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Fractional prey-predator model in biological Pest control

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Abstract

In developing countries, the agricultural industry is pivotal in economic development and sustaining rural livelihoods. However, one of the major challenges to achieving global food security is crop losses caused by pests. Pest control is a vital practice for safeguarding crops; it is often complicated by the need to balance pest reduction with the associated costs of operations, as well as the potential impacts on the environment and human health. This delicate balance is crucial for sustainable agriculture and long-term food security. In this paper, a mathematical model is developed to quantify the complex biological processes involved in the biological control of pests through prey-predator mechanisms. We provide trajectory analysis in the research discussion analysis.

Keywords. Fractional derivative, Prey-predator, Biological pest, Food Security, Stability.

2010 Mathematics Subject Classification. 92D45, 37N25, 34D20.

1. Introduction

The nature of the affected crop defines the nature of the impacts. Damage to food crops can lead reduced food availability, which may also reduce the economic sector. The severity of these initial impacts depends on the intensity and duration of the disease outbreak. These primary effects can also trigger further consequences: epidemics on food crops may result in a shortage in supply to storage systems, a decline in exports, or an inconsistent supply of food to markets. These outcomes are often a reflection of the socioeconomic conditions and market structures at various scales [22]. Food security is increasingly threatened by various factors worldwide. Among the most significant risks, especially in developing countries, are food damage caused by plant diseases and pests [9]. Pests are organisms that disrupt human activities or cause damage, loss, or irritation to crops, stored products, and livestock. In agriculture, pest control is predominantly reliant on chemical insecticides. However, this approach has several drawbacks, including diminished effectiveness over time due to pests developing resistance, adverse impacts on beneficial insect populations, and the disruption of natural pest control systems. Such disruptions can lead to the destruction of pests' natural enemies, the emergence of more destructive pest outbreaks, and the accumulation of chemical residues in crops. As global population growth heightens the need for effective pest management, developing sustainable techniques becomes crucial. Educating farmers about agricultural practices and pest control methods is equally important to enhance these efforts [23]. Effective pest control strategies have a profound societal impact, making it essential to adopt integrated approaches. The use of natural predators, in combination with the judicious application of chemical pesticides, is strongly recommended for managing pest populations. Biological control [21], in particular, is defined as the use of parasitoids, predators, or pathogens to maintain the population density of a target organism at levels lower than would occur in the absence of these natural enemies.

Mathematical models are being utilized more frequently to analyze agricultural issues, especially in the context of pest and natural enemy interactions. By employing simulation tools, these models enhance the understanding of system dynamics, offering researchers a comprehensive view of the environment. This capability allows them to

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create and execute computational experiments that accurately reflect real-world systems. Modelling in biological pest control facilitates both qualitative and quantitative assessments of the relationships between pest populations and their natural enemies. These models are instrumental in the development of stable prey-predator or host-parasitoid system as they help identify natural enemies possessing traits that enhance the stability of these systems.

The predator-prey system is a fundamental model in ecological dynamics, often used to describe the relationship between two species: the predator and their prey. The predator population consumes the prey population, which, in turn, depends on the prey for survival. Understanding the interplay between these populations is crucial for studying the stability and dynamics of ecological communities. This topic remains a dominant theme in ecology and mathematical ecology due to its universal relevance and importance [5]. The classical predator-prey model [6], often referred to as the Lotka-Volterra model, has been studied extensively since its inception in the early 20th century. Numerous mathematical models in the literature investigate prey-predator dynamics under different conditions. For instance, [24] introduced a prey-diseased predator model that incorporates prey refuge. The study established the necessary and sufficient conditions for the stability of interactions among infected prey, non-infected prey, and predators within an ecosystem. In a subsequent study, [26] analyzed a three-species food web system involving multi-species interactions. In this model, two species act as competitors, while the third serves as both a predator and a host to the competitors. This predator-host species is an ally to the first competitor but an adversary to the second, eliminating it through predation. The analytical results, supported by simulations, indicated that such a multi-interaction system is unlikely to persist in nature for an extended period. In contrast, [25] studied a different three-species food web model, where two competing species engage in mutualism governed by a Holling type II functional response. The third species, a predator, preys on the second mutualistic species. Although the findings were comparable to those in [26], this interaction exhibited greater stability and persistence in natural ecosystems compared to the earlier model.

Significant research has been conducted on the Lotka-Volterra predator-prey system, yet the simple Lotka-Volterra model has notable limitations. For instance, it assumes that prey populations grow indefinitely in the absence of predators, which is unrealistic. This issue can be addressed by incorporating logistic growth, as proposed by biologist Pearl in [18], to model prey population dynamics. The logistic growth function is defined as $f(x) = rx\left(1 - \frac{x}{k}\right)$, where r represents the maximum relative growth rate, and k denotes the ecosystem's carrying capacity. However, introducing logistic growth alone does not fully account for the sustained predator-prey oscillations observed in nature. Predator growth rates are influenced by additional factors, such as the predator's maximal fecundity and "handling time," which refers to the period required to process prey. During this time, the predator cannot attack another prey item [27]. To address these complexities, researchers have developed various functional responses. In [10, 11], Holling introduced three functional responses, commonly known as Holling type I, Holling type II, and Holling type III. The Holling type II functional response is expressed as $\frac{mx}{a+x}$, where m is the maximum predation rate and a is the saturation constant. This response accounts for the predator's finite handling time, reflecting the realistic behavior that predators cannot continuously capture prey without interruption.

Traditionally, predator-prey models have been formulated using integer-order differential equations, often representing simplified population dynamics. For example, [4] developed a mathematical model to quantify intricate biological procedures in the context of biological control using prey-predator mechanisms. However, this new approach introduces the *Caputo fractional derivative* of order ϑ, which generalizes the classic model to incorporate fractional calculus [7, 13, 17, 20]. In recent years, researchers have turned to fractional calculus to enhance traditional predator-prey models by incorporating fractional derivatives, refer to [28] and references therein, which are known to better capture memory effects and complex dynamics that arise in biological systems. To overcome these limitations, researchers began exploring modifications to the classical model, including the use of fractional calculus. Fractional derivatives provide a powerful tool to model systems with memory, as they allow for a more accurate representation of processes that depend on the history of the system as compared to ordinary derivatives. Caputo fractional derivative provides a way to introduce these memory effects into the model. Fractional-order models are particularly valuable in situations where the past states of a system significantly influence its present dynamics, making them more suitable for representing real-world ecological systems.

