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Deriving Novel wave solutions to the (2+1)-dimensional fractional Paraxial wave dynamical equation with Kerr law using the (G'/G^2)-expansion function technique

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Abstract

The method is used to find traveling wave solutions of nonlinear evolution equations (NLEEs), not "certain types". Specify "to find exact traveling wave solutions to certain types of nonlinear partial differential equations (NLPDEs)". In this case, the Paraxial Wave Dynamical Equation with Kerr law (PWDE) in the sense of the truncated M-fractional derivative. This equation is important in the study of wave propagation and optical phenomena. By employing this method, the researchers were able to obtain new, previously unknown exact solutions to this equation. These solutions represent different types of wave solutions, each with their own unique characteristics and properties. The significance of these novel wave solutions lies in their potential applications in physics and engineering. Wave phenomena play a crucial role in various fields, such as optics, photonics, and electromagnetics. The researchers indicate that these specific wave solutions have important practical applications in these domains. Moreover, we provide visual representations of the obtained solutions in the form of 3D, contour, and 2D plots. These graphical illustrations serve to demonstrate the feasibility and reliability of our proposed technique, showcasing its ability to capture the essential characteristics and behaviors of the solutions.

Keywords. Fractional calculus, Truncated M-fractional derivative, (G'/G^2)-expansion function method, Paraxial Wave Dynamical Equation (PWDE), Exact solutions.

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1. Introduction

Fractional Partial differential equations (FPDEs) have gained significant attention from researchers and practitioners across diverse fields in recent years. Several studies have demonstrated the benefits of employing Fractional differential equations (FDEs) over conventional differential equations for more accurate modelling of complex events, including those in the domains of biology, economics, electricity and electronic devices, quantum mechanics, and plasma physics [1–4, 30]. The ability to find solutions for nonlinear partial differential equations (NPDEs) has become increasingly important for accurately modeling a wide range of real-world phenomena. Consequently, the development of efficient techniques to derive exact traveling wave solutions for NPDEs has garnered significant attention from researchers in various fields. This is due to the crucial role that such solutions play in enhancing our understanding and representation of complex systems and processes. Some of the efficient methods that have been developed for extracting exact solutions of NPDEs are the modified (G'/G^2)-expansion method[8], the extended generalized (G'/G)-expansion method[18], the Bäcklund transformations, [9], the improved auxiliary the formula method, [31], the two-variables $(G'/G, \frac{1}{G})$-expansion method [22] and more see [10–12, 14, 16]. The paraxial wave dynamical equation (PWDE) is one of the fascinating mathematical models used in scientific research and engineering [28]. This model provides a powerful framework

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for analyzing the nonlinear propagation of light in optical fibers with Kerr media, having many important practical applications in modern photonics [5, 20]. The PWE can be written as[21, 27]:

$$i\frac{\partial w}{\partial y}+\frac{p}{2}\frac{\partial^2 w}{\partial \mathfrak{t}^2}+\frac{g}{2}\frac{\partial^2 w}{\partial x^2}+k|w|^2 w=0, \tag{1.1}$$

where $w(x, y, \mathfrak{t})$ represents the complex envelope of the electric field, and p, g, k are real constants. This equation is a form of the nonlinear Schrödinger equation (NLSE), specifically adapted for describing light propagation in optical fibers exhibiting the Kerr effect. If the parameters p and g in Eq. (1.1) satisfy $p, g < 0$, then the equation becomes a hyperbolic NLSE. However, if $p, g > 0$, then Eq. (1.1) becomes an elliptic NLSE. In recent years, researchers have utilized various analytical techniques to solve Eq. (1.1) and obtain its exact traveling wave solutions. To address Eq. (1.1), Arshad and his team have developed three distinct methods: the updated extended direct mathematical methodology, the $\exp(-\phi(\xi))$-expansion technique, and the newly enhanced simple equation method. These methods have enabled the derivation of various solutions, such as solitons, elliptic functions, and other solutions for Eq. (1.1) [5]. The modified extended mapping method was applied in [6] to produce rational solutions for Eq. (1.1), as well as solitary waves and solitons. Modulational instability (MI) analysis was used to evaluate the stability of Eq. (1.1). The results showed that all of the soliton solutions of the equation are accurate and stable. Moreover, in addition to other research, different solutions were obtained for Eq. (1.1); for details, see [20, 21]. Recently, other techniques have also been employed to solve the fractional PDWE using the conformable derivative and M-fractional derivative. Various solutions have been obtained, which can be found in [13] and [15], respectively. In this article, we change Eq. (1.1) by using the simplified M-fractional derivative in place of its traditional partial derivatives with regard to time. This yields a revised equation that can be expressed as:

$$i\frac{\partial w}{\partial y}+\frac{p}{2}\mathfrak{D}^{\mu,\nu}_{M,t}w+\frac{g}{2}\frac{\partial^2 w}{\partial x^2}+k|w|^2 w=0, \quad 0<\mu\leq 1, \;\; \nu>0, \tag{1.2}$$

where the M-truncated derivatives of order μ with regard to $\mathfrak{t}$ are represented by the expressions $\mathfrak{D}^{\mu,\nu}_{M,t}$.

The main goal of this paper is to find accurate traveling wave solutions for Eq. (1.2) by employing the (G'/G^2)-expansion function approach. This will reveal new solutions and their as-yet-unexplored physical behavior. The structure that follows constitutes the way the paper is established: The fundamentals of the M-truncated derivative are covered in section 2. Section 3 outlines the essential procedures for the (G'/G^2)-expansion function approach. The application of this expansion method to Eq. (1.2) is covered in section 4. Graphical depictions of a few chosen precise solutions are shown in section 5. The study's conclusion and closing observations are presented in section 6 at last.

2. Mathematical preliminaries

This section provides a brief description of the basic properties of the truncated M-fractional derivative, which was first introduced by Sulaiman et al.[26].

Definition 2.1. The extension of the exponential function is the one-parameter truncated Mittag-Leffler function, $E_\nu(Z)$. It is defined as[24, 25, 29]:

$$E_\nu(Z)=\sum_{\mathfrak{i}=0}^{\mathfrak{j}}\frac{Z^{\mathfrak{i}}}{\Gamma(\mathfrak{i}\nu+1)},$$

where $\nu > 0$ and $Z \in \mathbb{C}$.

