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The double ramadan group accelerated Adomian decomposition method for solving nonlinear partial differential equations

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Abstract

This paper investigates an advanced method for solving partial differential equations (PDEs) by integrating the Double Ramadan Group Transform (DRGT) with a faster version of the Adomian Decomposition Method (ADM). Initially, the DRGT is applied to transform the PDEs, which simplifies the management of boundary conditions and linear elements. The resulting transformed PDEs are subsequently solved using the enhanced ADM, which is specially tailored to efficiently handle the nonlinear terms that typically make solutions more difficult. The acceleration of the ADM is achieved by utilizing improved decomposition techniques and optimized series expansion methods, leading to significant gains in both the speed of convergence and the accuracy in addressing nonlinearities. The effectiveness of this combined approach is illustrated through several examples involving complex PDEs with challenging nonlinear aspects. The findings demonstrate significant improvements in computational efficiency and solution accuracy, underscoring the potential of this method for solving a wide variety of PDE problems in scientific and engineering applications.

Keywords. Double Ramadan group transform, Adomian decomposition method, Double Sumudu Integral transform, Double Laplace integral transform, Nonlinear partial differential equations, Accuracy, Efficiency.
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1. Introduction

Partial differential equations (PDEs) are valuable tools for exploring various natural phenomena and are frequently used to describe numerous physical laws [8, 14, 23]. In addition, PDEs are applied to model and investigate a broad array of engineering and technical problems, such as wake turbulence, optical fiber communication, atmospheric pollution dispersion, and others [7, 13, 32]. As a result, developments in partial differential equations are frequently important in a variety of industries, including aerospace and numerical weather prediction [2, 5, 15, 21, 29]. Partial differential equations, like many other disciplines, are becoming more interconnected and mutually supportive. Therefore, the study of partial differential equations is very important. Recently, some papers investigated the solution of different important types of partial differential equations. Interested readers are referred to [17–19, 22, 27] where in [22] Nawzad et al. focused on the traveling wave solutions and the analytical analysis of the simplified MCH equation and the combined KdV–mKdV equations. They examined rational solutions by using a polynomial expansion in terms of x and t. While in [27], Seyyedeh et al. investigated the initial-boundary value problem for the one-dimensional wave equation with a non-classical condition. They applied the reduced differential transform method to find the solution to the problem. Also, Mehrdad et al. [18] derived exact solutions for the $(1+1)$-dimensional and $(2+1)$-dimensional fifth-order integrable equations (FOIEs). These solutions were obtained using the enhanced $\tanh(\phi(\xi)/2)$ expansion method (IThEM) with the help of Maple software. In addition, Jalil et al. in [17] applied the generalized Hirota bilinear method to identify the knot structure and their interaction patterns for a precise analysis of the generalized fifth-order KdV-like equation. Additionally, Mehdi et al. in [19] applied the homotopy analysis method to solve

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nonlinear fractional partial differential equations. Using this approach, they created a method to approximate the solutions for the fractional $KdV, K(2,2)$, Burgers, BBM-Burgers, cubic Boussinesq, coupled KdV, and Boussinesq-like $B(m,n)$ equations with specified initial conditions.

Therefore, various numerical methods for solving partial differential equations have been proposed by researchers, including the finite difference method, which approximates derivatives using finite differences; the finite element method, a robust technique for solving PDEs by discretizing the domain into finite elements; and the finite volume method, among others [4, 24]. Numerical approaches have tremendously aided the investigation of partial differential equations. They also suggested, analytical methods such as separation of variables is a technique that simplifies the problem by reducing it to ordinary differential Equations (ODEs), method of characteristics is a technique particularly useful for first-order PDEs, integral transform methods such as Fourier and Laplace transforms for converting PDEs into algebraic Equations. Also, spectral methods that use global basis functions to achieve high accuracy. Currently, these methods are extensively used and continuously improving. Simultaneously, researchers are striving to create new techniques and tools for solving partial differential equations [31]. A range of integral transforms have been developed and used to solve both partial and integral differential equations. These transforms enable the exact solutions of the equations without the need for linearization or discretization. They can convert partial differential equations into ordinary differential equations through a single transformation, or into algebraic equations through a double integral transformation. Double integral transformations, in particular, have gained significant popularity for solving PDEs that involve unknown functions of two variables, offering greater efficiency compared to other numerical methods in addressing PDEs [1]. Additionally, variations of the double transform have been introduced in the literature, such as the double Laplace Transform, which is frequently employed to solve ODEs and PDEs, especially for initial value problems. It transforms differential equations into algebraic equations. The double Laplace transform [11] is one such example. Another recent addition is the double Sumudu Transform, which provides an efficient method for solving differential and integral equations, offering a viable alternative to traditional techniques [20]. The double Sumudu transform [12]. The connection between the double Sumudu transform and the double Laplace transform is established, showing that they are theoretical counterparts. The practical use of this relatively new transform is illustrated through several special functions, which appear in the solutions of evolution equations in population dynamics and partial differential equations. In this paper, the main goal is to introduce a double Ramadan group transform is a lesser-known but effective transform for certain types of PDEs, focusing on simplifying the equation structure to facilitate solution. This study introduces a novel method that combines the Double Ramadan Group Transform with the accelerated Adomian Decomposition Method to solve nonlinear PDEs. It addresses challenges like slow convergence and complex nonlinear terms, improving solution accuracy and computational efficiency. The integration of these techniques provides a new tool for tackling complex PDEs, with potential applications in various scientific and engineering fields, marking a significant contribution to mathematical methods for PDEs. The Double Ramadan Group Accelerated Adomian Decomposition Method offers several advantages for solving nonlinear partial differential equations in this study. Compared to other existing methods, it provides improved accuracy and faster convergence. Its ability to handle complex nonlinearities effectively makes it more reliable for solving intricate problems. Additionally, the method's flexibility and efficiency reduce computational costs, making it a more practical choice in comparison to traditional approaches in the literature.

2. Preliminaries

In this section, we discuss key definitions, lemmas, results, and properties of double integral transforms, including the Laplace, Sumudu, and Ramadan group integral transforms. These concepts will be crucial during the computational phase of our main study. Moreover, we present the basic idea of accelerated Adomian polynomials which are to be consisted modification of the regular Adomian polynomials.

2.1. **Double Laplace Integral Transform** $(DLIT)_2$**.** The double Laplace Transform that widely used for solving ODEs and PDEs, particularly for initial value problems. It converts differential Equations into algebraic Equations.

Definition 2.1. [31] Let $f(x,\ t)$ be a function of two independent variables 'x' and 't'. The $(DLIT)_2$ of $f(x,\ t)$ is denoted by $F(p,\ s)$ and defined as

$$\mathcal{L}_x\mathcal{L}_t\{f(x,\ t)\} = F(p,s) = \int_0^\infty\int_0^\infty e^{-(px+st)}f(x,t)dtdx, \tag{2.1}$$

we can also express it in the form of

$$\mathcal{L}_2\{f(x,t);(p,s)\} = F(p,s) = \int_0^\infty\int_0^\infty e^{-(px+st)}f(x,t)dtdx. \tag{2.2}$$

Remark 2.2. Equations (2.1) and (2.2) are equivalent; Equation (2.2) is written to clarify the order of variables before and after the transformation. This notation explicitly shows the sequential application of the double Laplace transform to both independent variables. Both forms lead to the same transformed function $F(p,s)$.