Ayoade and Thota [1] introduced a model to analyze pest management and food security, focusing on locusts as prey and wasps as their natural enemies. Building on this foundation, the present study theoretically explores

pest management through biological control using a well-established prey-predator framework. The proposed model incorporates logistic growth for the prey population and a Holling type II functional response for the predator. Also see [3] for a modeling of the treatment accessibility and treatment compliance on the dynamics of HIV. As discussed earlier, fractional-order systems provide a more accurate representation of predator-prey dynamics, particularly in scenarios where interactions depend on the system's historical states rather than being instantaneous. To enhance the model, we integrate the Caputo fractional derivative into the pest control framework. This choice is motivated by the fractional derivative's ability to retain and utilize system information over time, which is crucial for studying dynamic systems in computational biology. By incorporating the fractional-order derivative, the model achieves a more flexible and precise depiction of predator-prey interactions, making it particularly suited for capturing complex dynamics in ecosystems. This approach highlights the potential of fractional-order systems in advancing the understanding of biological control mechanisms in pest management. Recently, the authors in [8] discussed an analysis on tobacco smoking dynamics in fractional and fractal-fractional approach, and [2] presented a mathematical modeling and analysis of the influence of family background on the spread of crime.

The paper is organized as follows: Section 2 presents the preliminaries, followed by a detailed explanation of the proposed model in section 3. The equilibrium and stability analysis of the model are discussed in section 4. Numerical simulations are provided in section 5, and the concluding remarks are summarized in section 6.

2. Preliminaries

Here some basic Preliminaries are presented [12, 19].

Definition 2.1. The Caputo fractional derivative of a function $g \in C^n$ with order ϑ is defined as:

$$ {}^{C}D_t^{\vartheta} g(t) = \frac{1}{\Gamma(n-\vartheta)} \int_{t_0}^{t} \frac{g^{(n)}(\xi)}{(t-\xi)^{\vartheta-n+1}} d\xi, \tag{2.1} $$

where $t_0 < t$, $\vartheta \in (n-1, n)$, $n \in N$ and $\Gamma(.)$ is the Gamma function. In particular, when $0 < \vartheta \leq 1$, then

$$ {}^{C}D_t^{\vartheta} g(t) = \frac{1}{\Gamma(1-\vartheta)} \int_{t_0}^{t} \frac{g'(\xi)}{(t-\xi)^{\vartheta}} d\xi. \tag{2.2} $$

Definition 2.2. The corresponding fractional integral is described as:

$$ I_t^{\vartheta} g(t) = \frac{1}{\Gamma(\vartheta)} \int_{0}^{t} (t-\xi)^{\vartheta-1} g(\xi) d\xi. \tag{2.3} $$

Lemma 2.3. *[16] Suppose that $\psi(t) \in C[c,d]$ and $D^{\vartheta}\psi(t) \in C[c,d]$ for $0 < \vartheta \leq 1$, then*

$$ \psi(t) = \psi(c) + \frac{1}{\Gamma(\vartheta)} D^{\vartheta}\psi(\xi)(t-c)^{\vartheta}, \ c < \xi < t, \ \forall t \in (c,d]. $$

Lemma 2.4. *[16] Let $0 < \vartheta \leq 1$, $\psi(t) \in C[c,d]$, and ${}_{t_0}^{C}D_t^{\vartheta}\psi(t) \in C[c,d]$. If ${}_{t_0}^{C}D_t^{\vartheta}\psi(t) \leq 0$ for $c < t < d$, then $\psi(t)$ is a non-increasing function on $[c,d]$. Conversely, if ${}_{t_0}^{C}D_t^{\vartheta}\psi(t) \geq 0$ for $c < t < d$, then $\psi(t)$ is a non-decreasing function on $[c,d]$.*

Lemma 2.5. *Let $z(t)$ be a continuous function on $[t_0, \infty)$ that satisfies*

$$ {}_{t_0}^{C}D_t^{\vartheta} z(t) \leq -\nu z(t) + \kappa, \quad z(t_0) = z_{t_0}, \quad t_0 \geq 0, \quad \nu, \kappa \in \Re, \tag{2.4} $$

where $\nu \neq 0$ and $\vartheta \in (0,1)$. Then,

$$ z(t) \leq \left(z_0 - \frac{\kappa}{\nu} \right) E_{\vartheta}[-\nu(t-t_0)^{\vartheta}] + \frac{\kappa}{\nu}. $$

Lemma 2.6. *[15] Consider a fractional-order system of order $\vartheta \in (0,1]$ defined as*

$$ {}_{t_0}^{C}D_t^{\vartheta} X(t) = \Psi(X), \quad X_{t_0} = (x_{t_0}^1, x_{t_0}^2, \ldots, x_{t_0}^n), \quad x_{t_0}^i > 0, \ i = 1, 2, \ldots, n, $$

where $X(t) = (x^1(t), x^2(t), \ldots, x^n(t))$. The equilibrium points of system (2.4) are locally asymptotically stable (LAS) if and only if $|\arg(\lambda_j)| > \frac{\vartheta\pi}{2}$ for all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial \Psi}{\partial x}$, evaluated at the corresponding equilibrium points.

3. Model description

Every pest has a natural enemy, and even in the absence of deliberate intervention, the population of pests can be influenced by their natural enemies. For example, locusts, which are significant agricultural pests in sub-Saharan Africa, have wasps as their natural predators. To understand how effectively wasps can reduce the locust population below the economic damage threshold, a mathematical model is developed.