Definition 2.2. Consider a function f such that $f : (0, +\infty) \to R$. Then, for $0 < leq\mu \leq 1$, the M-fractional derivative of f of order μ is defined as [24–26, 29]:

$$\mathfrak{D}^{\mu,\nu}_{M,t}f(t)=\lim_{\epsilon\to 0}\frac{f(tE_\nu(\epsilon t^{1-\mu}))-f(t)}{\epsilon}, \quad 0<\mu\leq 1, \;\; \nu>0, \tag{2.1}$$

Thus, we are able to demonstrate that t is μ-truncated M-fractionally differentiable at $t > 0$ if the limit in Eq. (2.1) exists. Moreover, if f is μ-differentiable in some $(0, \upsilon)$ with $\upsilon > 0$ and the limit $\lim_{t\to 0+}\mathfrak{D}^{\mu,\nu}_{M,t}f(t)$ exists, then we define $\mathfrak{D}^{\mu,\nu}_{M,t}f(0) = \lim_{t\to 0+}\mathfrak{D}^{\mu,\nu}_{M,t}f(t)$.

The following are some helpful characteristics of the truncated-fractional derivative,[24–26].

Theorem 2.3. *For every $t > 0$, $\mu \in (0,1]$, and $\nu > 0$, let $F(t), \Xi(t)$ be μ-differentiable functions. Next, we have*

(1) $\mathfrak{D}^{\mu,\nu}_{M,t}(\Upsilon) = 0$ *where* Υ=*constant.*
(2) $\mathfrak{D}^{\mu,\nu}_{M,t}(\upsilon F(t) + \sigma \Xi(t)) = \upsilon \mathfrak{D}^{\mu,\nu}_{M,t} F(t) + \sigma \mathfrak{D}^{\mu,\nu}_{M,t}\Xi(t)$, *for all* $\upsilon, \sigma \in R$.
(3) $\mathfrak{D}^{\mu,\nu}_{M,t}(F(t)\Xi(t)) = F(t)\mathfrak{D}^{\mu,\nu}_{M,t}\Xi(t) + \Xi(t)\mathfrak{D}^{\mu,\nu}_{M,t}F(t)$.
(4) $\mathfrak{D}^{\mu,\nu}_{M,t}\left(\dfrac{F(t)}{\Xi(t)}\right) = \dfrac{\Xi(t)\mathfrak{D}^{\mu,\nu}_{M,t}F(t) - F(t)\mathfrak{D}^{\mu,\nu}_{M,t}\Xi(t)}{\Xi(t)^2}$.
(5) $\mathfrak{D}^{\mu,\nu}_{M,t}(f(t)) = \frac{t^{1-\mu}}{\Gamma(\nu+1)}\dfrac{df(t)}{dt}$, *provided that $f(t)$ is differentiable.*
(6) $\mathfrak{D}^{\mu,\nu}_{M,t}(F \circ \Xi)(t) = F^{'}(\Xi(t))\mathfrak{D}^{\mu,\nu}_{M,t}\Xi(t)$, *where the prime notation ($'$) denotes the classical derivative.*

Based on Definition 2.2, we can extend the concept to define the M-derivative for a function $\psi = \psi(x,t)$ with respect tot of order $\mu \in (0,1]$ can be defined as

$$\mathfrak{D}^{\mu,\nu}_{M,t}\psi(x,t) = \underset{\epsilon\to 0}{Lim}\frac{\psi(x, tE_\nu(\epsilon t^{1-\mu})) - \psi(x,t)}{\epsilon}, \quad 0 < \mu \le 1, \quad \nu{>}0,$$

The concept of the higher-order truncated derivative and its associated properties have been explored in previous studies[24–26, 29].

3. Analysis of (G'/G^2)-expansion method

This section presents a brief overview of the (G'/G^2)-expansion technique, which has been explored and refined in multiple prior studies [17, 19, 32]. The (G'/G^2)-expansion method is a technique used to find an exact traveling wave solutions of both classical and the truncated M-fractional derivative NPDEs. The key steps of this method are:

$$\mathcal{F}(w, \mathfrak{D}^{\mu,\nu}_{y,M}w, \mathfrak{D}^{\mu,\nu}_{t,M}w, \mathfrak{D}^{\mu,\nu}_{x,M}w, \mathfrak{D}^{2\mu,\nu}_{x,M}w, \mathfrak{D}^{2\mu,\nu}_{t,M}w, ...),\ 0 < \mu \le 1,\ \nu > 0. \tag{3.1}$$

In this context, the function $w(x,y,t)$ represents an unknown variable, while $\mathcal{F}$ denotes a polynomial comprising w and its associated partial fractional operators. The essential steps of this method are:
Step 1. Through the use of the subsequent travelling wave transformation

$$w(x,y,t) = W(\zeta), \quad \zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}. \tag{3.2}$$

The original nonlinear fractional evolution PDE can be transformed into an ordinary differential equation (ODE) in the variable $w = W(\zeta)$, where ζ is a new independent variable. This transformation is achieved by assuming a traveling wave solution in the form:

$$\mathcal{G}(W, W^{'}, W^{''}, W^{'''}, ...), \tag{3.3}$$

where a_1, a_2, and a_3 are constants to be determined later.
Step 2. For NPDEs, precise traveling wave solutions can be found using the (G'/G^2)-expansion method. According to this approach, the formal solution can be expressed as follows:

$$W(\zeta) = \beta_0 + \sum_{\mathfrak{n}=0}^{N}\beta_{\mathfrak{n}}\left(\frac{G'}{G^2}\right)^{\mathfrak{n}} + \delta_{\mathfrak{n}}\left(\frac{G'}{G^2}\right)^{-\mathfrak{n}}, \tag{3.4}$$

$$\left(\frac{G'}{G^2}\right)^{'} = \vartheta + \lambda\left(\frac{G'}{G^2}\right)^{2}, \tag{3.5}$$

in which, $\vartheta \neq 1$ and $\lambda \neq 0$ are integers, while β_0, $\beta_{\mathfrak{n}}$, and $\delta_{\mathfrak{n}}$ (for $\mathfrak{n} = 1, 2, ..., N$) are constants to be determined. The homogeneous balance concept can be used to easily determine the positive integer N. This principle involves balancing the highest-order derivative term with the nonlinear terms present in the reduced ordinary differential equation (ODE).

More specifically, the following formulas [23] can be used to calculate the degree of the following terms if the degree of $W(\zeta)$ is $\text{Deg}[W(\zeta)] = N$,

$$Deg\left[\frac{dW(\zeta)^{\eta}}{d\zeta^{\eta}}\right] = N + \eta, \;\; Deg\left[W(\zeta)^{\rho}\left(\frac{dW(\zeta)}{d\zeta}\right)^{\alpha}\right] = N\rho + \alpha(N+\eta). \tag{3.6}$$

Step 3. By introducing Equations (3.4) and (3.5) into Eq. (3.3), we generate a polynomial in terms of (G'/G^2). We then group the coefficients of like powers of $(G'/G^2)^i$, where i ranges from 0 to $N($ N being a positive integer). These grouped coefficients can be set to zero to produce an algebraic structure of nonlinear equations.The unspecified parameters $\beta_0, \beta_n, \delta_n(n = 1, 2, ..., N), a_1, a_2,$ and a_3 are involved in these equations. We assume that this resulting system of equations can be resolved for these unknowns using symbolic computation software such as Mathematica.
Step 4. The standard solutions for (G'/G^2) in Eq. (3.5) can be formulated as follows: When $\vartheta\lambda > 0$, Eq. (3.5) yields a solution in terms of trigonometric functions, which can be expressed as:

$$(G'/G^2) = \sqrt{\frac{\vartheta}{\lambda}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right), \tag{3.7}$$

when the arbitrary positive parameters E, F are used.