2.2. **Double Sumudu Integral Transform** $(DSIT)_2$ **[28].** The Sumudu transform simplifies the introduction of the double sumudu transforms $(DSIT)_2$, assuming the function has a power series transformation with respect to its variables.

Definition 2.3. Let $f(x,t);\ t,\ x\ \in\ \mathbb{R}_+$ For a function defined in the positive quadrant of the xt-plane and stated as a convergent infinite series, the double Sumudu transform is as follows:

$$\begin{aligned} G(s,\ p) &= S_2[f(x,\ t);\ (s,\ p)] \\ &= S[S\{f(x,\ t);x\ \to\ s\};\ t\ \to\ p] \\ &= \mathrm{S}[\{\frac{1}{p}\int_0^\infty e^{\frac{-t}{p}}f(x,\ t)\mathrm{dt}\};\ x \to\ s] \\ &= \frac{1}{sp}\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{s}+\frac{t}{p}\right)}f(x,t)dxdt, \end{aligned} \tag{2.3}$$

where p and s are the transform variables for t and x, respectively.

2.3. **Double Ramadan Group Integral Transform** $(DRGT)_2$ **[25].** A new integral RG transform has been defined for exponential-order functions [26]. We consider functions in the set A that are defined as follows:

$$A = \left\{f\,(x,t)\,\text{such that}\ \exists\ M\ ,\ \tau_1\ ,\ \tau_2 > 0\ , |f\,(x,t)| \le Me^{\frac{x+t}{\tau_i^2}}\ ,\ i = 1,2\ if\,(x,t)\,\epsilon R_+^2\right\}.$$

Definition 2.4. The double Ramadan group integral transform of the function $f\,(x,t)$ formed in the set A in the positive quadrant of the xt- plane is denoted by$(\mathrm{DRGT})_2$, and is defined as:

$$\begin{aligned} \mathrm{K}\,(\mathrm{s},\ \mathrm{p,u,v}) &= (\mathrm{DRGT})_2\left[\ f\,(x,t)\ \ ;(\mathrm{s},\ \mathrm{p,u,v})\right] \\ &= RG\left[\ RG\,[f\,(x,t)\ \ ;(\mathrm{s},\ \mathrm{p,u,v})]\,;(p,v)\right] \\ &= RG\left[\frac{1}{u}\int_0^\infty e^{\frac{-sx}{u}}f\,(x,t)\,dx\ ;(p,v)\right] \\ &= \frac{1}{v}\int_0^\infty e^{\frac{-pt}{v}}\left[\frac{1}{u}\int_0^\infty e^{\frac{-sx}{u}}f\,(x,t)\,dx\ dt\ \right] \\ &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)}\ dxdt\ , \end{aligned} \tag{2.4}$$

where s,p,u and v are complex variables, with s and p being the transform variables for x and t, respectively and u,v $\in(-\ \tau_1\ ,\ \tau_2)$ where $\tau_1\ ,\ \tau_2 > 0$ and Re(s), Re(p) >0.

By applying the duality between the double Laplace $(DLIT)_2$ and double Sumudu $(DSIT)_2$ transforms, presented by Dhunde and Waghmare [9], and Tchunche and Mbare [28] respectively we can readily prove the following relationships among double Ramadan group, double Laplace and double Sumudu integral transforms.

i) $\mathrm{K}\ (\mathrm{s}\ ,\ \mathrm{p}\ ,\ \mathrm{u}\ ,\ \mathrm{v}) = \frac{1}{uv}\ F(\frac{s}{u}\ ,\frac{p}{v})$

Proof. Since,

$$F(s,p) = L_2\left[f(x,t);(s,p)\right] = \int_0^\infty \int_0^\infty e^{-(sx+pt)} f(x,t)dxdt.$$

Then, we have

$$\begin{aligned} F\left(\frac{s}{u}, \frac{p}{v}\right) &= \int_0^\infty \int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)}\ dxdt \\ &= uv\frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{sx}{u}+\frac{pt}{v}\right)}\ dxdt \\ &= uv(\mathrm{DRGT})_2\ \left[\ f(x,t)\ \ ;(\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right] \\ &= uv\ K(s\ ,\ p\ ,u,\ v)\,. \end{aligned}$$

□

ii) $K(1\ ,\ 1, u\ ,\ v) = G(u\ ,\ v)$

Proof. For the double Ramadan group we have

$$\mathrm{K}\ \ (\mathrm{s}\ ,\ \mathrm{p}\ ,\ \mathrm{u}\ ,\ \mathrm{v}) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{sx}{u}+\frac{py}{v}\right)}\ f(x,y)\ \ dxdt\ .$$

Setting $s = p = 1$, we get

$$\mathrm{K}\ \ (1,1\ ,\ \mathrm{u}\ ,\ \mathrm{v}) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u}+\frac{y}{v}\right)}\ f(x,y)\ \ dxdt.$$

Let $\frac{x}{u} = r$; $\frac{t}{v} = t$, we get

$x = ur$, $dx = udr$, $y = vt$, $dy = vdt$.

Then

$$\mathrm{K}\ \ (1,1\ ,\ \mathrm{u}\ ,\ \mathrm{v}) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(r+t)}\ f(ur\ ,\ vt)\,uv\ drdt = \int_0^\infty \int_0^\infty e^{-(r+t)}\ f(ur\ ,\ vt)\,drdt.$$

Thus , $K(1\ ,\ 1, u\ ,\ v) = G(u\ ,\ v)$.

□

2.3.1. *Properties of double Ramadan group integral transform* $(DRGT)_2$. The $(\mathrm{DRGT})_2$ can be utilized to simplify differential equations involving multiple variables. by transforming them into algebraic Equations. Specifically, when applied to partial derivatives, the transform is particularly useful in solving partial differential Equations (PDEs): The double transform can simplify the complexity of PDEs by turning them into easier-to-solve algebraic forms. For instance, if you have a partial derivative, for example, of the form $\frac{\partial f(x,t)}{\partial x}$ or $\frac{\partial^2 f(x,t)}{\partial x^2}$ you can apply the $(\mathrm{DRGT})_2$ and use its properties to find a solution for the transformed function K (s , p , u , v). Once the solution is found in the transform domain, inverse transforms can be used to obtain the solution in the original domain.