In the proposed model, the densities of the prey (locusts) and predator (wasps) populations at any time $t > 0$ are denoted by $u(t)$ and $v(t)$, respectively. To represent the prey-predator dynamics within a hypothetical crop field, it is assumed that locusts (u) and wasps (v) co-exist within this agroecosystem. The locusts act as pests feeding on the crops, while the wasps serve as natural enemies that prey on the locusts. Based on these assumptions, the following model is proposed:

(1) The prey population exhibits logistic growth, represented by the term $ru\left(1-\frac{u}{k}\right)$, where r is the intrinsic growth rate, and k is the environment's carrying capacity.
(2) The prey-predator interaction is captured by the term $\frac{wuv}{a+u}$, where w denotes the predation rate, and a is the half-saturation constant, reflecting the predator's functional response to changes in prey abundance.
(3) The predator population grows through predation, represented by $\frac{w_1uv}{a+u}$, and declines due to natural mortality. The decline is modeled by βv, accounting for linear mortality, and γv^2, representing quadratic mortality caused by intra-species competition.

The system is governed by a set of fractional differential equations that describe the growth and interaction of these populations. The Caputo fractional derivative of order ϑ modifies the traditional time derivatives, allowing the model to account for more accurate representation of complex dynamics. This fractional model provides a more accurate representation of the population dynamics by incorporating the history of the system, offering deeper insights into the long-term behavior of predator-prey interactions. The model system is expresses as follows:

$$\begin{aligned} {}_0^C D_t^{\vartheta} u(t) &= ru\left(1-\frac{u}{k}\right) - \frac{wuv}{a+u}, \\ {}_0^C D_t^{\vartheta} v(t) &= \frac{w_1uv}{a+u} - \beta v - \gamma v^2, \end{aligned} \tag{3.1}$$

with initial conditions $u(t_0) = u_0 > 0,\ v(t_0) = v_0 > 0$.

3.1. Existence and Uniqueness.

Next, we demonstrate the existence and uniqueness of the solution for system (3.1).

Lemma 3.1. *[14] Consider the system*

$$ {}_{t_0}^C D_t^{\vartheta} x(t) = \psi(t,x), \quad t_0 > 0, \tag{3.2}$$

with the initial condition $x(t_0) = x_{t_0}$, where $\vartheta \in (0,1]$, $\psi : [t_0,\infty) \times \Omega \to \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$. If $\psi(t,x)$ fulfills the local Lipschitz condition concerning x, then a unique solution to (3.2) exists over $[t_0,\infty) \times \Omega$.

To investigate the system (3.1) existence and uniqueness, consider the region $\Omega \times [t_0, T]$, where $\Omega = \{(u,v) \in \mathbb{R}^2 : max(\mid u \mid, \mid v \mid) \le \eta\}$ and $T < +\infty$.

Theorem 3.2. *For each $X_0 = (u_0, v_0) \in \Omega$, there exists a unique solution $X(t) \in \Omega$ to the system (3.1) with initial condition X_0, defined for all $t \ge 0$.*

Proof. Let us denote $X = (u,v)$ and $\bar{X} = (\bar{u},\bar{v})$. Consider a mapping $M : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $M(X) = (M_1(X), M_2(X))$, where

$$\begin{aligned} M_1(X) &= ru\left(1-\frac{u}{k}\right) - \frac{wuv}{a+u}, \\ M_2(X) &= \frac{w_1uv}{a+u} - \beta v - \gamma v^2 = 0. \end{aligned}$$

For any $X, \bar{X} \in \Omega$, it follows that

$$\begin{aligned}\|M(X)-M(\bar{X})\| &= |M_1(X)-M_1(\bar{X})|+|M_2(X)-M_2(\bar{X})|\\ &=|ru\left(1-\frac{u}{k}\right)-\frac{wuv}{a+u}-r\bar{u}\left(1-\frac{\bar{u}}{k}\right)+\frac{w\bar{u}\bar{v}}{a+\bar{u}}|+|\frac{w_1uv}{a+u}-\beta v-\gamma v^2-\frac{w_1\bar{u}\bar{v}}{a+\bar{u}}+\beta\bar{v}+\gamma\bar{v}^2|\\ &= r|u-\bar{u}|+\frac{r}{k}|u^2-\bar{u}^2|+w|\frac{uv}{a+u}-\frac{\bar{u}\bar{v}}{a+\bar{u}}|w_1|\frac{uv}{a+u}-\frac{\bar{u}\bar{v}}{a+\bar{u}}|+\beta|v-\bar{v}|+\gamma|v^2-\bar{v}^2|\\ &\leq [r+\frac{2\eta r}{k}+a\eta(w+w_1)]|u-\bar{u}|+[\beta+2\eta\gamma+\eta(w+w_1)]|v-\bar{v}|\\ &\leq L\|X-\bar{X}\|,\end{aligned}$$

where $L=max\{r+\frac{2\eta r}{k}+a\eta(w+w_1),\beta+2\eta\gamma+\eta(w+w_1)\}$.

As a result, $M(X)$ satisfies the Local Lipschitz's condition. This implies that system (3.1) have unique solution. □

3.2. **Non-negativity and Boundedness.** Let $\mathbb{R}_+=\{x\in\mathbb{R}:x\geq 0\}$ and $\Omega_+=\{(x,y)\in\Omega:x,y\in\mathbb{R}_+\}$.

Theorem 3.3. *Any solution of the system (3.1) that starts in $\mathbb{R}^2_+$ remains non-negative for all time.*

Proof. We need to show that the domain $\mathbb{R}^2_+$ is positively invariant. For the Caputo fractional differential system (3.1), the following conditions hold:

$$\begin{aligned}&{}_0^C D_t^{\vartheta}u(t)|_{u=0}=0,\\ &{}_0^C D_t^{\vartheta}v(t)|_{v=0}=0.\end{aligned}\tag{3.3}$$

By Lemmas 2.3 and 2.4, it follows that the solution of (3.1) remains non-negative. As a result, for each hyperplane that bounds the non-negative orthant, the vector field points into $\mathbb{R}^2_+$. Therefore, the solutions of (3.1) stay within $\mathbb{R}^2_+$, making this domain positively invariant. □

Theorem 3.4. *Every solution of the system (3.1) that starts in $\mathbb{R}^2_+$ is uniformly bounded within π.*

Proof. Let the function $W(t)=u(t)+\frac{w}{w_1}v(t)$, which gives us

$$\begin{aligned}{}_{t_0}^C D_t^{\upsilon}W(t)+\eta W(t) &= ru\left(1-\frac{u}{k}\right)-\frac{wuv}{a+u}+\frac{wuv}{a+u}-\frac{w\eta}{w_1}v-\frac{w\gamma}{w_1}v^2+\eta u+\frac{\eta w}{w_1}v\\ &\leq \Big(-\frac{r}{k}u^2+(r+\eta)u\Big).\end{aligned}$$

