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A novel high-order approximation method for higher-dimensional time-fractional reaction-diffusion problems with weak initial singularity

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Abstract

The objective of this manuscript is to construct and analyze a fully discrete method to approximate one and two dimensional time-fractional reaction-diffusion equations defined in Caputo sense. The current approach combines Alikhanov's L2-1_θ formula on a non-uniform graded mesh to discretize the time-fractional Caputo derivative and the discretization of the space variables using a cubic spline difference scheme. The two-dimensional problem is then separated into two one-dimensional problems using the alternating direction implicit (ADI) approach. The theoretical analysis which consists of both stability and convergence has been provided for both one and two-dimensional problems. Further, in order to illustrate the accuracy and efficiency of the proposed method, numerical results for two test examples have been presented.

Keywords. Cubic spline difference scheme, Caputo derivative, L2-1_θ formula, Graded mesh, ADI scheme, Convergence analysis.

1991 Mathematics Subject Classification. 65M06, 65M12, 65M70, 35R11.

1. Introduction

The use of fractional-order derivatives in physical and chemical equations has become increasingly popular in recent years. Fractional differential equations are attracting a lot of interest because the special features of fractional derivatives enhance the accuracy of models by incorporating memory and hereditary properties. Applications of these equations extend across a wide range of fields, encompassing finance, control theory, biological systems, material science, viscoelasticity, nuclear reactor dynamics, acoustics, electrical networks, physics, electromagnetics, fluid mechanics, and signal processing [4, 16, 26, 38, 39]. In particular, the use of time-fractional reaction-diffusion equations, where the first-order derivative is replaced with a fractional-order derivative, has become an important tool for modeling various phenomena, such as transport in porous media, anomalous diffusion, and non-Fickian behavior in chemical reactions [1, 23, 45]. These equations have been instrumental in the extensive research conducted on the reaction and diffusion processes of components in porous catalysts, as indicated in references [8, 11]. By employing such problems, we can study the pollution caused by industrial waste material entering the atmosphere [15]. Additionally, reaction-diffusion equations prove versatile in modeling real-world issues like chemical reactions [37], logistic population growth [2], branching Brownian motion processes, and nuclear reactor theory.

The numerical solution of time-fractional reaction-diffusion equations is often necessary due to the difficulty in finding analytic solutions. However, numerical methods for solving these equations can be computationally expensive, particularly in higher dimensions, as the solution at each time level depends on the previous time levels. Therefore, the development of stable and efficient numerical schemes for time-fractional reaction-diffusion equations, particularly in two or higher dimensions, is an important and active area of research.

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Fractional models with variable coefficients are more flexible than fractional models with constant coefficients in simulating some real-life phenomena. Therefore, In this paper, we consider the following two-dimensional time-fractional variable coefficient reaction-diffusion equation (TFRDE) [35, 46]:

$$\partial_t^\alpha w(\varkappa,y,t) = p_1(t)\frac{\partial^2 w(\varkappa,y,t)}{\partial \varkappa^2} + p_2(t)\frac{\partial^2 w(\varkappa,y,t)}{\partial y^2} - q(t)w(\varkappa,y,t) + F(\varkappa,y,t),$$
$$(\varkappa,y,t) =: \Omega\times(0,T],\ \Omega\subset\mathbb{R}^2, \tag{1.1}$$

with initial and boundary conditions

$$\begin{cases} w(\varkappa,y,0)=\phi(\varkappa,y),\ (\varkappa,y)\in\bar{\Omega}=\Omega\cup\partial\Omega, \\ w(\varkappa,y,t)=0,\ (\varkappa,y)\in\partial\Omega,\ 0<t\le T, \end{cases} \tag{1.2}$$

where $\partial\Omega$ denotes the boundary of $\Omega=(0,\text{Ł})\times(0,\text{Ł})$ and

$$\partial_t^\alpha w(\varkappa,y,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial w(\varkappa,y,\zeta)}{\partial\zeta}(t-\zeta)^{-\alpha}\,d\zeta\ , \tag{1.3}$$

defines the Caputo derivative of fractional order $\alpha\in(0,1)$. Also, $p_1,p_2>0$, $q\ge 0$, F and ϕ are sufficiently smooth functions. Here, we employ the Caputo fractional derivative in place of others, as it naturally incorporates classical initial conditions, making it more suitable for physical and engineering applications. Unlike the Riemann-Liouville (RL) derivative, which requires fractional-order initial conditions with limited physical interpretation. Other derivatives, such as Grünwald-Letnikov (GL), provide alternative formulations. While GL is useful for numerical computations, it lacks smoothness for analytical approaches.