Let $f(x,t)$ be a function defined in the x-plane's positive quadrant. The $(\mathrm{DRGT})_2$ transformation of the first and second order partial derivatives of $f(x,t)$is given as

$$(\mathrm{DRGT})_2\left[\frac{\partial f(x,t)}{\partial x}\ ;\ (\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right] = \frac{s}{u}\ K(s,p,u,v) - \ \frac{1}{u}\ K(0,p,0,v)\,, \tag{2.5}$$

where $s, p, u,$ and v are complex variables, with s and p being the transform variables for x and t, respectively and $u, v \in (-\ \tau_1\ ,\ \tau_2)$ where $\tau_1\ ,\ \tau_2 > 0$ and $\mathrm{Re}(s)$,$\mathrm{Re}(p)$ >0 , and K $(0,\mathrm{p}\ ,\ 0\ ,\ \mathrm{v})$ is defined as K $(0,\mathrm{p}\ ,\ 0\ ,\ \mathrm{v}) = \frac{1}{v}\int_0^\infty e^{-\left(\frac{pt}{v}\right)} f(0,t)dt$.

To prove (2.5), we use the definition of the $(DRGT)_2$ transform defined in (2.4), which yields

$$
\begin{aligned}
(\mathrm{DRGT})_2\left[\frac{\partial f(x,t)}{\partial x};(\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right] &= RG\left[RG\left[\frac{\partial f(x,t)}{\partial x};(\mathrm{s},\ \mathrm{u})\right];(p,v)\right]\\
&= RG\left[\frac{1}{u}\int_0^\infty e^{\frac{-sx}{u}}\frac{\partial f(x,t)}{\partial x}dx\ ;(p,v)\right]\\
&= RG\left[\frac{1}{u}\ [[e^{\frac{-sx}{u}}\ f(x,t)]_{x=0}^{\infty}+\frac{s}{u}\int_0^\infty e^{\frac{-sx}{u}}f(x,t)dx]\ ;(p,v)\right]\\
&= RG\left[\frac{1}{u}\ [-f(0,t);(p,v)]+RG[\frac{s}{u^2}\int_0^\infty e^{\frac{-sx}{u}}f(x,t)dx]\ ;(p,v)\right]\\
&= -\frac{1}{u}\frac{1}{v}\int_0^\infty e^{\frac{-pt}{v}}f(0,t)\,dt+\frac{s}{vu^2}\int_0^\infty\int_0^\infty e^{\frac{-pt}{v}}e^{\frac{-sx}{u}}f(x,t)\,dx\\
&= -\ \frac{1}{u}\ K(0,p,0,v)+\frac{s}{u}\left[\frac{1}{uv}\ \int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{py}{v}\right)}\ f(x,y)\ \ dxdt\ \right]\\
&= -\ \frac{1}{u}\ K(0,p,0,v)+\frac{s}{u}\ K(s,p,u,v)\,.
\end{aligned}
$$

Then,

$$(\mathrm{DRGT})_2\left[\frac{\partial f(x,t)}{\partial x}\ ;\ (\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right]=\frac{s}{u}\ K(s,p,u,v)-\ \frac{1}{u}\ K(0,p,0,v)\,.$$

Similarly, we can demonstrate that

$$(\mathrm{DRGT})_2\left[\frac{\partial f(x,t)}{\partial t}\ ;\ (\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right]=\frac{p}{v}\ K(s,p,u,v)-\ \frac{1}{v}\ K(s,0,u,0)\,.$$

Now for the second partial derivatives $\frac{\partial^2 f(x,t)}{\partial x^2}$ one can prove easily that

$$(\mathrm{DRGT})_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right]=\frac{s^2}{u^2}K(s,p,u,v)-\frac{s}{u^2}K(0,p,0,v)-\frac{s}{u}\ \frac{\partial K(0,p,0,v)}{\partial x}.$$

Overall, we can conclude for the n^{th} partial derivatives $\frac{\partial^n f(x,t)}{\partial x^n}$, its $(\mathrm{DRGT})_2$ is as:

$$
\begin{aligned}
&(\mathrm{DRGT})_2\left[\frac{\partial^n f(x,t)}{\partial x^n};(\mathrm{s},\ \mathrm{p},\mathrm{u},\mathrm{v})\right]\\
&=\frac{s^n}{u^n}\ K(s,p,u,v)-\frac{s^{n-1}}{u^n}K(0,p,0,v)-\frac{s^{n-2}}{u^{n-1}}\ K(0,p,0,v)-\cdots-\frac{1}{u}\ \frac{\partial^n K(0,p,0,v)}{\partial x^n}.
\end{aligned}
$$

Table 1. Double Ramadan Group Transform of some function.

$f(x,t)$	$(\mathrm{DRGT})_2 = [\ f(x,t\ )\ ] = K(s,p,u,v)$
1	$\frac{uv}{p\mathrm{suv}}$
t	$\frac{uv^2}{p^2\mathrm{suv}}$
x	$\frac{u^2v}{ps^2\mathrm{uv}}$
t^n	$\frac{n!uv^{n+1}}{p^{n+1}\mathrm{suv}}$
$x^m t^n$	$\frac{m!n!\ u^{m+1}v^{n+1}}{\mathrm{uv}\ p^{n+1}s^{m+1}}$
e^{ax+bt}	$\frac{1}{\mathrm{uv}(-a+\frac{s}{u})(-b+\frac{p}{v})}$
$u_t(x,t)$	$\frac{p}{v}\ K(s,p,u,v)-\ \frac{1}{v}\ K(s,0,u,0)$
$u_x(x,t)$	$\frac{s}{u}\ K(s,p,u,v)-\ \frac{1}{u}\ K(0,p,0,v)$
$u_{xx}(x,t)$	$\frac{s^2}{u^2}\ K(s,p,u,v)-\frac{s}{u^2}\ K(0,p,0,v)-\frac{1}{u}\ \frac{\partial K(0,p,0,v)}{\partial x}$

2.3.2. *The Convergence Theorem of* $(DRGT)_2$. In this subsection, we state the convergence of the double Ramadan group integral transform; for the proof, refer to [25].

Theorem 2.5. *Convergence of double Ramadan group integral transform* $(\mathrm{DRGT})_2$
Let $g(x,t)$ *be a two-variable function that is continuous in the first region of the x-plane. If the integral, defined for*$(\mathrm{DRGT})_2$ *of the form* $\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{sx}{u}+\frac{pt}{v})}g(x,t)\,dxdt$ *converges at* $s=s_0$, $u=u_0$, $p=p_0$, $v=v_0$, *then the integral converges for* $s>s_0$, $u>u_0$, $p>p_0$ *and* $v>v_0$, *where we assume* $\frac{s}{u}-\frac{s_0}{u_0}>0$ *and* $\frac{p}{p_0}-\frac{v}{v_0}>0$.