Therefore, the maximum value of $\Big(-\frac{r}{k}u^2+(r+\eta)u\Big)$ is $\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2$. So,

$${}_{t_0}^C D_t^{\vartheta}W(t)+\eta W(t)\leq\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2.$$

Now, by Lemma 2.4, we get

$$\begin{aligned}W(t) &\leq \Big(W(t_0)-\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2\Big)E_{\upsilon}[-\Lambda(t-t_0)^{\upsilon}]+\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2,\\ &\to\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2,\ as\ t\to\infty.\end{aligned}$$

So, all solution of the system (3.1) remain bounded in π, where $\pi=\{(u,v)\in\Omega^+:W\leq\frac{rk}{4}\Big(1+\frac{\eta}{r}\Big)^2+\epsilon,\epsilon>0\}$. □

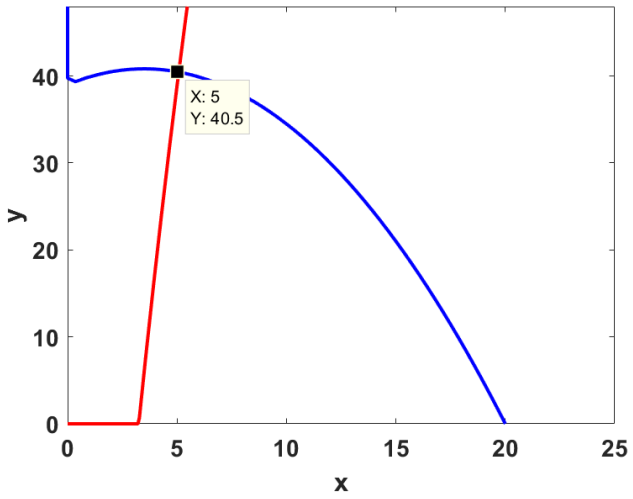


FIGURE 1. The figure depicts that the isoclines (4.1) and (4.2) uniquely intersect in the interior of the positive quadrant of the $x-y$ plane. The values of parameters are taken from Table 1 except $a = 13$.

4. Equilibria and stability

The system (3.1) has three possible equilibria points:

(i) $E^0(0,0)$ is trivial equilibrium point.
(ii) $E^1(k,0)$ is predator free equilibrium point.
(iii) The coexistence equilibria $E^*(u^*,v^*)$ is the interior point of the positive quadrant of state space satisfying the system of equations

$$ru\left(1-\frac{u}{k}\right)-\frac{wuv}{a+u}=0, \tag{4.1}$$

and

$$\frac{w_1uv}{a+u}-\beta v-\gamma v^2=0. \tag{4.2}$$

We verify this numerically in Figure 1 in the xy-plane.

The standard linearization technique, along with the Jacobian matrix, can be used to analyze the local stability of equilibrium points. At the equilibrium states, the Jacobian matrix of system 3.1 is derived as follows:

$$J(u,v)=\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix},$$

where

$$\varphi_{11}=r-\frac{2ru}{k}-\frac{awv}{(a+u)^2},\ \varphi_{12}=-\frac{wu}{a+u},$$
$$\varphi_{21}=\frac{aw_1v}{(a+u)^2},\ \varphi_{22}=\frac{w_1u}{a+u}-\beta-2\gamma v.$$

Theorem 4.1. *The trivial equilibrium point $E^0(0,0)$ of the system (3.1) is saddle point if without any parametric condition.*

Proof. The jacobian matrix at E^0, is given by

$$J(0,0)=\begin{pmatrix}r & 0\\ 0 & -\beta\end{pmatrix}.$$

The eigenvalues of the system $\lambda_1 = r$ and $\lambda_2 = -\beta$, implies $\mid arg(\lambda_1) \mid = 0 < \frac{\vartheta\pi}{2}$ and $\mid arg(\lambda_2) \mid = \pi > \frac{\vartheta\pi}{2}$. Thus, from Lemma 2.6, we say that the equilibrium point $E^0(0,0)$ is a saddle point. □

Theorem 4.2. *The predator free equilibrium point $E^1(k,0)$ of the system (3.1) is found to be locally asymptotically stable if $w_1 k < \beta(a+k)$.*

Proof. The jacobian matrix at E^1, is given by

$$J(k,0) = \begin{pmatrix} -r & -\frac{wk}{a+k} \\ 0 & \frac{w_1 k}{a+k} - \beta \end{pmatrix}.$$

The eigenvalues of the system $\lambda_1 = -r$ and $\lambda_2 = \frac{w_1 k}{a+k} - \beta$ implies $\mid arg(\lambda_1) \mid = \pi > \frac{\vartheta\pi}{2}$ and $\mid arg(\lambda_2) \mid = 0 < \frac{\vartheta\pi}{2}$, if $\frac{w_1 k}{a+k} - \beta > 0$ i.e., $\frac{w_1 k}{a+k} > \beta$, otherwise $\mid arg(\lambda_2) \mid = \pi > \frac{\vartheta\pi}{2}$, when $\frac{w_1 k}{a+k} - \beta < 0$. Thus, the equilibrium point E^1 will be asympotically stable iff $w_1 k < \beta(a+k)$. □

Theorem 4.3. *The system (3.1) is locally asymptotically stable around E^*, If any of the following conditions is satisfied:*

(i) $T \leq 0$, $D > 0$.

(ii) $T > 0$, $\sqrt{4D - T^2} > T\ tan\frac{\vartheta\pi}{2}$.

Proof. The jacobian matrix at E^*, is given as

$$J(u^*, v^*) = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix},$$

where

$$\chi_{11} = r - \frac{2ru}{k} - \frac{awv}{(a+u)^2},\ \chi_{12} = -\frac{wu}{a+u},$$
$$\chi_{21} = \frac{aw_1 v}{(a+u)^2},\ \chi_{22} = \frac{w_1 u}{a+u} - \beta - 2\gamma v.$$

The characteristic equation corresponding to the matrix $E^*(u^*, v^*)$ is given by $\lambda^2 - (\chi_{11}+\chi_{22})\lambda + (\chi_{11}\chi_{22} - \chi_{12}\chi_{21}) = 0$, where $T = \chi_{11}\chi_{22}$ and $D = \chi_{11}\chi_{22} - \chi_{12}\chi_{21}$. The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{T}{2} \pm \frac{\sqrt{T^2 - 4D}}{2}. \tag{4.3}$$

Case I. Assume $T \leq 0$ and $D > 0$. We have three sub-cases here.