A typical solution to Eqs. (1.1)-(1.2) is commonly known to display a singularity near the initial time $t=0$. Moreover, its derivatives adhere to specified regularity conditions, as outlined in [43]

$$\left|\frac{\partial^i w}{\partial t^i}\right| \le \bar{c}\left(1+t^{\alpha-i}\right),\ for\ i=0,1,2,3, \tag{1.4}$$

where $\bar{c}$ is a positive constant. The expression in Eq. (1.4) suggests that $w(x,y,t)$ demonstrates a weak singularity at $t=0$, resulting in the unbounded behavior of the time derivative $\left|\frac{\partial w}{\partial t}\right|$ as $t\to 0^+$. The existence of a weak initial singularity poses significant challenges, both practically and theoretically, for conventional numerical techniques. This is due to their inherent limitations in accurately capturing the solution's behavior in the proximity of singular points. Consequently, the development of effective numerical methods that can adeptly handle the singularity at $t=0$ becomes an intriguing and demanding task.

Several numerical approaches have been developed to handle the challenges associated with solving one and two-dimensional time-fractional diffusion equations [3, 5, 7, 9, 12, 14, 20, 22, 24, 25, 30, 40, 42, 44, 47, 48]. Despite its popularity, solutions to time-fractional differential equations often exhibit a lack of smoothness near the initial time $t=0$. This lack of smoothness poses challenges when employing a temporal standard uniform mesh, leading to a loss of full accuracy. Achieving high accuracy typically requires a high regularity of the solution. In addressing this issue, non-uniform meshes, such as those presented in [13, 18, 20, 21], have proven to be effective. These non-uniform meshes concentrate more mesh points as time approaches zero. Consequently, these approaches have garnered significant interest in the numerical analysis of time-fractional differential equations in recent years. For linear subdiffusion equations involving Caputo derivatives of order $\alpha\in(0,1)$, a non-uniform mesh technique (also known as non-uniform L1 formula) was developed by Stynes et al. [43] and Liao et al. [20]. They successfully established convergence of $\mathcal{O}(K^{-\{2-\alpha\}})$, with an optimal grading parameter $r=\frac{2-\alpha}{\alpha}$ based on reasonable regularities. Further improvements were made in [13], where a fitted scheme with the same convergence order was constructed to enhance the grading parameter to $r=\max\{1,\frac{2-\alpha}{2\alpha}\}$. Alikhanov's L2-$1_\theta$ formula [3], combined with a non-uniform mesh, has also been explored in more recent works [6].

Previously various spline collocation methods have been used to approximate partial differential equations [10, 17, 34, 36, 38, 39]. In recent decades, the development of cubic spline difference schemes has greatly increased. Papamichael et al. [31] solved a one-dimensional heat conduction equation using cubic spline technique having lower order accuracy. Later, Raggett et al. [32] performed the same for a one-dimensional wave equation. Mohanty and Jain

[28] developed a solution with higher accuracy for one-dimensional quasi-linear parabolic equations. Various partial differential equations of integer order and integral equations are solved using cubic splines in [27–29, 33]. However, it has not yet been used much for fractional partial differential equations. Recently, Singh et al. [41] considered a cubic spline difference scheme in space and the classical L1 scheme in time for approximating a one-dimensional fractional reaction-diffusion equation. Note that the work in [41] is restricted to a one-dimensional problem and the lower order L1 scheme is used to discretize the time-fractional derivative.

The motivation for this work is to construct and analyze a novel fully discrete method to approximate one and two-dimensional time-fractional reaction-diffusion equations using Alikhanov's L2-1_θ formula on non-uniform graded mesh for temporal discretization and cubic spline difference scheme for spatial discretization. We have conducted a comprehensive theoretical analysis encompassing both stability and convergence aspects for problems in both one and two dimensions.

Further, to show the method's accuracy and efficiency, numerical results are provided which agree with the theoretical results.

