$u_{25}(x,t) = \operatorname{arccosh}\left[\frac{\alpha+\beta}{10(c^2-k)r}\,\frac{\frac{25(c^2-k)^2r^2}{(\alpha+\beta)^2}\operatorname{csch}^4\left[\frac{\sqrt{5}}{2}(x-ct)\right] + \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}(x-ct)\right]\right)^4}{\operatorname{csch}^2\left[\frac{\sqrt{5}}{2}(x-ct)\right]\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}(x-ct)\right]\right)^2}\right], \tag{3.42}$$

$$u_{26}(x,t) = \operatorname{arccosh}\left[-\frac{\alpha+\beta}{10(c^2-k)r}\,\frac{\frac{25(c^2-k)^2r^2}{(\alpha+\beta)^2}\sec^4\left[\frac{\sqrt{5}i}{2}(x-ct)\right] + \left(\frac{1}{2} - \frac{\sqrt{5}i}{2}\tan\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\right)^4}{\sec^2\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\left(\frac{1}{2} - \frac{\sqrt{5}i}{2}\tan\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\right)^2}\right], \tag{3.43}$$

$$u_{27}(x,t) = \operatorname{arccosh}\left[\frac{\alpha+\beta}{10(c^2-k)r}\,\frac{\frac{25(c^2-k)^2r^2}{(\alpha+\beta)^2}\csc^4\left[\frac{\sqrt{5}i}{2}(x-ct)\right] + \left(\frac{1}{2} + \frac{\sqrt{5}i}{2}\cot\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\right)^4}{\csc^2\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\left(\frac{1}{2} + \frac{\sqrt{5}i}{2}\cot\left[\frac{\sqrt{5}i}{2}(x-ct)\right]\right)^2}\right]. \tag{3.44}$$

(III) The third set for Eq. (3.16)

By using **Case 1** we have

$$u_{28}(x,t) = \operatorname{arccosh}\left[\frac{(\alpha+\beta)}{2\sqrt{\beta^2-\alpha^2}}\,\frac{1 + \frac{\beta-\alpha}{\beta+\alpha}\;\tan^4\left[\frac{\sqrt{-\Delta}}{2}\left(x - \sqrt{k + \frac{\sqrt{\alpha^2+\beta^2}}{\Delta}t}\right) + C\right]}{\tan^2\left[\frac{\sqrt{-\Delta}}{2}\left(x - \sqrt{k + \frac{\sqrt{\alpha^2+\beta^2}}{\Delta}t}\right) + C\right]}\right], \tag{3.45}$$

$$u_{29}(x,t)=\operatorname{arccosh}\left[-\frac{(\alpha+\beta)}{2\sqrt{\beta^2-\alpha^2}}\frac{1+\frac{\beta-\alpha}{\beta+\alpha}\ \tanh^4\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}{\tanh^2\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}\right], \tag{3.46}$$

$$u_{30}(x,t)=\operatorname{arccosh}\left[\frac{(\alpha+\beta)}{2\sqrt{\beta^2-\alpha^2}}\frac{1+\frac{\beta-\alpha}{\beta+\alpha}\cot^4\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}{\cot^2\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}\right], \tag{3.47}$$

$$u_{31}(x,t)=\operatorname{arccosh}\left[-\frac{(\alpha+\beta)}{2\sqrt{\beta^2-\alpha^2}}\frac{1+\frac{\beta-\alpha}{\beta+\alpha}\coth^4\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}{\coth^2\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k+\frac{\sqrt{\alpha^2+\beta^2}}{\Delta}}t\right)+C\right]}\right]. \tag{3.48}$$

(IV) The fourth set for Eq. (3.17)