When $\vartheta\lambda < 0$, Eq. (3.5) yields a solution in terms of exponential function, which can be expressed as:

$$(G'/G^2) = -\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right), \tag{3.8}$$

where E, F are arbitrary nonzero constants. When $\vartheta = 0, \lambda \neq 0$, Eq. (3.5) yields a solution in terms of rational function, which can be expressed as:

$$(G'/G^2) = -\frac{E}{\lambda(E\zeta + F)}, \tag{3.9}$$

where E, F are arbitrary nonzero constants.

We obtain three types of solution for Eq. (3.1) by using the values of $\beta_0, \beta_n, \delta_n(n = 1, 2, ..., N)$ and Eqs. (3.7)–(3.9) in Eq. (3.4).

4. Benefits for the (G'/G^2)-Expansion Technique

n this section, we will solve Eq. (1.2) and derive its precise traveling wave solution utilizing the (G'/G^2)-Expansion Technique.

Through the use of the subsequent traveling wave transformation

$$w(x, y, t) = W(\zeta)e^{i\psi}, \quad \zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^{\mu}}{\mu}, \quad \psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^{\mu}}{\mu} + \theta, \tag{4.1}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ and θ are real constants. By substituting the trial solution form given in Eq. (4.1) into Eq. (1.2), we arrive at the following (ODE) in the new variable ζ:

$$\textbf{Re}: \quad (a_3^2p + a_1^2g)W(\zeta)'' - (2b_2 + pb_3^2 + b_1^2g)W(\zeta) + 2kW(\zeta)^3 = 0, \tag{4.2}$$

$$\textbf{Im}: \quad (2a_2 + 2pa_3b_3 + 2gb_1a_1)W(\zeta)' = 0. \tag{4.3}$$

From Eq. (4.3), we have

$$a_2 = (-pa_3b_3 - gb_1a_1), \tag{4.4}$$

after balancing the terms W^3 and $W(\zeta)''$ in Eq. (4.2), we determine that $N = 1$. As a result, the following is an expression for the solution:

$$W(\zeta) = \beta_0 + \beta_1\left(\frac{G'}{G^2}\right) + \delta_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{4.5}$$

where $\beta_0, \beta_1, \delta_1$ are constants to be computed.

Applying the third step outlined in the section "Analysis of (G'/G^2)-expansion method " generates the following system of algebraic equations:

$$\begin{aligned}
&(G'/G^2)^{-3} : 2a_1^2\delta_1 g\vartheta^2 + 2a_3^2\delta_1\vartheta^2 p + 2\delta_1^3 k = 0,\\
&(G'/G^2)^{-2} : 6\beta_0\delta_1^2 k = 0,\\
&(G'/G^2)^{-1} : 2a_1^2\delta_1 g\lambda\vartheta + 2a_3^2\delta_1\lambda\vartheta p - b_1^2\delta_1 g - 2b_2\delta_1 - b_3^2\delta_1 p + 6\beta_0^2\delta_1 k + 6\beta_1\delta_1^2 k,\\
&(G'/G^2)^{0} : -2b_2\beta_0 - b_1^2\beta_0 g - b_3^2\beta_0 p + 2\beta_0^3 k + 12\beta_1\delta_1\beta_0 k,\\
&(G'/G^2) : 2a_1^2\beta_1 g\lambda\vartheta + 2a_3^2\beta_1\lambda\vartheta p - b_1^2\beta_1 g - b_3^2\beta_1 p - 2b_2\beta_1 + 6\beta_0^2\beta_1 k + 6\beta_1^2\delta_1 k,\\
&(G'/G^2)^2 : 6\beta_0\beta_1^2 k,\\
&(G'/G^2)^3 : 2a_1^2\beta_1 g\lambda^2 + 2a_3^2\beta_1\lambda^2 p + 2\beta_1^3 k.
\end{aligned} \tag{4.6}$$

By resolving the algebraic system presented in Eq. (4.6) using the Mathematica software package, we obtain the following three distinct sets of results:

Result 1:

$$\begin{aligned}
&\beta_0 = 0, \beta_1 = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}},\\
&\delta_1 = 0, b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}},
\end{aligned} \tag{4.7}$$

where ϑ, λ are arbitrary values and $a_1, a_2, a_3, , b_2, g, p$ are arbitrary constants.The solution $W(\zeta)$ may be obtained by putting the values from Eq. (4.7) into the solution form Eq. (4.5) and incorporating the ratio (G'/G^2) from equations Eqs. (3.7)–(3.9) based on the particular values of ϑ and λ.

The answer to Eq. (4.5) can be stated using trigonometric functions if $\vartheta\lambda > 0$. The solution is presented as follows:

$$W_{1,1}(\zeta) = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\sqrt{\frac{\vartheta}{\lambda}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right). \tag{4.8}$$

The analytical solution of Eq. (1.2), expressed in terms of trigonometric functions, can be derived from Eq. (4.1):

$$w_{1,1}(x,y,t) = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\sqrt{\frac{\vartheta}{\lambda}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)e^{i\psi}, \tag{4.9}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form:

$$W_{1,2}(\zeta) = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right). \tag{4.10}$$

The analytical solution of Eq. (1.2), described in terms of exponential functions, can be derived from Eq. (4.1):

$$w_{1,2}(x,y,t) = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)e^{i\psi}, \tag{4.11}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

Result 2:

$$\begin{aligned} &\beta_0 = 0, \beta_1 = \pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}, \delta_1 = \pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}, \\ &b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta - 6\sqrt{-\lambda^2\left(a_1^2 g + a_3^2 p\right)}\sqrt{-\vartheta^2\left(a_1^2 g + a_3^2 p\right)} + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}}. \end{aligned} \tag{4.12}$$

The solution to Eq. (4.5) can be stated using trigonometric functions if $\vartheta\lambda > 0$. This solution is presented as follows:

$$\begin{aligned} W_{2,1}(\zeta) = &\pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right) \\ &\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}. \end{aligned} \tag{4.13}$$