The paper is arranged as follows: In section 2, we first present some auxiliary lemmas and then develop numerical methods for solving one and two-dimensional time-fractional reaction-diffusion equations. In section 3, we give a comprehensive theoretical analysis encompassing both stability and convergence of the developed numerical methods. In section 4, numerical illustrations are given to demonstrate the accuracy and effectiveness of the proposed methods. In section 5, some conclusions are given about the paper.

2. Fully discrete numerical schemes

This section provides the fully discrete numerical schemes for problem (1.1)-(1.2) and its one-dimensional analogue.

2.1. **Temporal discretization.** Denoting a positive integer as K and a grading parameter as r (where r is greater than or equal to 1), we define a graded mesh for $j = 0, 1, 2, \cdots, K$ as $t_j = T(j/K)^r$. The corresponding time step is given by $\tau_j = t_j - t_{j-1}$ for $j = 1, 2, \cdots, K$. Additionally, for $j = 0, 1, 2, \cdots, K-1$, and for a parameter θ where $\theta = 1 - \alpha/2$, we define time point $t_{j+\sigma} = t_j + \theta\tau_{j+1}$ for $j = 0, 1, 2, \cdots, K-1$.

Defining the maximum time-step as $\tau = \max\limits_{1\le j\le K} \tau_j$, we introduce the time step ratio $\mathcal{B}_j = \frac{\tau_j}{\tau_{j+1}}$ for $j = 1, 2, \cdots, K-1$, and designate the maximum time step ratio as $\mathcal{B} = \max\limits_{1\le j\le K-1} \mathcal{B}_j$. The L2-$1_\theta$ formula on non-uniform graded mesh approximation of Caputo time-fractional derivative of a function $v(t) \in \mathcal{C}[0,T] \cap \mathcal{C}^3(0,T]$, at point $t_{j+\theta}$, $j = 0, 1, \cdots, K-1$ is given following [19]

$$\partial_t^\alpha v(t_{j+\theta}) = \left[r^\alpha_{j,j} v(t_{j+1}) - \sum_{l=1}^{j} (r^\alpha_{j,l} - r^\alpha_{j,l-1}) v(t_l) - r^\alpha_{j,0} v(t_0) \right] + \mathcal{R}_t^{j+\theta}, \tag{2.1}$$

where the coefficients are defined as $r^\alpha_{0,0} = \tau_1^{-1} b^\alpha_{0,0}$ having $b^\alpha_{j,j} = \frac{\theta^{1-\alpha}}{\Gamma(2-\alpha)} \tau_{j+1}^{1-\alpha}$ for $j \ge 0$,

$$\text{for } j \ge 1, \; r^\alpha_{j,i} = \begin{cases} \tau_{i+1}^{-1} \left(b^\alpha_{j,0} + a^\alpha_{j,0} \right), & i = 0, \\ \tau_{i+1} \left(b^\alpha_{j,i} + a^\alpha_{j,i-1} - a^\alpha_{j,i} \right), & 1 \le i \le j-1, \\ \tau_{i+1} \left(b^\alpha_{j,j} - a^\alpha_{j,j-1} \right), & i = j, \end{cases}$$

with

$$a^\alpha_{j,i} = \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_{i+1}} (t_{j+\sigma} - \zeta)^{-\alpha} d\zeta, \; 0 \le i \le j-1,$$

$$b^\alpha_{j,i} = \frac{1}{\Gamma(1-\alpha)} \frac{2}{(\tau_{i+2} - \tau_i)} \int_{t_i}^{t_{i+1}} (t_{j+\sigma} - \zeta)^{-\alpha} (\zeta - t_{j+1/2}) d\zeta, \; 0 \le i \le j-1,$$

and $\mathcal{R}_t^{j+\theta}$ is the local truncation error term, which is bounded by the following lemma.

Lemma 2.1. *[6] If $v \in C[0,T] \cap C^3(0,T]$ and satisfies the conditions specified in (1.4). The local truncation error $\mathcal{R}_t^{j+\theta}$, of the approximation (2.1) is bounded as follows*

$$\left|\mathcal{R}_t^{j+\theta}\right| \leq t_{j+\theta}^{-\alpha} K^{-\min\{3-\alpha, r\alpha\}}, \ \text{for } 0 \leq j \leq K-1, \tag{2.2}$$

where $0 \leq \theta \leq 1$.