By using **Case 1** we have

$$u_{32}(\xi)=\operatorname{arccosh}\left[\frac{(\alpha+\beta)\Delta}{2\sqrt{\beta^2-\alpha^2}}\frac{\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^4+\frac{\beta-\alpha}{(\alpha+\beta)\Delta^2}\left(\frac{-\Delta}{2q}+\frac{p\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2}{\left(\frac{-\Delta}{2q}+\frac{p\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2}\right], \tag{3.49}$$

$$u_{33}(\xi)=\operatorname{arccosh}\left[-\frac{(\alpha+\beta)\Delta}{2\sqrt{\beta^2-\alpha^2}}\frac{\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^4+\frac{\beta-\alpha}{(\alpha+\beta)\Delta^2}\left(\frac{\Delta}{2q}+\frac{p\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2}{\left(\frac{\Delta}{2q}+\frac{p\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2}\right], \tag{3.50}$$

$$u_{34}(\xi)=\operatorname{arccosh}\left[\frac{(\alpha+\beta)\Delta}{2\sqrt{\beta^2-\alpha^2}}\frac{\left(\frac{p}{2q}+\frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^4+\frac{\beta-\alpha}{(\alpha+\beta)\Delta^2}\left(\frac{\Delta}{2q}+\frac{p\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2}{\left(\frac{\Delta}{2q}+\frac{p\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2\left(\frac{p}{2q}+\frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}+C\right)\right)^2}\right], \tag{3.51}$$

$$u_{35}(\xi)=\operatorname{arccosh}\left[-\frac{(\alpha+\beta)\Delta}{2\sqrt{\beta^2-\alpha^2}}\frac{\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^4+\frac{\beta-\alpha}{(\alpha+\beta)\Delta^2}\left(\frac{\Delta}{2q}+\frac{p\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2}{\left(\frac{\Delta}{2q}+\frac{p\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}+C\right)\right)^2}\right], \tag{3.52}$$

which are the exact solutions of the combined sinh–cosh-Gordon equation. We obtain solitary wave and periodic wave solution for the combined sinh–cosh-Gordon equation. Can be seen that the solutions developed by aforementioned method and also, results are the same, with comparing results [48].

4. The double combined sinh-cosh-Gordon equation

In this section we study the double combined sinh–cosh–Gordon equation with the generalized tanh–coth method as the following

$$u_{tt}-ku_{xx}+\alpha\sinh u+\alpha\cosh u+\beta\sinh 2u+\beta\cosh 2u=0. \tag{4.1}$$

Using the wave variable as follow $\xi=x-ct$, is carried to an ODE

$$(c^2-k)u''+\alpha\sinh u+\alpha\cosh u+\beta\sinh 2u+\beta\cosh 2u=0. \tag{4.2}$$

We use the Painlevé property

$$v=e^u, \tag{4.3}$$

or equivalently

$$\mathrm{u} = \ln \mathrm{v}, \tag{4.4}$$

where we have

$$\mathrm{u}' = \frac{\mathrm{v}'}{\mathrm{v}}, \tag{4.5}$$

$$\mathrm{u}'' = \left(\frac{\mathrm{v}''}{\mathrm{v}} - \frac{(\mathrm{v}')^2}{\mathrm{v}^2}\right).$$

The transformation (4.3) also gives

$$\sinh \mathrm{u} = \frac{\mathrm{v} - \mathrm{v}^{-1}}{2}, \qquad \cosh \mathrm{u} = \frac{\mathrm{v} + \mathrm{v}^{-1}}{2}, \tag{4.6}$$

that also gives

$$\mathrm{u} = \operatorname{arccosh}\left[\frac{\mathrm{v} + \mathrm{v}^{-1}}{2}\right]. \tag{4.7}$$

Substituting these transformations namely (4.5) and (4.6) into Eq. (4.2) we obtain

$$2(\mathrm{c}^2 - \mathrm{k})(\mathrm{v}\mathrm{v}'' - (\mathrm{v}')^2) + 2\beta \mathrm{v}^4 + 2\alpha \mathrm{v}^3 = 0. \tag{4.8}$$