(a) If $T = 0$, the system yields two complex conjugate eigenvalues with a real part of zero, satisfying the condition $\mid arg(\lambda_{1,2}) \mid = \frac{\vartheta\pi}{2}, \vartheta \in (0,1)$. Then by Lemma 2.6, the interior equilibrium is asymptotically stable.

(b) If $T_2 \geq 4D$ and $T < 0$, then the eigenvalues are negative real values $\lambda_{1,2} < 0$. Since $\mid arg(\lambda_i) \mid = \pi > \frac{\vartheta\pi}{2}$, $i = 1$ or 2. Then by Lemma 2.6, the interior equilibrium is locally asymptotically stable.

(c) For $T_2 < 4D$ and $T < 0$, the system has complex conjugate eigenvalues ($\lambda_1 = \bar{\lambda}_2$). Given that $T < 0$, we have $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = T < 0$, and $|\arg(\lambda_i)| > \frac{\vartheta\pi}{2}$ for $i = 1$ or 2. Therefore, E^* is locally asymptotically stable when $T < 0$.

Case II. For $T > 0$, the eigenvalues λ_1 and λ_2 satisfy $\lambda_1 = \lambda_2 = \tan^{-1}\left(\frac{\sqrt{4D-T^2}}{T}\right) > \frac{\nu\pi}{2}$, when $\sqrt{4D-T^2} > T\tan\left(\frac{\vartheta\pi}{2}\right)$. Under this condition, the system has a pair of complex conjugate eigenvalues λ_1 and λ_2. The real and imaginary parts of these eigenvalues have the following properties

$$Im(\lambda_1) = Im(\lambda_2) = \frac{1}{2}\sqrt{4D-T^2} > 0, \text{ and}$$
$$Re(\lambda_1) = Re(\lambda_2) = \frac{T}{2}.$$

Based on the assumptions, we have $|\mathrm{Im}(\lambda_{1,2})| > \mathrm{Re}(\lambda_{1,2})\tan\frac{\vartheta\pi}{2}$, which satisfies the condition $|\arg(\lambda_{1,2})| = \frac{\pi}{2} > \frac{\vartheta\pi}{2}$. Thus, by Lemma 2.6, the equilibrium point E^* is locally asymptotically stable. □

Theorem 4.4. *The model system (3.1) shows unstable behavior near E^* under the following conditions:*

(i) $T > 0$ *and* $T^2 - 4D > 0$.

(ii) $T > 0$ *and* $\sqrt{4D-T^2} < T\tan\left(\frac{\vartheta\pi}{2}\right)$.

Proof. (*i*) For $T^2 - 4D \geq 0$ and $T > 0$, we obtain $\lambda_1 > 0$ and $\lambda_2 > 0$, which implies $|\arg(\lambda_i)| = 0 < \frac{\vartheta\pi}{2}$ for $i = 1$ or 2. Therefore, by Lemma 2.6, the interior equilibrium is unstable.

(*ii*) If $T > 0$, $T^2 - 4D < 0$, and $\sqrt{4D-T^2} < T\tan\dfrac{\vartheta\pi}{2}$, then $\lambda_1 = \bar{\lambda_2}$ and $\lambda_2 = \bar{\lambda_1}$. This implies that $\mathrm{Im}(\lambda_1) = -\mathrm{Im}(\lambda_2) = \dfrac{1}{2}\sqrt{4D-T^2} > 0$. Therefore, $|\arg(\lambda_{1,2})| = \tan^{-1}\dfrac{\sqrt{4D-T^2}}{T} < \dfrac{\vartheta\pi}{2}$. By Lemma 2.6, the interior equilibrium is unstable. □

5. Numerical simulation

In this section, the numerical simulations are conducted via the Adamas Bashforth predictor-corrector scheme to confirm the theoretical results. The simulations are performed in MATLAB. For numerical simulation, we consider the set of hypothetical values of the model parameters in Table 1. The role of prey carrying capacity κ, predator attack rate β, half saturation constant a, and fractional order ϑ are discussed to validate our fractional-order model.

Table 1. Parametric values for numerical simulation.

r	k	w	w_1	a	β	γ
1.8	20	0.6	0.5	9	0.1	0.001

Figure 2 presents the time series and phase portraits for the model at different fractional order values. Initially, the population trajectories exhibit rapid growth, but after reaching a saturation point, the growth rate slows significantly. The numerical approximations agree that oscillatory behavior for both populations reflects the natural predator-prey interactions over time. As the fractional order values decrease, the trajectories gradually approach a stable equilibrium, indicating a balanced coexistence between the prey and predator populations. This phenomenon highlights the resilience of both species, as neither faces extinction under these conditions. The phase portrait in Figure 2(e) and Figure 2(f) displays a stable limit cycle at higher fractional-order values, transitioning to stable equilibrium as the fractional-order decreases. This suggests a shift from oscillatory coexistence to a more balanced, steady-state interaction between species.