Lemma 2.2. *[6] Assuming $1 - \frac{\alpha}{2} \leq \theta \leq 1$ and that the local mesh ratio $\mathcal{B}_j = \frac{\tau_{j+1}}{\tau_j}$ for $1 \leq j \leq K-1$ satisfies $\frac{3}{4} \leq \mathcal{B}_j \leq 62$, then*

$$(1).\ r_{j,0}^{\alpha} > \frac{t_{j+\theta}^{-\alpha}}{\Gamma(1-\alpha)} > 0, \ j \geq 0.$$

$$(2).\ (2\theta - 1)\, r_{1,1}^{\alpha} - \theta r_{1,0}^{\alpha} > 0.$$

$$(3).\ r_{j,1}^{\alpha} > r_{j,0}^{\alpha}, \ j \geq 1.$$

$$(4).\ \text{If } \mathcal{B}_{i-1}^2 \left(\mathcal{B}_{i-1} + 1\right) \geq \frac{\mathcal{B}_i}{\mathcal{B}_i + 1} \text{ for } 2 \leq i \leq j \text{ with } j \geq 2 \text{ then } r_{j,i-1}^{\alpha} < r_{j,i}^{\alpha}.$$

$$(5).\ \text{If } \mathcal{B}_{j-1}^2 \left(2 - \frac{1}{\theta} + \mathcal{B}_j\left(\mathcal{B}_j + 2\right)\right) \geq \frac{\mathcal{B}_j\left(\mathcal{B}_j + 1\right)}{\left(\mathcal{B}_{j-1} + 1\right)} \text{ for } 2 \leq j \leq K, \text{ then } (2\theta - 1) r_{j,j}^{\alpha} - \theta r_{j,j-1}^{\alpha} > 0.$$

At point $t = t_{j+\theta}$, problem (1.1) becomes

$$\partial_{t_{j+\theta}}^{\alpha} w^{j+\theta}(\varkappa, y) = p_1^{j+\theta} \frac{\partial^2 w^{j+\theta}(\varkappa, y)}{\partial \varkappa^2} + p_2^{j+\theta} \frac{\partial^2 w^{j+\theta}(\varkappa, y)}{\partial y^2} - q^{j+\theta} w^{j+\theta}(\varkappa, y) + F^{j+\theta}(\varkappa, y), \tag{2.3}$$
$$(x, y) \in \Omega, \ j = 0, 1, \ldots, K-1,$$

where we denote $w^{j+\theta}(\varkappa, y) = w(\varkappa, y, t_{j+\theta})$, $p_1^{j+\theta} = p_1(t_{j+\theta})$, $p_2^{j+\theta} = p_2(t_{j+\theta})$, and $F^{j+\theta}(\varkappa, y) = F(\varkappa, y, t_{j+\theta})$.

Now using L2-1_θ approximation of Eq. (2.1) in Eq. (2.3) we get the following semi-discrete scheme

$$r_{j,j}^{\alpha} w^{j+1}(\varkappa, y) - \sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) w^l(\varkappa, y) - r_{j,0}^{\alpha} w^0(\varkappa, y) = p_1^{j+\theta} \frac{\partial^2 w^{j+\theta}(\varkappa, y)}{\partial \varkappa^2} + p_2^{j+\theta}$$
$$\times \frac{\partial^2 w^{j+\theta}(\varkappa, y)}{\partial y^2} - q^{j+\theta} w^{j+\theta}(\varkappa, y) + F^{j+\theta}(\varkappa, y) + \mathcal{R}_t^{j+\theta}, \quad (x, y) \in \Omega, \ 0 \leq j \leq K-1. \tag{2.4}$$

Further,

$$r_{j,j}^{\alpha} w^{j+1}(\varkappa, y) - \sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) w^l(\varkappa, y) - r_{j,0}^{\alpha} w^0(\varkappa, y) = p_1^{j+\theta} \frac{\partial^2 w^{j,\theta}(\varkappa, y)}{\partial \varkappa^2} + p_2^{j+\theta} \tag{2.5}$$
$$\times \frac{\partial^2 w^{j,\theta}(\varkappa, y)}{\partial y^2} - q^{j+\theta} w^{j,\theta}(\varkappa, y) + F^{j+\theta}(\varkappa, y) + \mathcal{R}_t^{j+\theta} + \mathcal{R}_\theta^{j,\theta}, \quad (x, y) \in \Omega, \ 0 \leq j \leq K-1,$$

where $w^{j,\theta} = \theta w^{j+1} + (1-\theta) w^j$ and a bound for $\mathcal{R}_\theta^{j,\theta}$ can be obtained by using the following Lemma 2.3.