The analytical solution of Eq. (1.2), represented by trigonometric functions, can be obtained from Eq. (4.1):

$$\begin{aligned} w_{2,1}(x,y,t) = \Bigg(&\pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right) \\ &\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}\Bigg)e^{i\psi}, \end{aligned} \tag{4.14}$$

where $\psi = b_1 x + b_2 y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1 x + a_2 y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form:

$$\begin{aligned} W_{2,2}(\zeta) = &\pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right) \\ &\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}. \end{aligned} \tag{4.15}$$

The analytical solution of Eq. (1.2), expressed in terms of exponential functions, can be determined using Eq. (4.1):

$$\begin{aligned} w_{2,2}(x,y,t) = \Bigg(&\pm\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right) \\ &\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}\Bigg)e^{i\psi}, \end{aligned} \tag{4.16}$$

where $\psi = b_1 x + b_2 y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1 x + a_2 y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

Result 3:

$$\begin{aligned} &\beta_0 = 0, \beta_1 = 0, \\ &\delta_1 = \pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}, b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}}. \end{aligned} \tag{4.17}$$

The solution to Eq. (4.5) can be stated using trigonometric functions if $\vartheta\lambda > 0$. The solution is presented as follows:

$$W_{3,1}(\zeta) = \pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}, \tag{4.18}$$

The analytical solution of Eq. (1.2), represented by trigonometric functions, can be obtained from Eq. (4.1):

$$w_{3,1}(x,y,t) = \left(\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}\right)e^{i\psi}, \tag{4.19}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form:

$$W_{3,2}(\zeta) = \pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}. \tag{4.20}$$

The analytical solution of Eq. (1.2), expressed in terms of exponential functions, can be determined using Eq. (4.1):

$$w_{3,2}(x,y,t) = \left(\pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}\right)e^{i\psi}, \tag{4.21}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

Result 4:

$$\begin{aligned} &\beta_0 = 0, \beta_1 = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}, \\ &\delta_1 = 0, b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}}, \end{aligned} \tag{4.22}$$

where ϑ, λ are arbitrary values and $a_1, a_2, a_3, , b_2, g, p$ are arbitrary parameters. The solution $W(\zeta)$ may be obtained by putting the values from Eq. (4.7) into the solution form Eq. (4.5) and incorporating the ratio (G'/G^2) from equations Eq. (3.7) and Eq. (3.9) based on the particular values of ϑ and λ.

The solution to Eq. (4.5) can be stated using trigonometric functions if $\vartheta\lambda > 0$. The solution is presented as follows

$$W_{4,1}(\zeta) = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\sqrt{\frac{\vartheta}{\lambda}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right). \tag{4.23}$$

Trigonometric functions is

$$w_{4,1}(x,y,t) = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\sqrt{\frac{\vartheta}{\lambda}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)e^{i\psi}, \tag{4.24}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form

$$W_{4,2}(\zeta) = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right). \tag{4.25}$$

Exponential functions is

$$w_{4,2}(x,y,t) = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)e^{i\psi}, \tag{4.26}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

Result 5:

$$\begin{aligned} &\beta_0 = 0, \beta_1 = \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}, \delta_1 = \pm\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}, \\ &b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta - 6\sqrt{-\lambda^2\left(a_1^2 g + a_3^2 p\right)}\sqrt{-\vartheta^2\left(a_1^2 g + a_3^2 p\right)} + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}}, \end{aligned} \tag{4.27}$$

Trigonometric functions can be used to define the solution of Eq. (4.5) if $\vartheta\lambda > 0$. This resolution manifests as follows

$$\begin{aligned} W_{5,1}(\zeta) &= \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right) \\ &\pm \frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}. \end{aligned} \tag{4.28}$$

Trigonometric functions is

$$\begin{aligned} w_{5,1}(x,y,t) = \Bigg(&\mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right) \\ &\pm \frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}\Bigg)e^{i\psi}, \end{aligned} \tag{4.29}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form:

$$\begin{aligned} W_{5,2}(\zeta) &= \mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right) \\ &\pm \frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}. \end{aligned} \tag{4.30}$$

Exponential functions is

$$\begin{aligned} w_{5,2}(x,y,t) = \Bigg(&\mp\frac{\sqrt{a_1^2(-g)\lambda^2 - a_3^2\lambda^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right) \\ &\pm \frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}\Bigg)e^{i\psi}, \end{aligned} \tag{4.31}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

Result 6:

$$\beta_0 = 0, \beta_1 = 0, \delta_1 = \mp\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}, b_1 = \pm\frac{\sqrt{2a_1^2 g\lambda\vartheta + 2a_3^2\lambda\vartheta p - b_3^2 p - 2b_2}}{\sqrt{g}}. \tag{4.32}$$

A solution that may be described using trigonometric functions is produced by Eq. (4.5) if $\vartheta\lambda > 0$. Here is how this solution is implemented:

$$W_{6,1}(\zeta) = \mp\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}. \tag{4.33}$$

Trigonometric functions

$$w_{6,1}(x,y,t) = \left(\mp\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(\frac{Ecos\sqrt{\vartheta\lambda}\zeta + Fsin\sqrt{\vartheta\lambda}\zeta}{Fcos\sqrt{\vartheta\lambda}\zeta - Esin\sqrt{\vartheta\lambda}\zeta}\right)^{-1}\right)e^{i\psi}, \tag{4.34}$$

where $\psi = b_1x + b_2y + b_3\frac{\Gamma(\nu+1)t^\mu}{\mu} + \theta$ and $\zeta = a_1x + a_2y + a_3\frac{\Gamma(\nu+1)t^\mu}{\mu}$.

If $\vartheta\lambda < 0$, Eq. (4.5) yields a solution that can be expressed using exponential functions. This solution takes the following form

$$W_{6,2}(\zeta) = \mp\frac{\sqrt{a_1^2(-g)\vartheta^2 - a_3^2\vartheta^2 p}}{\sqrt{k}}\left(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\left(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) + F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta) + Ecosh(2\sqrt{|\vartheta\lambda|}\zeta) - F}\right)\right)^{-1}. \tag{4.35}$$

Exponential functions

$$w_{6,2}(x,y,t)=\Bigg(\mp\frac{\sqrt{a_1^2(-g)\vartheta^2-a_3^2\vartheta^2p}}{\sqrt{k}}\Bigg(-\frac{\sqrt{|\vartheta\lambda|}}{\lambda}\Big(\frac{Esinh(2\sqrt{|\vartheta\lambda|}\zeta)+Ecosh(2\sqrt{|\vartheta\lambda|}\zeta)+F}{Esinh(2\sqrt{|\vartheta\lambda|}\zeta)+Ecosh(2\sqrt{|\vartheta\lambda|}\zeta)-F}\Big)\Bigg)^{-1}\Bigg)e^{i\psi}, \tag{4.36}$$

where $\psi=b_1x+b_2y+b_3\frac{\Gamma(\nu+1)t^{\mu}}{\mu}+\theta$ and $\zeta=a_1x+a_2y+a_3\frac{\Gamma(\nu+1)t^{\mu}}{\mu}$.