Figure 3 aligns with ecological observations related to the half-saturation constant. The time series plot with different fractional orders for prey illustrated in Figures 3(a) and 3(b) exhibits periodic behavior for smaller values of the half-saturation constant a. However, after a certain threshold value of a, the prey population stabilizes after few days because predators become less effective when the half-saturation constant is high. As a result, predation pressure decreases, leading to a more stable prey population. Similar behavior can be observe in the predator population in Figures 3(c) and 3(d). This is because predators can efficiently consume prey, leading to population fluctuations. However, as a increases beyond a certain threshold, the predator population stabilizes. This occurs because a higher half-saturation constant reduces the predator's efficiency at lower prey densities, making it harder for predators to

sustain their population through feeding that stabilizes the predator population. The phase portraits are presented in Figures 3(e) and 3(f). The findings indicate that the proposed system transitions from a periodic state to a stable state as the value of fractional order decreases. Changes in carrying capacity directly impact the balance between prey and predator populations, influencing the long-term stability and dynamics of the ecosystem. Figure 4 illustrates the effect of environmental carrying capacity and fractional order on prey and predator populations. For $k = 10$, we observe a decrease in the prey population, which eventually stabilizes. In this case, the environment cannot support a large prey population. This limitation leads to increased competition for resources, which slows prey growth and further reduces their overall population. In response, the predator population also stabilizes after an initial increase. This explains the ecological system, as a smaller prey population provides less food for predators, which could lead to the predator population's potential decline or extinction over time. On the other hand, a larger carrying capacity means that the environment can support a larger population due to an abundance of resources such as food, space, and shelter. This can be seen in population dynamics: as the carrying capacity increases, the populations of