Lemma 2.3. *For $v(t) \in \mathcal{C}^2[0,T]$, subsequent condition follows*

$$\left|\sigma v(t_{j+1}) + (1-\sigma) v(t_j) - v(t_{j+\theta})\right| \leq \frac{1}{8} \tau_{j+1}^2 \max_{1 \leq j \leq K} \left|v''(t_j)\right|.$$

2.2. **Spatial discretization.** In this subsection we give spatial discretization of both one and two-dimensional problems using cubic spline finite difference scheme.

2.2.1. **One-dimensional problem.** The one-dimensional analogue of problem (1.1)-(1.2) can be written as

$$\partial_t^{\alpha} w(\varkappa,t) = p(t)\frac{\partial^2 w(\varkappa,t)}{\partial \varkappa^2} - q(t)w(\varkappa,t) + F(\varkappa,t), \ (\varkappa,t)\in(0,\text{Ł})\times(0,T], \tag{2.6}$$

with initial and boundary conditions

$$w(\varkappa,0) = \phi(\varkappa), \ \varkappa\in[0,\text{Ł}], \tag{2.7}$$

$$w(0,t) = w(\text{Ł},t) = 0, \ t\in(0,T]. \tag{2.8}$$

Similarly, one-dimensional time-fractional analogue of Eq. (2.5) is given as

$$r_{j,j}^{\alpha} w^{j+1}(\varkappa) - \sum_{l=1}^{j}(r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha})w^{l}(\varkappa) - r_{j,0}^{\alpha} w^{0}(\varkappa) = p^{j+\theta}\frac{\partial^2 w^{j,\theta}(\varkappa)}{\partial\varkappa^2} - q^{j+\theta}w^{j,\theta}(\varkappa) + F^{j+\theta}(\varkappa) + \mathcal{R}_t^{j+\theta} + \mathcal{R}_{\theta}^{j,\theta},$$

$$\varkappa\in(0,\text{Ł}), \quad 0\le j\le K-1, \tag{2.9}$$

$$w^0(\varkappa) = \phi(\varkappa), \quad \varkappa\in[0,\text{Ł}], \tag{2.10}$$

$$w^j(0) = w^j(\text{Ł}) = 0, \ 1\le j\le K. \tag{2.11}$$

To discretize problem (2.9), take a uniform mesh of $(M+1)$ points for spatial domain $[0,L]$. Defining $\varkappa_m = mh$, $0\le m\le M$, where $h=\frac{\text{Ł}}{M}$. Let $w(\varkappa,t)$ denote the exact solution of problem (2.6). For $0\le m\le M$ and $0\le j\le K$, we shall use the notation $w(\varkappa_m,t_j) = w_m^j$. Defining $D_h = \{w|(w_0,w_1,\cdots,w_m), w_0 = w_M = 0\}$.

Suppose ${}_mS_{j,\theta}(\varkappa)$ is the cubic spline interpolate defined on $[\varkappa_m,\varkappa_{m+1}]$, $m=0,1,\ldots,M-1$ and at time $t_{j+\theta}$, for given approximation $\left(W_m^{j,\theta}\right)_{m=0}^{M}$ of the function $w^{j,\theta}(\varkappa)$ at the nodal points $\varkappa_0,\varkappa_1,\ldots,\varkappa_M$. We have

$$\begin{aligned}{}_mS_{j,\theta}(\varkappa) &= P_m^{j,\theta}\frac{(\varkappa_{m+1}-\varkappa)^3}{6h} + P_{m+1}^{j,\theta}\frac{(\varkappa-\varkappa_m)^3}{6h} + \left(W_m^{j,\theta} - \frac{h^2}{6}P_m^{j,\theta}\right)\frac{(\varkappa_{m+1}-\varkappa)}{h}\\ &\quad + \left(W_{m+1}^{j,\theta} - \frac{h^2}{6}P_{m+1}^{j,\theta}\right)\frac{(\varkappa-\varkappa_m)}{h}, \quad \forall\ \varkappa\in[\varkappa_m,\varkappa_{m+1}], \ \ 0\le m\le M-1,\end{aligned} \tag{2.12}$$