3. Double Ramadan Group Transform combined with the Accelerated Adomian Method for Nonlinear Partial Differential Equations

To explain the fundamental principle of this method, we consider a general partial differential equation with the initial conditions given below

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \tag{3.1}$$

$$u(x,0)=h(x), \qquad u_t(x,0)=f(x). \tag{3.2}$$

Where L is the second - order linear differential operator $L=\frac{\partial^2}{\partial t^2}$, where R is the differential operator of linear order with lower orders than L, N is the general nonlinear differential operator, and $g(x,t)$ is the source term. Applying the double Ramadan group integral transform to both sides of Equations (3.1) and (3.2), we obtain:

$$(DRGT)_2\left[Lu(x,t)\right]+(DRGT)_2\left[Ru(x,t)\right]+(DRGT)_2\left[Nu(x,t)\right]=(DRGT)_2\left[g(x,t)\right], \tag{3.3}$$

$$\begin{aligned} &RG\left[u(x,0)\right]=RG\left[h(x)\right]=K(s,0,u,0),\\ &RG\left[\,u_t(x,0)\right]=RG\left[f(x)\right]=\frac{\partial}{\partial t}K(s,0,u,0), \end{aligned} \tag{3.4}$$

where

$$(DRGT)_2\left[\frac{\partial^2 f}{\partial t^2}\right]=\frac{p^2}{v^2}\,K(s,p,u,v)-\frac{p}{v^2}\,K(s,0,u,0)-\frac{1}{v}\,\frac{\partial K(s,0,u,0)}{\partial t}. \tag{3.5}$$

To substitute Eq. (3.4) in (3.3), after using Eq. (3.5), we get:

$$\begin{aligned} (DRGT)_2\left[u(x,t)\right]=\frac{1}{p}\,RG\left[h(x)\right]+\frac{p}{v^2}\,RG\left[f(x)\right]+\frac{v^2}{p^2}(DRGT)_2\left[g(x,t)\right]\\ -\frac{v^2}{p^2}(DRGT)_2\left[Ru(x,t)\right]-\frac{v^2}{p^2}(DRGT)_2\left[Nu(x,t)\right]. \end{aligned} \tag{3.6}$$

Now, using the inverse double Ramadan group integral transform on both sides of Eq. (3.6) we get:

$$u(x,t)=G(x,t)-(DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2\left[Ru(x,t)\right]+\frac{v^2}{p^2}(DRGT)_2\left[Nu(x,t)\right]\right], \tag{3.7}$$

where $G(x,t)$ reflects the terms resulting from the source term and the specified initial circumstances.
Subsequently, we express the solution as an infinite series, as demonstrated below.

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)\;. \tag{3.8}$$

The nonlinear term can be expressed as follows,

$$Nu(x,t)=\sum_{n=0}^{\infty}\overline{A_n}(u), \tag{3.9}$$

where, $\overline{A_n}(u)$ are Accelerated Adomian polynomials and they can be calculated by formula given below:

$$\overline{A_n} = N(s_n) - \sum_{i=0}^{n-1} \overline{A_i}, \qquad n = 1, 2, \ldots, \tag{3.10}$$

where, the partial sum $s_n = \sum_{n=0}^{\infty} u_n(x,t)$ and $\overline{A_0} = N(s_0) = N(u_0)$

To substitute (3.8) and (3.9) in (3.7), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - (DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2\left[R\sum_{n=0}^{\infty} u_n(x,t)\right] + \frac{v^2}{p^2}(DRGT)_2\left[\sum_{n=0}^{\infty}\overline{A_n}\right]\right]. \tag{3.11}$$

Then from Eq. (3.11) we get:

$$u_0(x,t) = G(x,t), \tag{3.12}$$

$$u_1(x,t) = -(DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2[Ru_0(x,t)] + \frac{v^2}{p^2}(DRGT)_2[\overline{A_0}]\right], \tag{3.13}$$

$$u_2(x,t) = -(DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2[Ru_1(x,t)] + \frac{v^2}{p^2}(DRGT)_2[\overline{A_1}]\right]. \tag{3.14}$$

Typically, the recursive relation is defined as follows:

$$u_n(x,t) = -(DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2[Ru_{n-1}(x,t)] + \frac{v^2}{p^2}(DRGT)_2[\overline{A_{n-1}}]\right], \quad n \geq 1. \tag{3.15}$$

Ultimately, the solution $u(x,t)$ is approximated by the series:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t). \tag{3.16}$$

4. General procedure for solution of Poisson Equation Via Double Ramadan Group Transform

In this section, the goal is to develop a general procedure to solve the Poisson equation using the $(\mathrm{DRGT})_2$. This approach is an advanced method that extends the traditional Ramadan transform technique to handle more complex boundary conditions and source terms in a systematic way.

The Poisson partial differential equation (PDE) is a second-order linear differential equation commonly written as:

$$\nabla^2\psi = -f(r),$$

where ∇^2 is the Laplacian operator, $\psi(r)$ is the potential function to be determent, and $f(r)$ is the given source term.

Below is an outline for the development of this procedure.

Example 4.1. Consider the Poisson PDE, given by [3],

$$y_{xx}(x,t) + y_{tt}(x,t) = x^2 + t^2, \tag{4.1}$$

subjected to the conditions

$$y(x,0) = 0,\ y_x(0,t) = 0,$$

$$y(0,t) = 0,\ y_t(x,0) = 0.$$

Solution: Applying $(DRGT)_2$ on (4.1), we have

$$(DRGT)_2\,[y_{xx}(x,t)] + (DRGT)_2\,[y_{tt}(x,t)] = (DRGT)_2\,[x^2 + t^2].$$

By using properties of$(DRGT)_2$, we get

$$\frac{s^2}{u^2}\ K(s,p,u,v) - \frac{s}{u^2}\ K(0,p,0,v) - \frac{1}{u}\ \frac{\partial K(0,p,0,v)}{\partial x} + \frac{p^2}{v^2}\ K(s,p,u,v) - \frac{p}{v^2}\ K(s,0,u,0)$$
$$- \frac{1}{v}\ \frac{\partial K(s,0,u,0)}{\partial t} = (DRGT)_2\ \left[x^2+t^2\right],$$

$$\left[\frac{s^2}{u^2}+\frac{p^2}{v^2}\right] K(s,p,u,v) = (DRGT)_2\left[x^2+t^2\right] + \frac{s}{u^2}\ K(0,p,0,v) + \frac{1}{u}\ \frac{\partial K(0,p,0,v)}{\partial x}$$
$$+ \frac{p}{v^2}\ K(s,0,u,0) + \frac{1}{v}\ \frac{\partial K(s,0,u,0)}{\partial t}.$$

Using the original circumstances, we obtain:

$$K(0,p,0,v) = (RG)\left[y(0,t);(0,p,0,v)\right] = 0,$$
$$K(s,0,u,0) = (RG)\left[y(x,0);(s,0,u,0)\right] = 0,$$
$$\frac{\partial K(0,p,0,v)}{\partial x} = (RG)\left[y_x(0,t);(s,0,u,0)\right] = 0,$$
$$\frac{\partial K(s,0,u,0)}{\partial t} = (RG)\left[y_t(x,0);(s,0,u,0)\right] = 0.$$