Balancing vv'' with v^4 in Eq. (4.8), and by using Eq. (2.7) we obtain $\mathrm{m} = 1$. The solutions of Eq. (4.1) is of the form

$$\mathrm{v}(\xi) = \mathrm{a}_0 + \mathrm{a}_1 \Phi + \frac{\mathrm{b}_1}{\Phi}. \tag{4.9}$$

Substituting Eq. (4.9) into Eq. (4.8) and by using the well-known software Maple, we obtain the system of the following results

$$\mathrm{a}_0 = \frac{\alpha}{\beta}\,\frac{2\mathrm{qr}}{\Delta \pm \mathrm{p}\sqrt{\Delta}}, \qquad \mathrm{a}_1 = \pm\frac{\alpha}{\beta}\,\frac{\mathrm{q}}{\sqrt{\Delta}}, \qquad \mathrm{b}_1 = 0, \tag{4.10}$$

$$\mathrm{c} = \sqrt{\mathrm{k} - \frac{\alpha^2}{\beta}\,\frac{2(\mathrm{p}^2 - 2\mathrm{qr} \pm \mathrm{p}\sqrt{\Delta})}{\Delta(\sqrt{\Delta} \pm \mathrm{p})}}, \qquad \Delta = \mathrm{p}^2 - 4\mathrm{qr},$$

or

$$\mathrm{a}_0 = \frac{\alpha}{\beta}\,\frac{2\mathrm{qr}}{\Delta \pm \mathrm{p}\sqrt{\Delta}}, \qquad \mathrm{a}_1 = 0, \qquad \mathrm{b}_1 = \pm\frac{\alpha}{\beta}\,\frac{\mathrm{r}}{\sqrt{\Delta}}, \tag{4.11}$$

$$\mathrm{c} = \sqrt{\mathrm{k} - \frac{\alpha^2}{\beta}\,\frac{2(\mathrm{p}^2 - 2\mathrm{qr} \pm \mathrm{p}\sqrt{\Delta})}{\Delta(\sqrt{\Delta} \pm \mathrm{p})}}, \qquad \Delta = \mathrm{p}^2 - 4\mathrm{qr},$$

or

$$\mathrm{a}_0 = -\frac{\alpha}{\beta}, \qquad \mathrm{a}_1 = -\frac{\alpha}{\beta}\,\frac{\mathrm{q}}{\mathrm{p}}, \qquad \mathrm{b}_1 = -\frac{\alpha}{\beta}\,\frac{\mathrm{r}}{\mathrm{p}}, \qquad \mathrm{c} = \sqrt{\mathrm{k} - \frac{\alpha^2}{\beta \mathrm{p}^2}}, \tag{4.12}$$

where $\mathrm{p}, \mathrm{q}, \mathrm{r}$, and c are arbitrary constants. Substituting Eqs. (4.10)–(4.12) into expression Eq. (4.9) along with using Eq. (4.7) and using before section we obtain

$$\mathrm{u}(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha \mathrm{r}}\,\frac{1 + \frac{\alpha^2 \mathrm{r}^2}{\beta^2 \Delta}\left[\frac{2\mathrm{q}}{\sqrt{\Delta} \pm \mathrm{p}} \pm \Phi\right]}{\frac{2\mathrm{q}}{\sqrt{\Delta} \pm \mathrm{p}} \pm \Phi}\right], \tag{4.13}$$

and

$$\mathrm{u}(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha \mathrm{r}}\,\frac{(\sqrt{\Delta} \pm \mathrm{p})^2 \Phi^2 + [2\mathrm{q}\Phi \pm (\sqrt{\Delta} \pm \mathrm{p})]^2}{(\sqrt{\Delta} \pm \mathrm{p})\Phi[2\mathrm{q}\Phi \pm (\sqrt{\Delta} \pm \mathrm{p})]}\right], \tag{4.14}$$

and

$$u(\xi) = \operatorname{arccosh}\left[-\frac{\alpha}{2\beta}\left(\frac{r + p\Phi + q\Phi^2}{p\Phi}\right) - \frac{\beta}{2\alpha}\left(\frac{p\Phi}{r + p\Phi + q\Phi^2}\right)\right] = \operatorname{arccosh}\left[-\frac{\beta}{2\alpha p}\,\frac{p^2\Phi^2 + \frac{\alpha^2}{\beta^2}\Phi'^2}{\Phi\Phi'}\right]. \tag{4.15}$$