both prey and predator also increase, often exhibiting oscillatory behavior. In applications like wildlife management or pest control, understanding the carrying capacity is crucial. For instance, maintaining an appropriate balance between prey and predators often depends on managing the environment to ensure an optimal carrying capacity. Figure 5 demonstrates the influence of varying intrinsic growth rates and fractional orders $\vartheta = 1, 0.95$ on the dynamics of the system. When the growth rate r is low, the prey population experiences slow growth, which may result in predation causing the prey numbers to decline significantly or even face extinction. Under such conditions, the system demonstrates a limit cycle behavior. In cases of moderate growth rates, the prey population increases sufficiently to support the predator population, resulting in oscillatory dynamics characterized by fluctuations in both prey and predator populations over time. Conversely, when the growth rate r is sufficiently high, it can lead to system stabilization, as the prey population becomes less susceptible to predation fluctuations. Additionally, the system stabilizes more rapidly at fractional orders, as illustrated in the visual representation demonstrating the efficacy of fractional dynamics.

The parameters a (half-saturation constant), r (intrinsic growth rate), and k (carrying capacity) are critical to the model's dynamics, as they govern the ecological interactions and environmental constraints. Varying these parameters allows to explore their impact on pest control effectiveness and system stability. A high a reduces predation efficiency, while a low k limits resource availability, both promoting population stability. Conversely, a high r enhances prey persistence. These insights inform the development of targeted biological control strategies for sustainable pest management.

Based on the discussion above, we conclude that the model shows several changes in the dynamical behaviour both for Holling type II functional response and the fractional order. Therefore, both play a significant role in determining the stability of population growth. The lower fractional values stabilize quickly in both the prey and predator populations.

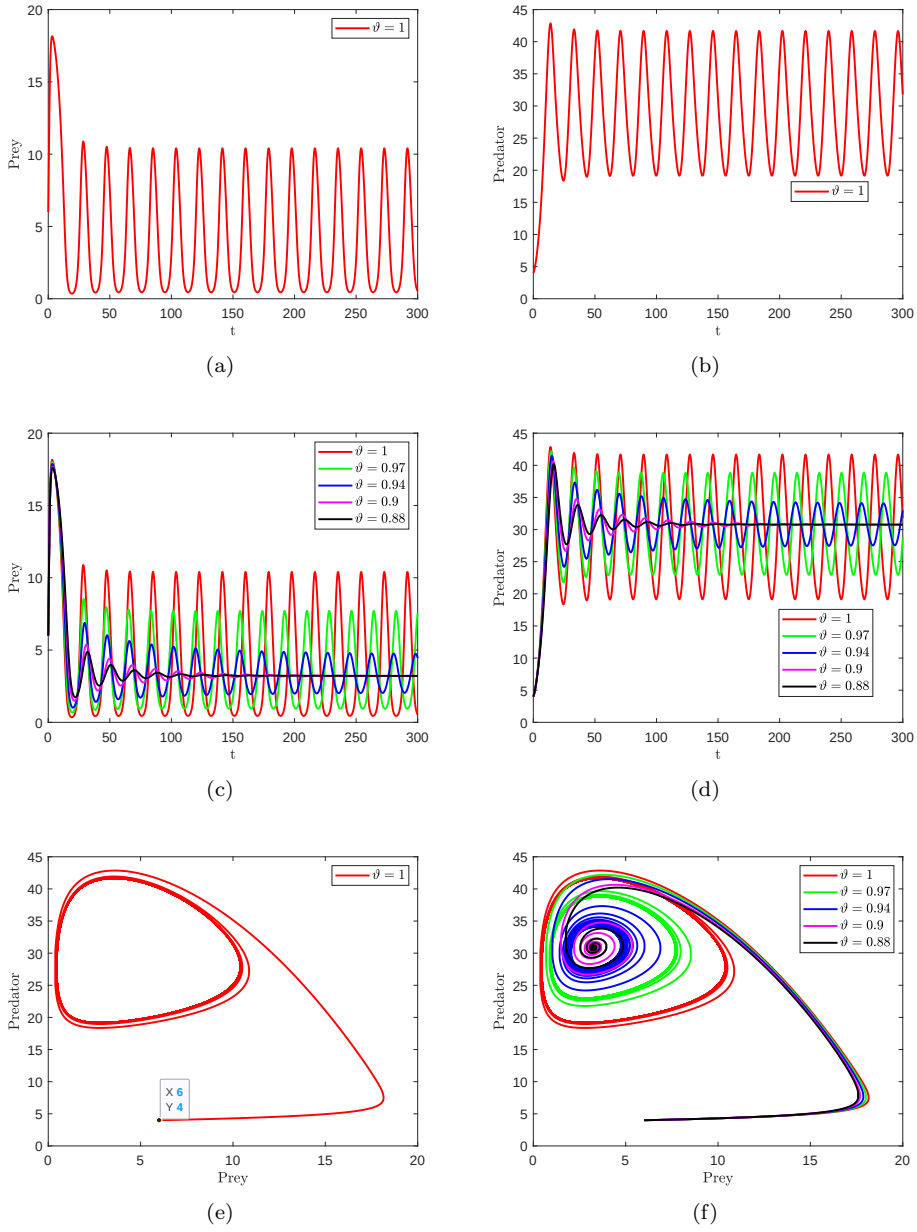
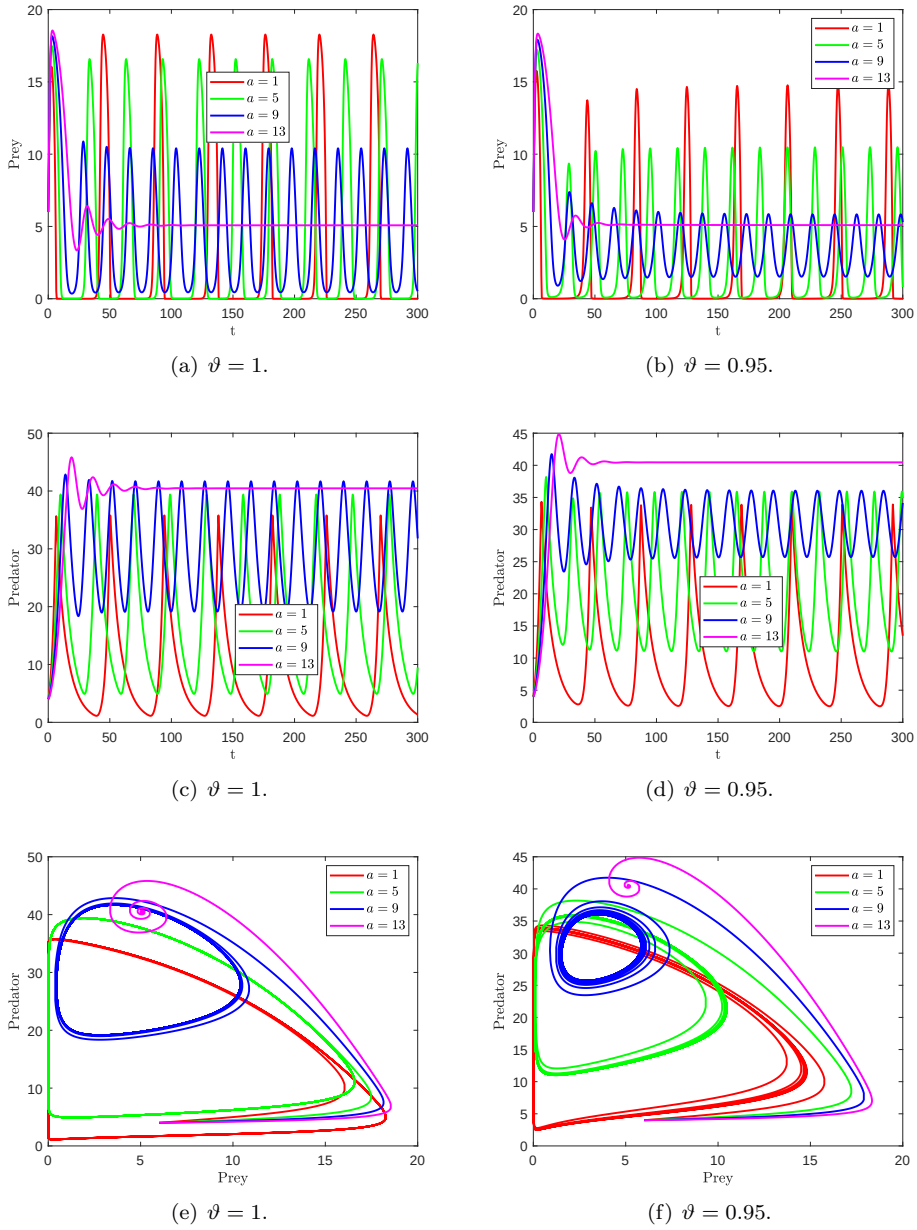