This yield,

$$\left[\frac{s^2}{u^2}+\frac{p^2}{v^2}\right] K(s,p,u,v) = (DRGT)_2\ \left[x^2+t^2\right] = \frac{\frac{2u^3v}{ps^3}+\frac{2uv^3}{p^3s}}{\text{uv}},$$

$$K(s,p,u,v) = \frac{\frac{2u^4v^2}{ps^3}+\frac{2u^2v^4}{p^3s}}{p^2u^2+s^2v^2},$$

$$K(s,p,u,v) = \frac{\frac{2u^4v^2p^3s+2u^2v^4ps^3}{p^4s^4}}{p^2u^2+s^2v^2} = \frac{\frac{(2u^2v^2ps)(u^2p^2+v^2s^2)}{p^4s^4}}{p^2u^2+s^2v^2},$$

$$K(s,p,u,v) = \frac{2u^2v^2}{p^3u^3}.$$

Take the inverse of double Ramadan group transform on both sides, we get:

$$y(x,t) = \frac{t^2x^2}{2}\ .$$

Example 4.2. Consider the following type of Poisson partial differential Eq. [3],

$$y_{xx}(x,t) + y_{tt}(x,t) = -x\text{cost}\ , \tag{4.2}$$

subjected to the conditions

$$y(x,0) = x\ ,\ y_x(0,t) = \text{cost}\ ,$$
$$y(0,t) = 0\ ,\ y_t(x,0) = 0.$$

Solution: Applying $(DRGT)_2$ on (4.2), we have

$$(DRGT)_2\ \left[y_{xx}(x,t)\right] + (DRGT)_2\ \left[y_{tt}(x,t)\right] = (DRGT)_2\ \left[-x\text{cost}\ \right].$$

By using properties of $(DRGT)_2$, we get

$$\frac{s^2}{u^2}\ K(s,p,u,v) - \frac{s}{u^2}\ K(0,p,0,v) - \frac{1}{u}\ \frac{\partial K(0,p,0,v)}{\partial x} + \frac{p^2}{v^2}\ K(s,p,u,v) - \frac{p}{v^2}\ K(s,0,u,0)$$
$$- \frac{1}{v}\ \frac{\partial K(s,0,u,0)}{\partial t} = -\frac{pu}{s^2(p^2+v^2)},$$

$$\left[\frac{s^2}{u^2}+\frac{p^2}{v^2}\right]K(s,p,u,v)=-\frac{pu}{s^2(p^2+v^2)}+\frac{s}{u^2}\ K(0,p,0,v)+\frac{1}{u}\ \frac{\partial K(0,p,0,v)}{\partial x}$$
$$+\frac{p}{v^2}\ K(s,0,u,0)+\frac{1}{v}\ \frac{\partial K(s,0,u,0)}{\partial t},$$

Using the original circumstances, we obtain:

$$K(0,p,0,v)=(RG)\left[y(0,t);(0,p,0,v)\right]=0,$$
$$K(s,0,u,0)=(RG)\left[y(x,0);(s,0,u,0)\right]=\frac{u}{s^2},$$
$$\frac{\partial K(0,p,0,v)}{\partial x}=(RG)\left[y_x(0,t);(0,p,0,v)\right]=\frac{p}{p^2+v^2},$$
$$\frac{\partial K(s,0,u,0)}{\partial t}=(RG)\left[y_t(x,0);(s,0,u,0)\right]=0.$$

This yield,

$$\left[\frac{s^2}{u^2}+\frac{p^2}{v^2}\right]K(s,p,u,v)=-\frac{pu}{s^2(p^2+v^2)}+\frac{s}{u}\left[\frac{p}{p^2+v^2}\right]+\frac{p}{v^2}\left[\frac{u}{s^2}\right],$$
$$\left[\frac{s^2}{u^2}+\frac{p^2}{v^2}\right]K(s,p,u,v)=-\frac{pu}{s^2(p^2+v^2)}+\frac{sp}{up^2+uv^2}+\frac{pu}{v^2s^2},$$
$$K(s,p,u,v)=\frac{u^2v^2}{p^2u^2+s^2v^2}\left[-\frac{pu}{s^2(p^2+v^2)}\right]+\frac{u^2v^2}{p^2u^2+s^2v^2}\left[\frac{sp}{up^2+uv^2}\right]+\frac{u^2v^2}{p^2u^2+s^2v^2}\left[\frac{pu}{v^2s^2}\right],$$
$$K(s,p,u,v)=\frac{1}{p^2u^2+s^2v^2}\left[-\frac{pu^3v^2}{s^2(p^2+v^2)}+\frac{pu^3}{s^2}+\frac{upv^2}{p^2+v^2}\right],$$
$$K(s,p,u,v)=\frac{pu(p^2u^2+s^2v^2)}{s^2(p^2+v^2)(p^2u^2+s^2v^2)},$$
$$K(s,p,u,v)=\frac{pu}{s^2(p^2+v^2)}.$$

Take the inverse of double Ramadan group transform to both sides, we get:

$$y(x,t)=x\cos t.$$

5. Numerical Results

In this part, a few examples of nonlinear partial differential Equations are considered to demonstrate the effectiveness of the double Ramadan group accelerated Adomian decomposition method (DRG-AADM) in comparison to the standard double Ramadan group Adomian decomposition method (DRG-ADM), highlighting its advantages:

Example 5.1. Take the following nonlinear partial differential Eq. [25]

$$y_t+yy_x=0, \tag{5.1}$$

with the given initial conditions: $y(x,0)=-x$, and exact solution: $y(x,t)=\frac{x}{t-1}$.
Applying the double Ramadan group transform $(DRGT)_2$ to both sides on (5.1), we obtain:

$$(DRGT)_2\left[y_t\right]+(DRGT)_2\left[yy_x\right]=0,$$
$$\frac{p}{v}K(s,p,u,v)-\frac{1}{v}K(s,0,u,0)=-\ (DRGT)_2\left[yy_x\right],$$

where,

$$K(s,0,u,0)=(RG)\left[y(x,0);(s,0,u,0)\right]=\frac{-u}{s^2}.$$

Then,

$$\frac{p}{v}K\left(s,p,u,v\right)=\frac{-u}{v\ s^2}-\ \left(DRGT\right)_2\left[yy_x\right],$$

$$K\left(s,p,u,v\right)=\frac{-u}{p\ s^2}\ -\ \frac{v}{p}\ \left(DRGT\right)_2\ \left[yy_x\right].$$

Using the inverse of the double Ramadan group transform on both sides, we obtain:

$$y\left(x,t\right)=\left(DRGT\right)_2^{-1}\left[\frac{-u}{p\ s^2}\right]-\left(DRGT\right)_2^{-1}[\ \frac{v}{p}\ \left(DRGT\right)_2\left[yy_x\right]].$$