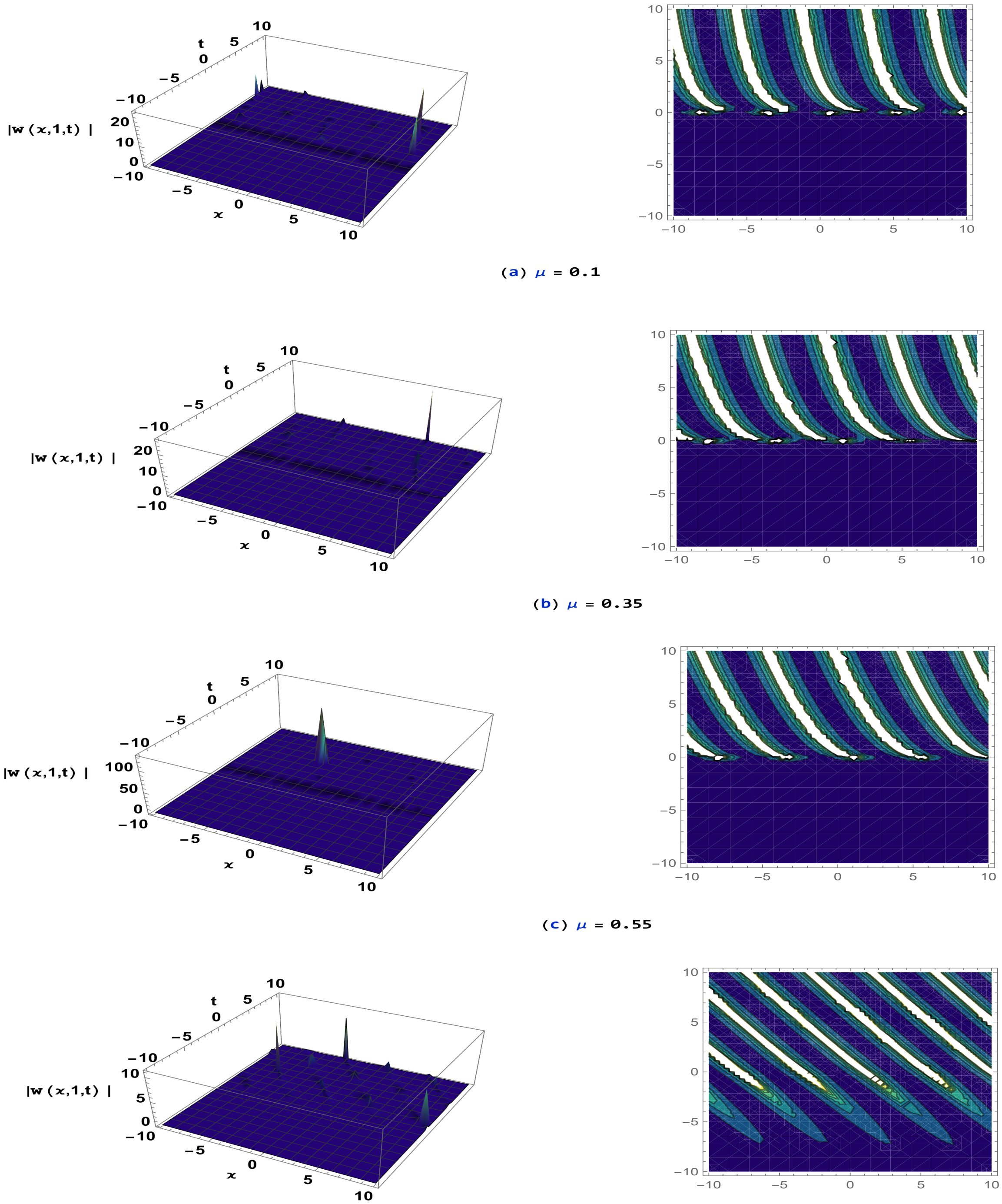
Remark 4.1. It is clear that some solutions to Eq. (4.1) are new and differ from those found by Bachar et al. in their study [7]. For example, the solutions resulting from results 3 and 6 are included.

5. Graphical explanations of the achieved solutions

This section provides a computational analysis of the solutions found using the suggested techniques, looking at how they behave at different parameter values. We illustrate these numerical solutions using 3D and 2D diagrams, as well as density plots. This dynamic model's characteristic wave forms-dark, bright, periodic, rogue, kink, double periodic, and unique solitary waves-are revealed by the study. A solution's singularity throws light on certain characteristics of nonlinear media. For example, materials can respond nonlinearly to high-intensity light in the field of non-linear optics. This suggests that there is a nonlinear connection between the polarization of the material and the electric field of the light. Optical self-focusing, a phenomenon where the material's refractive coefficient fluctuates with the amount of light, can happen at extremely bright light intensities. Spatial solitons, which are focused light waves that can sustain themselves, may arise from this process. The point where these effects become particularly intense or concentrated is referred to as a singularity in the solution. The wave pattern of the absolute component of solution $w_{1,1}$ is shown in Figure 1 for the following values of the parameters: $\vartheta=0.5,\lambda=1,a_1=1,a_3=1,b_2=1,b_3=1,g=-0.0005,\nu=0.6,p=0.001,F=1,E=0.6,k=1,\theta=0.5,\mu(0.1,0.35,0.55,0.95)$, and $y=1$ inside the range $-10\leq x,t\leq 10$.

The combination of an exponential function and a trigonometric function yields a periodic wave, as displayed in the illustration. Furthermore, the wave changes from being just one periodic wave to a dual periodic wave as the fractional order μ grows. The wave form of the absolute part of the solution $w_{2,2}$ is shown in Figure 2 for the following values of the parameters: $\vartheta=5,\lambda=-1,a_1=1,a_3=1,b_2=1,b_3=1,g=6,\nu=0.6,p=-3,F=1,E=0.6,k=1,\theta=0.5,\mu(0.1,0.35,0.55,0.95)$, and $y=1$ during a period $-10\leq x,t\leq 0$. Figure 2 illustrates a kink wave that is formed by the interaction between a hyperbolic function and an exponential function. A significant change in the shape of the waves is seen as the value of the fractional order μ is raised. Specifically, the wave undergoes a transition from a dark kink formation to a bright kink formation. This transition from a dark to a bright kink demonstrates how the fractional order μ significantly influences the wave's behavior and appearance. Such changes in wave characteristics can have important implications in various fields of physics and mathematics where kink waves are studied, such as nonlinear optics, plasma physics, or condensed matter systems.