By the manipulation as explained in the previous section, we have

(I) The first set for Eq. (4.13)

By using **Case 1** we obtain

$$u_1(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \pm \frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \pm \frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right)}\right], \tag{4.16}$$

$$u_2(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2} + C\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2} + C\right)}\right], \tag{4.17}$$

$$u_3(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2} + C\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2} + C\right)}\right], \tag{4.18}$$

$$u_4(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \mp \frac{p}{2q} \mp \frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2} + C\right)}\right], \tag{4.19}$$

where $\xi = x - \sqrt{k - \frac{\alpha^2}{\beta}\,\frac{2(p^2 - 2qr \pm p\sqrt{\Delta})}{\Delta(\sqrt{\Delta}\pm p)}}\,t.$

By using **Case 2** and $\Phi = e^{\xi} - 1$, before section we have

$$u_5(x,t) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \pm e^{x - \sqrt{k - \frac{2\alpha^2}{\beta}}t} - 1\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \pm e^{x - \sqrt{k - \frac{2\alpha^2}{\beta}}t} - 1}\right]. \tag{4.20}$$

By using **Case 3** and $\Phi = \coth(\xi) \pm \operatorname{csch}(\xi)$, before section we have

$$u_6(x,t) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \pm \coth\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right) \pm \operatorname{csch}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \pm \coth\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right) \pm \operatorname{csch}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)}\right]. \tag{4.21}$$

By using **Case 3** and $\Phi = \tanh(\xi) \pm i\operatorname{sech}(\xi)$, we get

$$u_7(x,t) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \pm \tanh\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right) \pm i\operatorname{sech}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \pm \tanh\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right) \pm i\operatorname{sech}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)}\right]. \tag{4.22}$$

By using **Case 3** and $\Phi = \tanh(\frac{\xi}{2})$ or $\Phi = \coth(\frac{\xi}{2})$, we have

$$u_8(x,t) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1 + \frac{\alpha^2 r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p} \pm \tanh\left[\frac{1}{2}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)\right]\right]}{\frac{2q}{\sqrt{\Delta}\pm p} \pm \tanh\left[\frac{1}{2}\left(x - \sqrt{k - \frac{\alpha^2}{\beta}}t\right)\right]}\right], \tag{4.23}$$

$$u_9(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\coth\left[\frac{1}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\coth\left[\frac{1}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]}\right]. \tag{4.24}$$

By using **Case 3** and $\Phi=-\mathrm{i}\tan(\frac{\mathrm{i}\xi}{2})$ or $\Phi=\mathrm{i}\coth(\frac{\mathrm{i}\xi}{2})$, we obtain

$$u_{10}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\mp\mathrm{i}\tan\left[\frac{\mathrm{i}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\mp\mathrm{i}\tan\left[\frac{\mathrm{i}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]}\right], \tag{4.25}$$

$$u_{11}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\mathrm{i}\cot\left[\frac{\mathrm{i}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\mathrm{i}\cot\left[\frac{\mathrm{i}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]}\right]. \tag{4.26}$$