This technique presents solutions as an infinite series given by

$$y\left(x,t\right)=\sum_{n=0}^{\infty}y_n(x,t).$$

The term y_n should be computed recursively, and the nonlinear term yy_x is decomposed as follows:

$$yy_x=\ \sum_{n=0}^{\infty}A_n,$$

$$\sum_{n=0}^{\infty}y_n\left(x,t\right)=\left(DRGT\right)_2^{-1}\left[\frac{-u}{p\ s^2}\right]-\left(DRGT\right)_2^{-1}[\ \frac{v}{p}\ \left(DRGT\right)_2\left[\sum_{n=0}^{\infty}A_n\right]],$$

By comparing both sides, we get:

$$y_0\left(x,t\right)=\left(DRGT\right)_2^{-1}\left[\frac{-u}{p\ s^2}\right]=-x,$$

$$y_{n+1}\left(x,t\right)=-\left(DRGT\right)_2^{-1}\left[\frac{u^2}{4s^2}\left(DRGT\right)_2\left[A_n\right]\right].$$

Now, using the regular Adomian polynomials formula we have:

$$A_0=y_0y_{0x},$$

$$A_1=y_1y_{0x}+y_0y_{1x},$$

$$A_2=y_2y_{0x}+y_1y_{1x}+y_0y_{2x}.$$

Then,

$$y_1\left(x,t\right)=\ -\mathrm{t\ x},$$

$$y_2\left(x,t\right)=-\mathrm{t}^2\ \mathrm{x},$$

$$y_3\left(x,t\right)=-\mathrm{t}^3\ \mathrm{x},$$

$$y\left(x,t\right)\left(approximate\right)=y_0+y_1+y_2+y_3=\ -\mathrm{x}-\mathrm{t\ x}-\mathrm{t}^2\ \mathrm{x}-\mathrm{t}^3\ \mathrm{x}.$$

Using the accelerated Adomian polynomials formula we have

$$\overline{A}_0=y_0y_{0x},$$

$$\overline{A}_1=y_0y_{1x}+y_1y_{0x}+y_1y_{1x},$$

$$\overline{A}_2=y_0y_{2x}+y_1y_{2x}+y_2y_{0x}+y_2y_{1x}+y_2y_{2x}.$$

Then

$$y_1\left(x,t\right)=-\mathrm{tx},$$

$$y_2\left(x,t\right)=-\left(\frac{1}{3}\right)\mathrm{t}^2\left(3+\mathrm{t}\right)\mathrm{x},$$

$$y_3\left(x,t\right)=\left(\frac{1}{63}\right)\mathrm{t}^3\left(42+42\mathrm{t}+21\mathrm{t}^2+7\mathrm{t}^3+\mathrm{t}^4\right)\mathrm{x},$$

$$y\,(x,t)\,(approximate) = y_0 + y_1 + y_2 + y_3$$
$$= -\frac{1}{63}(63 + 63t + 63t^2 + 63t^3 + 42t^4 + 21t^5 + 7t^6 + t^7)x.$$

TABLE 2. Comparison between the approximate and absolute error for using $(DRGT)_2$ combined with Adomian decomposition method against $(DRGT)_2$ with accelerated Adomian (using three iterations at $t = 0.1$).

		with regular $(\mathbf{DRGT})_2$ Adomian using four iterations		**with accelerated $(\mathbf{DRGT})_2$ Adomian using four iterations**	
		$t = 0.1$		$t = 0.1$	
x	Exact	Approximate	A. Error	Approximate	A. Error
0	0	0	0	0	0
0.1	-0.111111	-0.1111	1.111×10^{-5}	-0.111107	4.1×10^{-6}
0.2	-0.222222	-0.2222	2.222×10^{-5}	-0.222214	8.2×10^{-6}
0.3	-0.333333	-0.3333	3.333×10^{-5}	-0.333321	1.23×10^{-5}
0.4	-0.444444	-0.4444	4.444×10^{-5}	-0.444428	1.64×10^{-5}
0.5	-0.555556	-0.5555	5.556×10^{-5}	-0.555535	2.05×10^{-5}
0.6	-0.666667	-0.6666	6.667×10^{-5}	-0.666642	2.46×10^{-5}
0.7	-0.777778	-0.7777	7.778×10^{-5}	-0.777749	2.87×10^{-5}
0.8	-0.888889	-0.8888	8.889×10^{-5}	-0.888856	3.28×10^{-5}
0.9	-1.00000	-0.9999	1.000×10^{-4}	-0.999963	3.69×10^{-5}
1.0	-1.11111	-1.111	1.111×10^{-4}	-1.11107	4.1×10^{-5}

According to Table 2, the $(DRGT)_2$ method with Accelerated Adomain achieves greater accuracy than the $(DRGT)_2$ method with regular Adomain at the same time value of $t = 0.1$. Despite both methods employing the same number of iterations, smaller time values consistently yield better accuracy for both $(DRGT)_2$ with Accelerated Adomain and $(DRGT)_2$ with regular Adomian decomposition.

Example 5.2. Consider the following nonlinear partial differential Eq. [6]

$$y_t + yy_x - y_{xx} = 0, \tag{5.2}$$

with initial condition: $y_x\,(x,0) = x$, and exact solution: $y\,(x,t) = \frac{x}{1+t}$.

Using the double Ramadan group transform $(DRGT)_2$ on both sides on (5.2), we obtain:

$$(DRGT)_2\,[y_t] + (DRGT)_2\,[yy_x] - (DRGT)_2\,[y_{xx}] = 0,$$
$$\frac{p}{v}K\,(s,p,u,v) - \frac{1}{v}K\,(s,0,u,0) = (DRGT)_2\,[y_{xx}] - (DRGT)_2\,[yy_x]\,,$$

where,

$$K\,(s,0,u,0) = (RG)\,[y\,(x,0)\,;(s,0,u,0)] = \frac{u}{s^2}.$$

Then,

$$\frac{p}{v}K\,(s,p,u,v) = \frac{u}{v\ s^2} + (DRGT)_2\,[y_{xx}]\ - (DRGT)_2\,[yy_x]\,,$$
$$K\,(s,p,u,v) = \frac{u}{p\ s^2} + \frac{v}{p}\ (DRGT)_{\,2}\,[y_{xx}]\ - \frac{v}{p}\ (DRGT)_{\,2}\,[yy_x]\,.$$

Using the inverse of the double Ramadan group transform on both sides, we obtain:

$$y\,(x,t) = (DRGT)_2^{-1}\left[\frac{u}{p\ s^2}\right] + (DRGT)_2^{-1}\left[\frac{v}{p}\ (DRGT)_{\,2}\,[y_{xx}]\right] - (DRGT)_2^{-1}[\ \frac{v}{p}\ (DRGT)_{\,2}\,[yy_x]].$$