The wave pattern of the absolute component of the solution $w_{2,2}$ is shown in Figure 3 for the following values of the parameters: $\vartheta=2$, $\lambda=-1$, $a_1=1$, $a_3=1$, $b_2=1$, $b_3=1$, $g=-0.005,\nu=0.6$, $p=-5$, $F=1$, $E=2.6$, $k=-1$, $\theta=2$, $\mu=(0.1,0.35,0.55,0.95)$, and $y=1$ within the interval $-10\leq x,t\leq 10$. A singular kink wave produced by combining hyperbolic and exponential functions is seen in Figure 3. Figure 4 illustrates the wave structure of the absolute portion of solution $w_{3,2}$ for the following parameter values: $\vartheta=2$, $\lambda=-1$, $a_1=0.2$, $a_3=0.75$, $b_2=1$, $b_3=0.1$, $g=-1$, $p=1$, $F=E=0.1$, $\nu=0.6$, $k=1$, $\theta=0.5$, $\mu=(0.35,0.65,0.75,0.95)$, and $y=1$ within the interval $-10\leq x,t\leq 10$. In this figure, we observe a dark soliton solution. Figure 5 (a) and (b) examines the behavior of solutions for different values of t. The representation of the data is done in two dimensions, with an emphasis on the actual content of the solutions: the values of $w_{1,1}$ and $w_{2,2}$ at $\mu=0.95$. Figure 5 (c) and (d) focuses on the absolute value of solutions $w_{2,2}$ at $\mu=0.95$ and $w_{3,2}$ at $\mu=0.95$. As the fractional order μ increases, the 3D, 2D, and contour plot reveal an emerging interaction with a soliton solution. This visualization demonstrates how the wave structure evolves and interacts with soliton-like features as the fractional order parameter is adjusted.



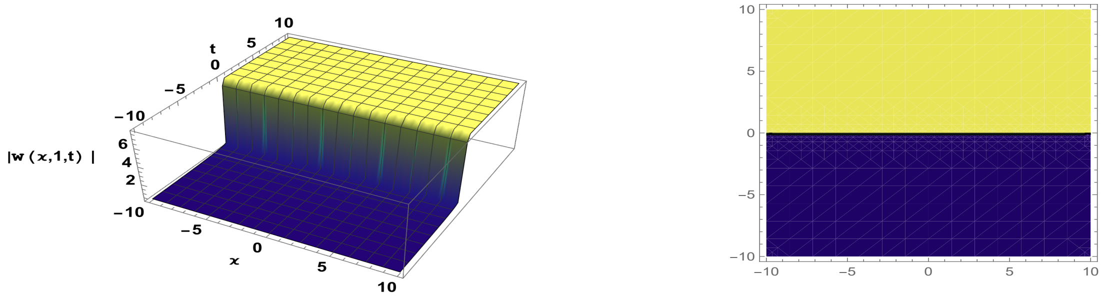
(a) μ = 0.1

(b) μ = 0.35

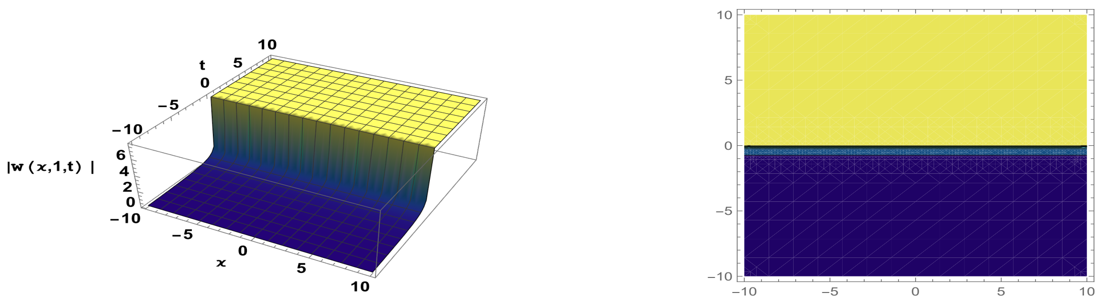
(c) μ = 0.55

(d) μ = 0.95

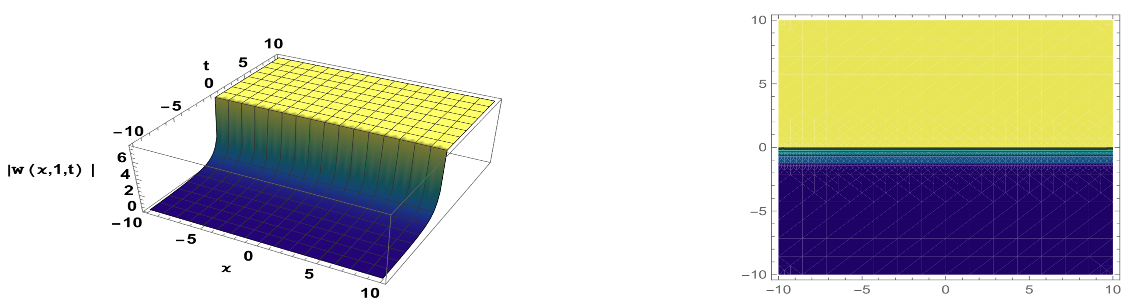
FIGURE 1. The absolute part of solution $w_{1,1}$ has a periodic wave characteristic. (a)–(d) represent 3D plot and contour plot at different values of μ.