By using **Case 4** we have

$$u_{12}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)}\right], \tag{4.27}$$

$$u_{13}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}-\frac{\sqrt{5}\mathrm{i}}{2}\tan\left[\frac{\sqrt{5}}{2}\mathrm{i}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}-\frac{\sqrt{5}\mathrm{i}}{2}\tan\left[\frac{\sqrt{5}}{2}\mathrm{i}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)}\right], \tag{4.28}$$

$$u_{14}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)}\right], \tag{4.29}$$

$$u_{15}(x,t)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{1+\frac{\alpha^2r^2}{\beta^2\Delta}\left[\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}\mathrm{i}}{2}\cot\left[\frac{\sqrt{5}}{2}\mathrm{i}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)\right]}{\frac{2q}{\sqrt{\Delta}\pm p}\pm\left(\frac{1}{2}+\frac{\sqrt{5}\mathrm{i}}{2}\cot\left[\frac{\sqrt{5}}{2}\mathrm{i}\left(x-\sqrt{k-\frac{(\sqrt{5}+1)\alpha^2}{\beta}}t\right)\right]\right)}\right]. \tag{4.30}$$

(II) The second set for Eq. (4.14)

By using **Case 1** we have

$$u_{16}(\xi)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{(\sqrt{\Delta}\pm p)^2\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2+\left[\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]^2}{(\sqrt{\Delta}\pm p)\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\left[\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]}\right], \tag{4.31}$$

$$u_{17}(\xi)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{(\sqrt{\Delta}\pm p)^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2+\left[\left(-p-\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]^2}{(\sqrt{\Delta}\pm p)\left(\frac{-p}{2q}-\frac{\sqrt{\Delta}}{2q}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\left[\left(-p-\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]}\right], \tag{4.32}$$

$$u_{18}(\xi)=\operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{(\sqrt{\Delta}\pm p)^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2+\left[\left(-p-\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]^2}{(\sqrt{\Delta}\pm p)\left(\frac{-p}{2q}-\frac{\sqrt{\Delta}}{2q}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\left[\left(-p-\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]}\right], \tag{4.33}$$

$$u_{19}(\xi) = \operatorname{arccosh}\left[\frac{\beta\sqrt{\Delta}}{2\alpha r}\,\frac{(\sqrt{\Delta}\pm p)^2\left(\frac{p}{2q}+\frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2+\left[\left(-p-\sqrt{-\Delta}\coth\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]^2}{(\sqrt{\Delta}\pm p)\left(\frac{-p}{2q}-\frac{\sqrt{-\Delta}}{2q}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\left[\left(-p-\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\pm(\sqrt{\Delta}\pm p)\right]}\right], \tag{4.34}$$

where $C=0$ and ξ is given above by

$$\xi = x-\sqrt{k-\frac{\alpha^2}{\beta}\,\frac{2(p^2-2qr\pm p\sqrt{\Delta})}{\Delta(\sqrt{\Delta}\pm p)}}t. \tag{4.35}$$

(III) The third set for Eq. (4.15)

By using **Case 1** we have

$$u_{20}(x,t) = \operatorname{arccosh}\left[\frac{2q\beta}{\alpha p\Delta}\,\frac{p^2\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)^2+\frac{\alpha^2\Delta^2}{4\beta^2q^2}\sec^4\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}{\left(\frac{-p}{2q}+\frac{\sqrt{-\Delta}}{2q}\tan\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)\sec^2\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}\right], \tag{4.36}$$

$$u_{21}(x,t) = \operatorname{arccosh}\left[\frac{2q\beta}{\alpha p\Delta}\,\frac{p^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\tanh\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)^2+\frac{\alpha^2\Delta^2}{4\beta^2q^2}\operatorname{sech}^4\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}{\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\tanh\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)\operatorname{sech}^2\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}\right], \tag{4.37}$$

$$u_{22}(x,t) = \operatorname{arccosh}\left[\frac{2q\beta}{\alpha p\Delta}\,\frac{p^2\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\coth\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)^2+\frac{\alpha^2\Delta^2}{4\beta^2q^2}\operatorname{csch}^4\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}{\left(\frac{p}{2q}+\frac{\sqrt{\Delta}}{2q}\coth\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)\operatorname{csch}^2\left[\frac{\sqrt{\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}\right], \tag{4.38}$$