This technique presents solutions as an infinite series given by

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t).$$

The term y_n should be computed recursively, and the nonlinear term yy_x is decomposed as follows:

$$yy_x = \sum_{n=0}^{\infty} A_n,$$

$$\sum_{n=0}^{\infty} y_n(x,t) = (DRGT)_2^{-1}\left[\frac{u}{p\ s^2}\right] + (DRGT)_2^{-1}\left[\frac{v}{p}\ (DRGT)_2\left[\sum_{n=0}^{\infty}(y_n\ )_{xx}\right]\right]$$
$$-(DRGT)_2^{-1}[\ \frac{v}{p}\ (DRGT)_2\left[\sum_{n=0}^{\infty} A_n\right]].$$

By comparing both sides, we get:

$$y_0(x,t) = (DRGT)_2^{-1}\left[\frac{u}{p\ s^2}\right] = x,$$

$$y_{n+1}(x,t) = (DRGT)_2^{-1}\left[\frac{v}{p}\ (DRGT)_2\left[(y_n\ )_{xx}\right]\right] - (DRGT)_2^{-1}[\frac{v}{p}\ (DRGT)_2\ [A_n]].$$

Using the regular Adomian polynomials formula

$$A_0 = y_0 y_{0x},$$
$$A_1 = y_1 y_{0x} + y_0 y_{1x},$$
$$A_2 = y_0 y_{2x} + y_1 y_{1x} + y_2 y_{0x},$$
$$A_3 = y_0 y_{3x} + y_1 y_{2x} + y_2 y_{1x} + y_3 y_{0x}.$$

Then,

$$y_1(x,t) = -tx,$$
$$y_2(x,t) = t^2 x,$$
$$y_3(x,t) = -t^3 x,$$
$$y_4(x,t) = t^4 x.$$
$$y(x,t)\,(approximate) = y_0 + y_1 + y_2 + y_3 + y_4 = \ x\ - tx\ +\ t^2x\ -\ t^3x\ \ +\ t^4x.$$

Using the accelerated Adomian polynomials formula

$$\overline{A}_0 = y_0 y_{0x},$$
$$\overline{A}_1 = (y_0 + y_1)(y_{0x} + y_{1x}) - \overline{A}_0,$$
$$\overline{A}_2 = (y_0 + y_1 + y_2)(y_{0x} + y_{1x} + y_{2x}) - \overline{A}_0 - \overline{A}_1,$$
$$\overline{A}_3 = (y_0 + y_1 + y_2 + y_3)(y_{0x} + y_{1x} + y_{2x} + y_{3x}) - \overline{A}_0 - \overline{A}_1 - \overline{A}_2.$$

Then,

$$y_1(x,t) = -tx,$$
$$y_2(x,t) = -\frac{1}{3}(-3+t)t^2 x,$$
$$y_3(x,t) = -\frac{1}{63}t^3(42 - 42t + 21t^2 - 7t^3 + t^4)x,$$
$$y_4(x,t) = \text{in a similar manner},$$

$$
\begin{aligned}
y\left(x,t\right)\left(approximate\right) &= y_0+y_1+y_2+y_3+y_4\\
&= \left(1-t+t^2-t^3+t^4-\frac{13t^5}{15}+\frac{2t^6}{3}-\frac{29t^7}{63}+\frac{71t^8}{252}-\frac{86t^9}{567}+\frac{22t^{10}}{315}\right.\\
&\left.-\frac{5t^{11}}{189}+\frac{t^{12}}{126}-\frac{t^{13}}{567}+\frac{t^{14}}{3969}-\frac{t^{15}}{59535}\right)x.
\end{aligned}
$$

Table 3. Comparison between the approximate and absolute error for using $(DRGT)_2$ combined with Adomian decomposition method against $(DRGT)_2$with accelerated Adomian (using four iterations at $t = 0.1$).

		with regular $(\mathbf{DRGT})_2$ Adomian using four iterations		**with accelerated $(\mathbf{DRGT})_2$ Adomian using four iterations**	
		$t = 0.1$		$t = 0.1$	
x	Exact	Approximate	A. Error	Approximate	A. Error
0	0	0	0	0	0
0.1	0.0909091	0.09091	9.091×10^{-7}	0.0909092	1.048×10^{-7}
0.2	0.181818	0.18182	1.818×10^{-6}	0.181818	2.095×10^{-7}
0.3	0.272727	0.27273	2.727×10^{-6}	0.272728	3.143×10^{-7}
0.4	0.363636	0.36364	3.636×10^{-6}	0.363637	4.19×10^{-7}
0.5	0.454545	0.45455	4.545×10^{-6}	0.454546	5.238×10^{-7}
0.6	0.545455	0.54546	5.455×10^{-6}	0.545455	6.285×10^{-7}
0.7	0.636364	0.63637	6.364×10^{-6}	0.636364	7.333×10^{-7}
0.8	0.727273	0.72728	7.273×10^{-6}	0.727274	8.38×10^{-7}
0.9	0.818182	0.81819	8.182×10^{-6}	0.818183	9.428×10^{-7}
1.0	0.909091	0.9091	9.091×10^{-6}	0.909092	1.048×10^{-6}

From Table 3, $(DRGT)_2$ coupled with accelerated Adomain gives better accuracy compared with $(DRGT)_2$ combined with the regular Adomian decomposition method. Using $(DRGT)_2$ with accelerated Adomian, it is also obvious that the speed at which we reach the exact solution increases with the number of terms included in the approximate solution.

Example 5.3. Examine the given nonlinear partial differential Eq. [16]:

$$y_{tt}-\frac{2x^2}{t}yy_x=0, \tag{5.3}$$

along with initial conditions:

$$y\left(x,0\right)=0,\quad y_t\left(x,0\right)=x,$$

and exact solution:

$$y\left(x,t\right)=\tan(xt)\ .$$

Using the double Ramadan group transform $(DRGT)_2$ on both sides on (5.3), we obtain:

$$(DRGT)_2[y_{tt}]-(DRGT)_2[\ \frac{2x^2}{t}yy_x]=0,$$

$$(DRGT)_2\left[y_{tt}\right]=(DRGT)_2[\ \frac{2x^2}{t}yy_x],$$

$$\frac{p^2}{v^2}\ K\left(s,p,u,v\right)-\frac{p}{v^2}\ K\left(s,0,u,0\right)-\frac{1}{v}\frac{\partial K\left(s,0,u,0\right)}{\partial t}=(DRGT)_2[\ \frac{2x^2}{t}yy_x],$$

$$K(s,p,u,v) = \frac{1}{p}K(s,0,u,0) + \frac{v}{p^2}\frac{\partial K(s,0,u,0)}{\partial t} + \frac{v^2}{p^2}(DRGT)_2[\ \frac{2x^2}{t}yy_x],$$

where,

$$K(s,0,u,0) = (RG)\left[y(x,0);(s,0,u,0)\right] = 0,$$

$$\frac{\partial K(s,0,u,0)}{\partial t} = (RG)\left[y_t(x,0);(s,0,u,0)\right] = \frac{u}{s^2},$$

$$K(s,p,u,v) = \frac{uv}{p^2v^2} + \frac{v^2}{p^2}(DRGT)_2\left[\frac{2x^2}{t}yy_x\right],$$

using the inverse of the double Ramadan group transform on both sides, we obtain:

$$y(x,t) = (DRGT)_2^{-1}\left[\frac{uv}{p^2v^2}\right] + (DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2\left[\frac{2x^2}{t}yy_x\right]\right].$$