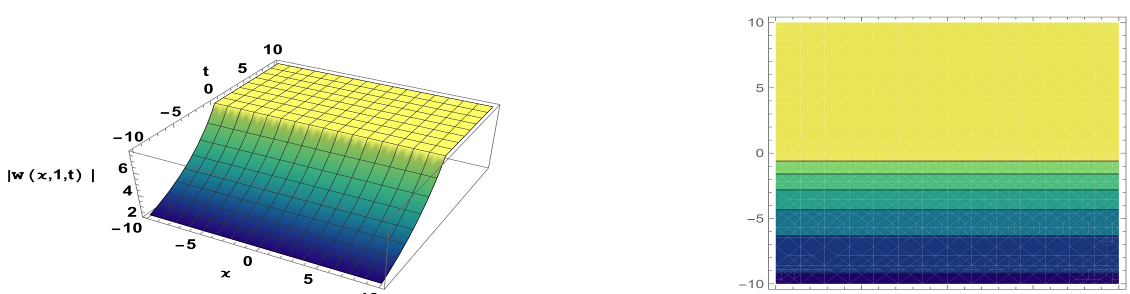
(a) μ = 0.1



(b) μ = 0.35

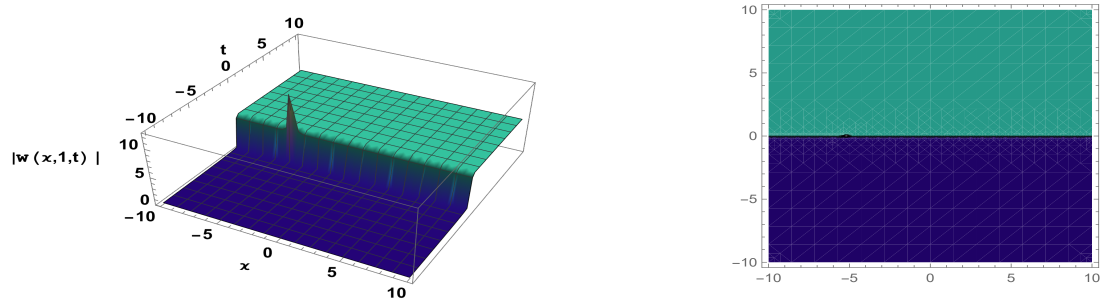


(c) μ = 0.55

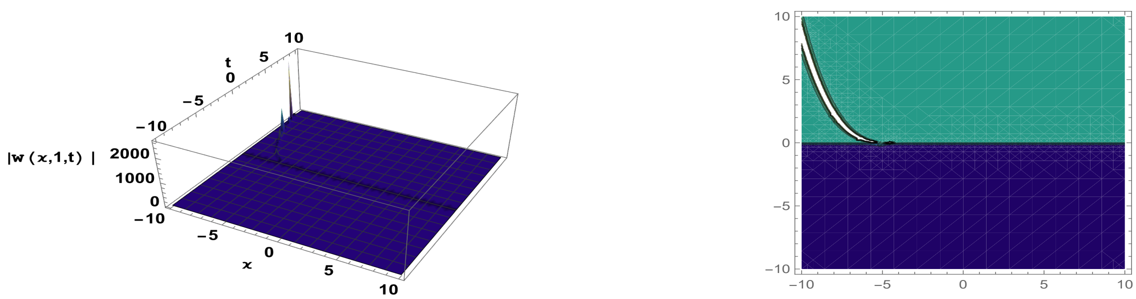


(d) μ = 0.95

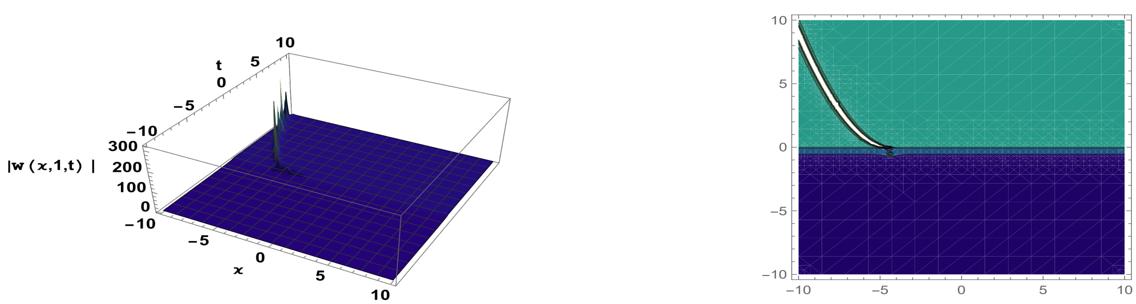
Figure 2. the kink wave characteristics of the absolute value component of solution $w_{2,2}$. (a)–(d) represent 3D plot and contour plot at different values of μ.



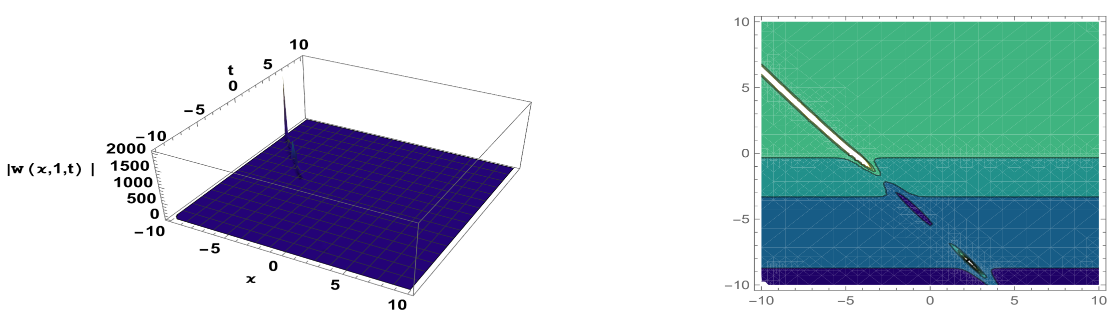
(a) $\mu = 0.1$



(b) $\mu = 0.35$



(c) $\mu = 0.55$



(d) $\mu = 0.95$

FIGURE 3. Singular kink in the absolute component of $w_{2,2}$ with interaction wave characteristic. (a)–(d) represent 3D plot and contour plot at different values of μ.