$$u_{23}(x,t) = \operatorname{arccosh}\left[-\frac{2q\beta}{\alpha p\Delta}\,\frac{p^2\left(\frac{p}{2q}+\frac{\sqrt{-\Delta}}{2q}\cot\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)^2+\frac{\alpha^2\Delta^2}{4\beta^2q^2}\csc^4\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}{\left(\frac{p}{2q}+\frac{\sqrt{-\Delta}}{2q}\cot\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]\right)\csc^2\left[\frac{\sqrt{-\Delta}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta p^2}t}\right)\right]}\right]. \tag{4.39}$$

By using **Case 2** we have

$$u_{24}(x,t) = \operatorname{arccosh}\left[-\frac{\beta}{2\alpha}\,\frac{\left(e^{x-\sqrt{k-\frac{\alpha^2}{\beta}}t}-1\right)^2+\frac{\alpha^2}{\beta^2}e^{\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)}}{e^{2\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)}-e^{\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)}}\right]. \tag{4.40}$$

By using **Case 4** we have

$$u_{25}(x,t) = \operatorname{arccosh}\left[-\frac{\beta}{\sqrt{5}\alpha}\,\frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)^2+\frac{25\alpha^2}{16\beta^2}\operatorname{sech}^4\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)\operatorname{sech}^2\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}\right], \tag{4.41}$$

$$u_{26}(x,t) = \operatorname{arccosh}\left[-\frac{\beta}{\sqrt{5}\alpha}\,\frac{\left(\frac{1}{2}-\frac{\sqrt{5}i}{2}\tan\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)^2+\frac{25\alpha^2}{16\beta^2}\sec^4\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}{\left(\frac{1}{2}-\frac{\sqrt{5}i}{2}\tanh\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)\sec^2\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}\right], \tag{4.42}$$

$$u_{27}(x,t) = \operatorname{arccosh}\left[\frac{\beta}{\sqrt{5}\alpha}\,\frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)^2+\frac{25\alpha^2}{16\beta^2}\operatorname{csch}^4\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\coth\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]\right)\operatorname{csch}^2\left[\frac{\sqrt{5}}{2}\left(x-\sqrt{k-\frac{\alpha^2}{\beta}t}\right)\right]}\right], \tag{4.43}$$

$$u_{28}(x,t) = \operatorname{arccosh}\left[-\frac{\beta}{\sqrt{5}\alpha}\,\frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}i\cot\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]\right)^2+\frac{25\alpha^2}{16\beta^2}\csc^4\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}i\cot\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]\right)\csc^2\left[\frac{\sqrt{5}}{2}i\left(x-\sqrt{k-\frac{\alpha^2}{\beta}}t\right)\right]}\right], \tag{4.44}$$

which are the exact solutions of the double combined sinh–cosh-Gordon equation. We obtain solitary wave and periodic wave solutions for the double combined sinh–cosh-Gordon equation. It can be seen that solutions formally developed for the aforementioned method also yield the same results, as compared to the results [48].

5. Conclusion

In this article, we obtained exact solutions for the combined and the double combined sinh-cosh-Gordon equations by using the generalized tanh-coth method. The generalized tanh-coth method is a useful method for finding traveling wave solutions of nonlinear evolution equations. This method has been successfully applied to obtain some new solitary wave and periodic wave solutions to the combined and the double combined sinh-cosh-Gordon equations. The generalized tanh-coth method is more powerful in searching for exact solutions of NLPDEs. Some of these results are in agreement with the results reported in the literature. Comparing our results and Wazwaz's [48] results, then it can be seen that the results are the same. Also, new results are formally developed in this article considerably. It can be concluded that this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

Conflict of interest

The authors declare that they have no conflict of interest.

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