FIGURE 2. Trajectories of the fractional model (3.1) taking different values of fractional order.

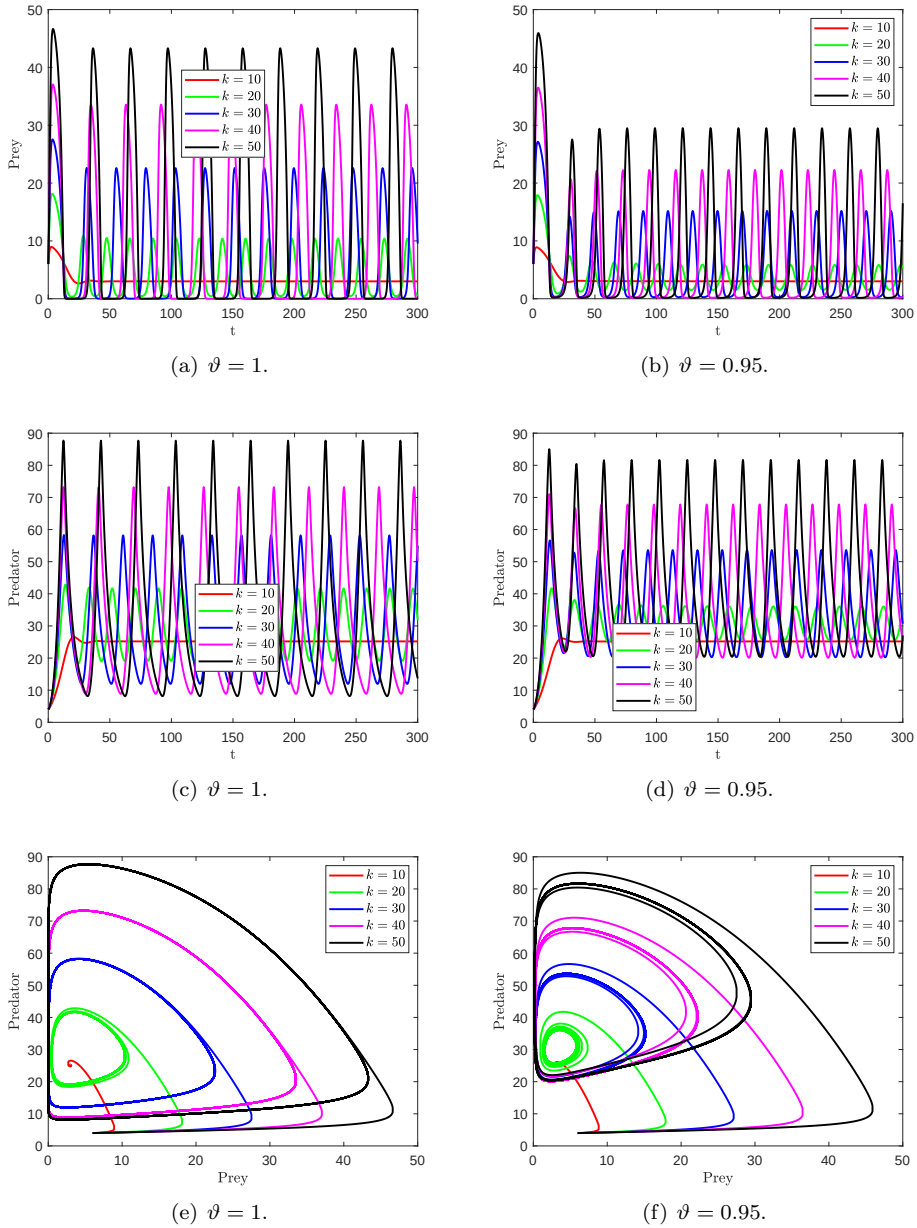


(a) $\vartheta = 1$. (b) $\vartheta = 0.95$.

(c) $\vartheta = 1$. (d) $\vartheta = 0.95$.

(e) $\vartheta = 1$. (f) $\vartheta = 0.95$.

FIGURE 3. Trajectories of the fractional model (3.1) at different rate of half saturation constant.

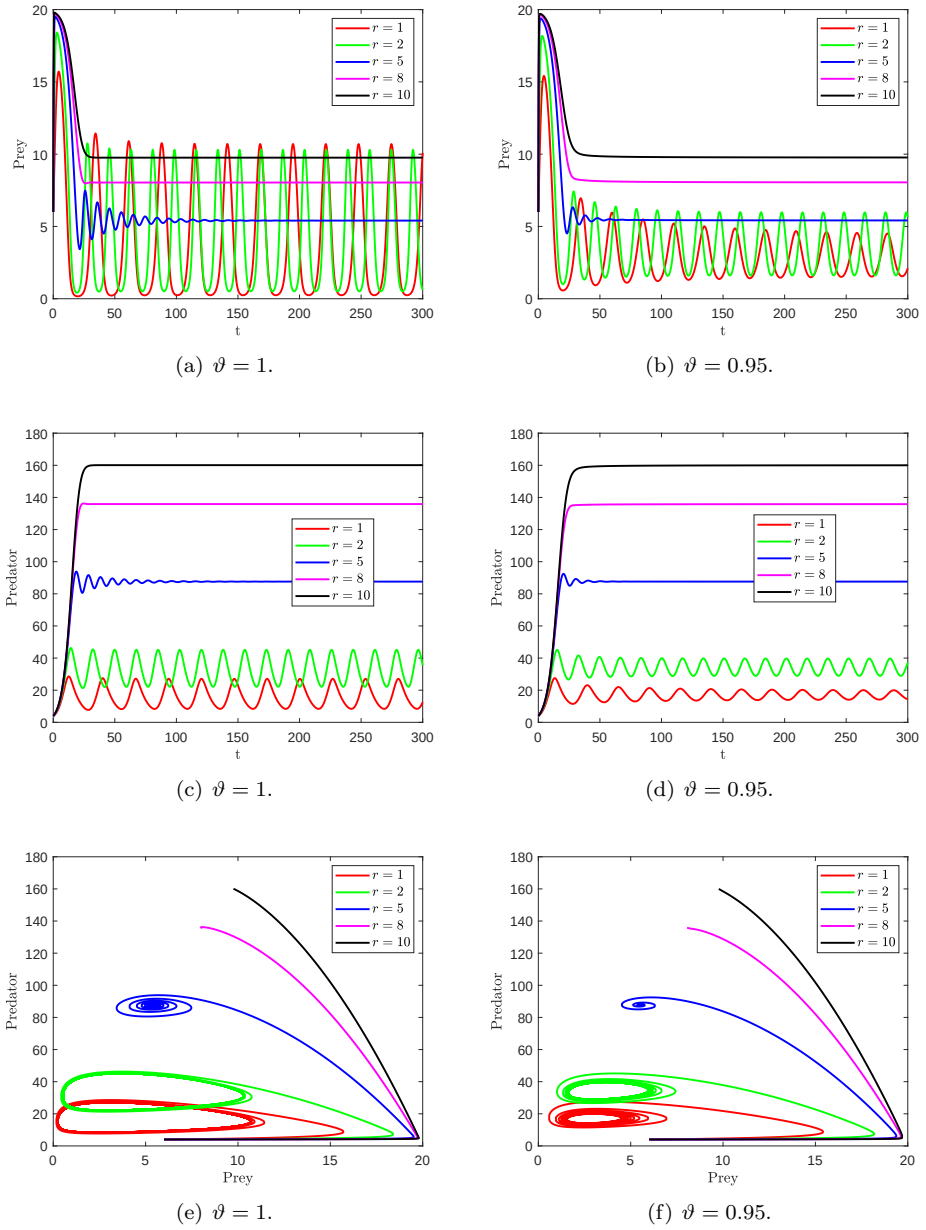


(a) $\vartheta = 1$. (b) $\vartheta = 0.95$.

(c) $\vartheta = 1$. (d) $\vartheta = 0.95$.

(e) $\vartheta = 1$. (f) $\vartheta = 0.95$.

FIGURE 4. Trajectories of the fractional model (3.1) at different carrying capacity.



(a) $\vartheta = 1$. (b) $\vartheta = 0.95$.

(c) $\vartheta = 1$. (d) $\vartheta = 0.95$.

(e) $\vartheta = 1$. (f) $\vartheta = 0.95$.

FIGURE 5. Trajectories of the fractional model (3.1) at different intrinsic growth rate.

6. Conclusions

In this paper, we theoretically explored the feasibility of biological pest control using a prey-predator model. We examined the existence and uniqueness of solutions, demonstrating that the solutions to the model are positive. The equilibrium points of the system were derived, and the stability at each equilibrium was analyzed. We also conducted numerical simulations to validate our theoretical results and to identify regions where biological control can be effectively applied. The proposed model successfully captures the essential oscillatory dynamics of predator-prey interactions. Our findings confirm that biological control has the potential to support sustainable harvests and enhance food security.

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