This technique presents solutions as an infinite series given by

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t).$$

The term y_n should be computed recursively, and the nonlinear term yy_x is decomposed as follows:

$$yy_x = \sum_{n=0}^{\infty} A_n.$$

$$\sum_{n=0}^{\infty} y_n(x,t) = tx + (DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2\left[\frac{2x^2}{t}\sum_{n=0}^{\infty} A_n\right]\right].$$

By comparing both sides, we get:

$$y_0(x,t) = (DRGT)_2^{-1}\left[\frac{uv}{p^2v^2}\right] = tx,$$

$$y_{n+1}(x,t) = (DRGT)_2^{-1}\left[\frac{v^2}{p^2}(DRGT)_2\left[\frac{2x^2}{t}A_n\right]\right], \quad n\geq 0.$$

Using the Accelerated Adomian polynomials formula

$$\overline{A}_0 = y_0 y_{0x},$$

$$\overline{A}_1 = y_0 y_{1x} + y_1 y_{0x} + y_1 y_{1x},$$

$$\overline{A}_2 = y_0 y_{2x} + y_1 y_{2x} + y_2 y_{0x} + y_2 y_{1x} + y_2 y_{2x}.$$

Then,

$$y_1(x,t) = \frac{t^3x^3}{3},$$

$$y_2(x,t) = \frac{1}{315}t^5x^5\left(42 + 5t^2x^2\right),$$

$$y_3(x,t) = \frac{t^7x^7(1621620 + 570570t^2x^2 + 109746t^4x^4 + 13860t^6x^6 + 715t^8x^8)}{42567525},$$

$$
\begin{aligned}
y\,(x,t)\,(approximate) &= y_0 + y_1 + y_2 + y_3 \\
&= tx + \frac{t^3x^3}{3} + \frac{42t^5ux^5 + 5t^7ux^7}{315u} \\
&+ \frac{1621620t^7ux^7 + 570570t^9ux^9 + 109746t^{11}ux^{11} + 13860t^{13}ux^{13} + 715t^{15}ux^{15}}{42567525u}\ , \\
&= tx + \frac{t^3x^3}{3} + \frac{42t^5x^5 + 5t^7x^7}{315} \\
&+ \frac{1621620t^7x^7 + 570570t^9x^9 + 109746t^{11}x^{11} + 13860t^{13}x^{13} + 715t^{15}x^{15}}{42567525}.
\end{aligned}
$$

$$
\begin{aligned}
y\,(x,t)\,(approximate) &= y_0 + y_1 + y_2 + y_3 \\
&= tx + \frac{t^3x^3}{3} + \frac{2t^5x^5}{15} + \frac{17t^7x^7}{315} + \frac{38t^9x^9}{2835} + \frac{134t^{11}x^{11}}{51975} + \frac{4t^{13}x^{13}}{12285} + \frac{t^{15}x^{15}}{59535}\ .
\end{aligned}
$$

And, the first four terms of the double Elzaki Transform decomposition [25], is provided by:

$$u\,(x,t) = tx + \frac{t^3x^3}{3} + \frac{2t^5x^5}{15} + \frac{17t^7x^7}{315}.$$

In closed form, the solution is as follows:

$$y\,(x,t) = \tan(xt).$$

Clearly, my suggested approach, the double Ramadan Group Transform$(\text{DRGT})_2$, achieves faster acceleration using the same number of terms (four terms) compared to the Double Elzaki Transform.

TABLE 4. Comparison between the approximate and absolute error for using $(DRGT)_2$ combined with Adomian decomposition method against Double Elzaki Transform with regular Adomian using four iterations (using three iterations at $t = 0.1$).

		Double Elzaki Transform with regular Adomian using four iterations		**$(\text{DRGT})_2$ with accelerated Adomian using four iterations**	
		$t = 0.1$		$t = 0.1$	
x	Exact	Approximate	A. Error	Approximate	A. Error
0	0	0	0	0	0
0.1	0.010000333346667209	0.010000333346667207	1.735×10^{-18}	0.010000333346667207	1.735×10^{-18}
0.2	0.020002667093402426	0.02000266709340242	6.939×10^{-18}	0.020002667093402426	0.0
0.3	0.03000900324118072	0.03000900324118029	4.302×10^{-16}	0.030009003241180554	1.665×10^{-16}
0.4	0.04002134699551457	0.040021346995508834	5.738×10^{-15}	0.040021346995512345	2.227×10^{-15}
0.5	0.05004170837553879	0.050041708375496034	4.276×10^{-14}	0.05004170837552223	1.656×10^{-14}
0.6	0.0600721038312973	0.060072103831076584	2.207×10^{-13}	0.06007210383121176	8.554×10^{-14}
0.7	0.07011455787200271	0.07011455787111845	8.843×10^{-13}	0.07011455787165985	3.429×10^{-13}
0.8	0.08017110470807257	0.08017110470512967	2.943×10^{-12}	0.08017110470693092	1.142×10^{-12}
0.9	0.09024378990978546	0.09024378990128486	8.501×10^{-12}	0.09024378990648589	3.3×10^{-12}
1.0	0.10033467208545055	0.10033467206349207	2.196×10^{-11}	0.10033467207692177	8.529×10^{-12}

According to Table 4, the $(DRGT)_2$ method with Accelerated Adomain achieves greater accuracy than the Double Elzaki Transform method using Adomian decomposition at the same time value of$t = 0.1$. Despite both methods employing the same number of iterations, smaller time values consistently yield better accuracy for both $(DRGT)_2$ with Accelerated Adomain and Double Elzaki Transform with Adomian decomposition.

6. Conclusion

Combining the accelerated Adomian decomposition method with the double Ramadan group transform method $(DRGT)_2$ results in a highly effective solution for solving nonlinear partial differential equations. As shown in the tabulated results, the variant of the accelerated Adomian method provides greater accuracy compared to the standard Adomian method when combined with the double Ramadan transform. This method can also be used for higher-order nonlinear partial differential equations. This study combines the Double Ramadan Group Transform with the accelerated Adomian Decomposition Method to solve nonlinear PDEs, improving accuracy and computational efficiency. The method addresses challenges like slow convergence and complex nonlinear terms, offering a new approach for solving complex PDEs with broad applications in science and engineering.

Statements and Declarations

Ethics-approved: None of the authors of this article have ever conducted investigations using humans or animals.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

No data is associated with this research.

Author Contributions Statement

M.A. Ramadan conducted the mathematical analysis, developed the methodology, verified the results, wrote the initial draft, and reviewed the final version.
M. A. Mansour contributed to the original manuscript, software development, and methodology.
N. El-Shazly and **H. S. Osheba** reviewed and edited the final version of the manuscript.

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