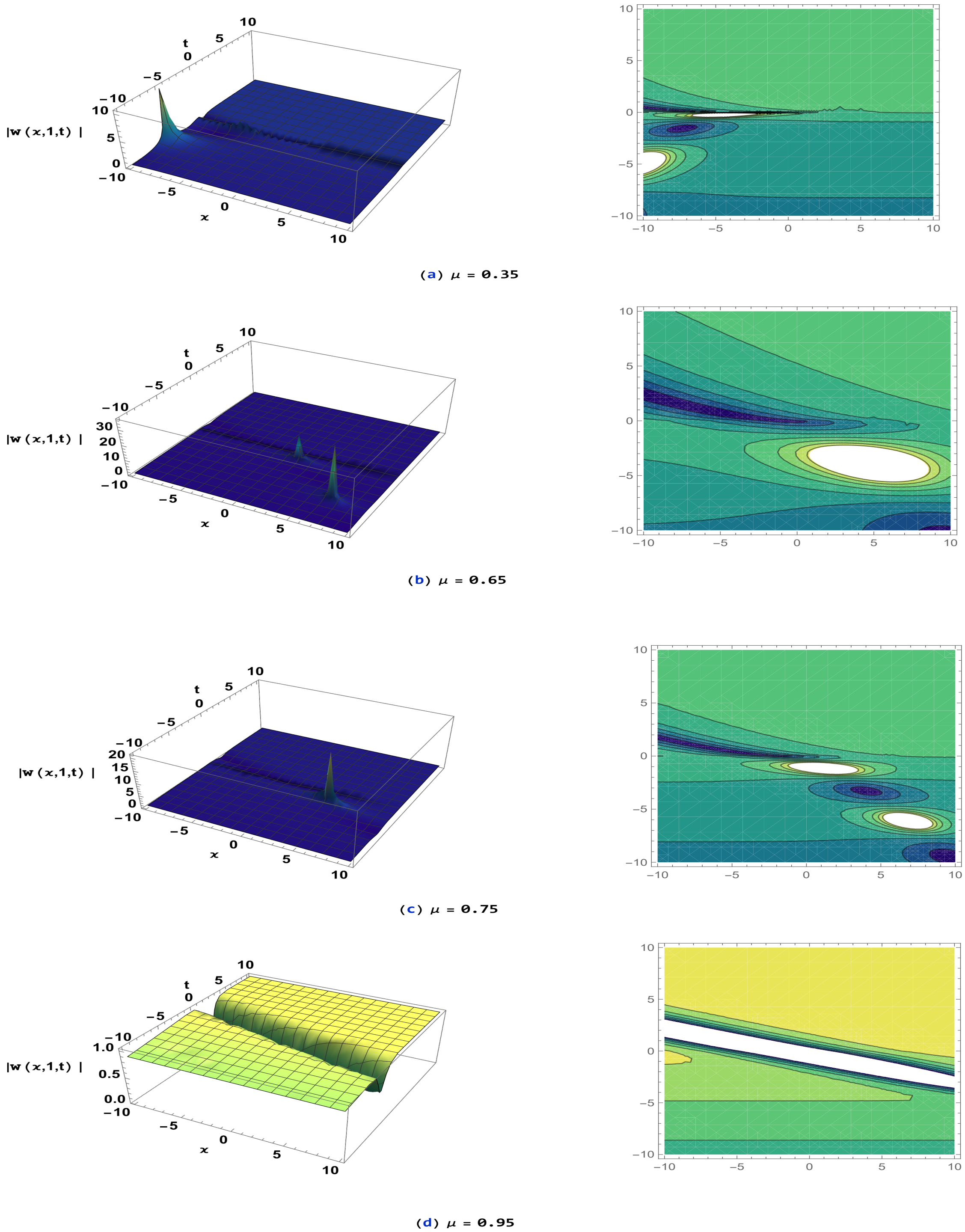


FIGURE 4. dark soliton solution of $w_{3,2}$. (a)–(d) represent 3D plot and contour plot at different values of μ.

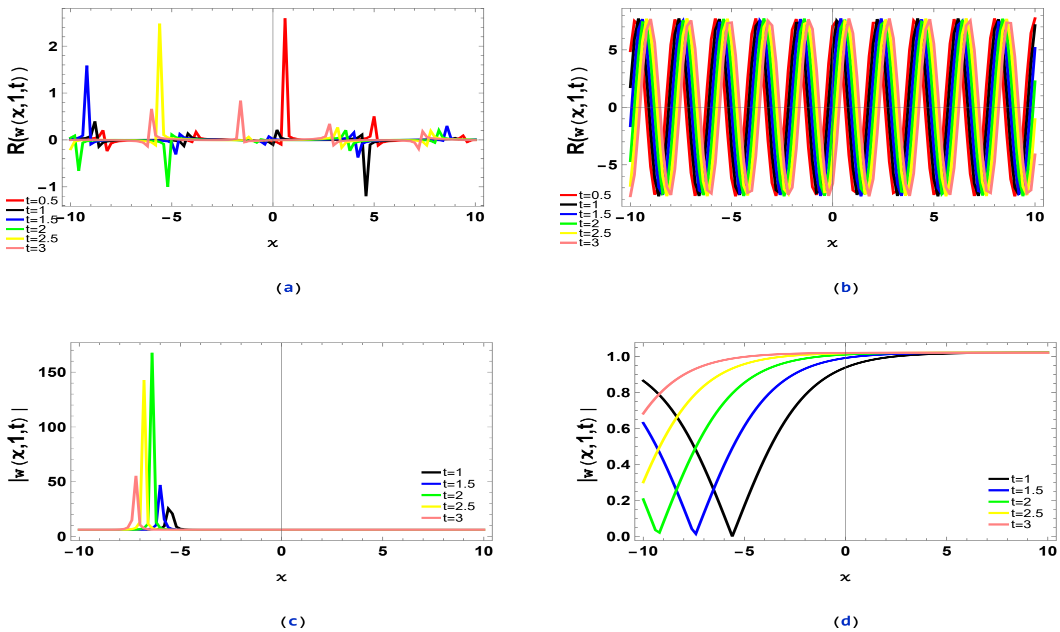


FIGURE 5. 2D plots of $w(x,y,t)$ (a) Re($w(x,y,t)$) for solution $w_{1,1}$ at several values of t for $\mu = 0.95$. (b) Re($w(x,y,t)$) for solution $w_{2,2}$ at several values of t for $\mu = 0.95$. (c) $|w(x,y,t)|$ for solution $w_{2,2}$ at several values of t for $\mu = 0.95$. (d)$|w(x,y,t)|$ for solution $w_{3,2}$ at several values of t for $\mu = 0.95$.

6. Conclusion

According to the M-fractional derivative, the study shows how well the (G'/G^2)-expansion function approach works for precisely solving the PWDE with Kerr law. This equation is of great importance in the study of wave propagation and optical phenomena, as it models the behavior of light in nonlinear media with the Kerr-type nonlinearity. By employing the (G'/G^2)-expansion function method, the researchers were able to derive new, previously unknown exact solutions to the PWDE. These solutions represent different types of wave solutions, including dark, bright, singular, bright-dark combined, and periodic solitons. The diversity of the obtained solutions highlights the versatility and power of the proposed technique in capturing the rich dynamics of the PWDE. The importance of these novel wave solutions lies in their potential applications across various fields of physics and engineering, particularly in optics, photonics, and electromagnetism. Wave phenomena play a crucial role in these disciplines, and the researchers emphasize that the specific solutions obtained in this study have important practical applications, including:

(1) Light propagation in nonlinear optical fibers and waveguides, which is essential for designing advanced photonic devices.
(2) Understanding soliton behavior, which aids in optical signal transmission, pulse shaping, and supercontinuum generation in fiber optics.
(3) Nonlinear laser dynamics and ultrafast optics, where the bright and dark solitons obtained in this study have potential applications.
(4) Quantum wave equations, particularly in Bose-Einstein condensates and quantum fluids, where nonlinear effects significantly influence wave propagation.

To further demonstrate the feasibility and reliability of the (G'/G^2)-expansion function method, the researchers provide visual representations of the obtained solutions in the form of 3D, contour, and 2D plots. These graphical illustrations serve to showcase the ability of the method to capture the essential characteristics and behaviors of the solutions, which is essential for understanding and predicting the dynamics of the PWDE in real-world applications.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Consent for publication

All authors agree to publish.

Competing interests

The authors declare no competing interests.

Author contributions

The authors affirm that the study was conducted through collaborative efforts with equal responsibility. All authors have thoroughly reviewed and endorsed the final manuscript.

Competing interests

There is no conflict of interest between the author or anyone else regarding this manuscript.

Additional information

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