where $P_m^{j,\theta} = {}_mS''_{j,\theta}(\varkappa_m)$, $0\le m\le M-1$. Using Eq. (2.12), we can obtain the cubic spline identity relation given by

$$\frac{W_{m+1}^{j,\theta} - 2W_m^{j,\theta} + W_{m-1}^{j,\theta}}{h^2} = \frac{1}{6}P_{m-1}^{j,\theta} + \frac{4}{6}P_m^{j,\theta} + \frac{1}{6}P_{m+1}^{j,\theta}, \qquad 1\le m\le M-1, \tag{2.13}$$

which ensures the continuity of ${}_mS'_{j,\theta}(\varkappa)$ at the interior points.

We rewrite the semi-discrete problem (2.9) at $\varkappa=\varkappa_m$ as follows

$$\begin{aligned}\frac{\partial^2 w^{j,\theta}(\varkappa_m)}{\partial\varkappa^2} &= \frac{1}{p^{j+\theta}}\Bigg[r_{j,j}^{\alpha}w^{j+1}(\varkappa_m) - \sum_{l=1}^{j}(r_{j,l}^{\alpha}-r_{j,l-1}^{\alpha})w^l(\varkappa_m) - r_{j,0}^{\alpha}w^0(\varkappa_m)\\ &\quad + q^{j+\theta}w^{j,\theta}(\varkappa_m) - F^{j+\theta}(\varkappa_m) + \mathcal{R}_t^{j+\theta} + \mathcal{R}_{\theta}^{j,\theta}\Bigg].\end{aligned} \tag{2.14}$$

As the solution $w(\varkappa,t)$ of problem (2.6) at $t=t_{j+\theta}$ is approximated by the cubic spline ${}_mS_{j,\theta}(\varkappa)$, it follows that ${}_mS''_{j,\theta}(\varkappa_m) = P_m^{j,\theta}$ is an approximation to $\frac{\partial^2 w^{j,\theta}(\varkappa_m)}{\partial\varkappa^2}$. Hence, from Eq. (2.14), we set

$$P_m^{j,\theta} = \frac{1}{p^{j+\theta}}\left[r_{j,j}^{\alpha}W_m^{j+1} - \sum_{l=1}^{j}(r_{j,l}^{\alpha}-r_{j,l-1}^{\alpha})W_m^l - r_{j,0}^{\alpha}W_m^0 + q^{j+\theta}W_m^{j,\theta} - F_m^{j+\theta}\right]. \tag{2.15}$$

Thus, we have

$$P_m^{j,\theta} = \frac{\partial^2 w^{j,\theta}(\varkappa_m)}{\partial\varkappa^2} + \mathcal{R}_t^1, \tag{2.16}$$

where $\mathcal{R}_t^1 = \mathcal{R}_t^{j+\theta} + \mathcal{R}_{\theta}^{j,\theta}$.

Now substitution of Eq. (2.15) in Eq. (2.13) gives

$$r^{\alpha}_{j,j}HW^{j+1}_m - \sum_{l=1}^{j}(r^{\alpha}_{j,l} - r^{\alpha}_{j,l-1})HW^{l}_m - r^{\alpha}_{j,0}HW^{0}_m - p^{j+\theta}\delta^2_{\varkappa}W^{j,\theta}_m + q^{j+\theta}HW^{j,\theta}_m = HF^{j+\theta}_m,$$

$$1 \le m \le M-1,\ 0 \le j \le K-1, \tag{2.17}$$

$$W^0_m = \phi(\varkappa_m),\ 0 \le m \le M, \tag{2.18}$$

$$W^j_0 = 0,\ W^j_M = 0,\ 1 \le j \le K, \tag{2.19}$$

where H is one-dimensional cubic spline operator given as

$$HW^j_m = \begin{cases} \frac{1}{6}W^j_{m-1} + \frac{4}{6}W^j_m + \frac{1}{6}W^j_{m+1}, & 1 \le m \le M-1, \\ W^j_m, & m = 0, M, \end{cases} \tag{2.20}$$

which can also be defined as

$$HW^j_m = \left(1 + \frac{h^2}{6}\delta^2_{\varkappa}\right)W^j_m, \tag{2.21}$$

where

$$\delta^2_{\varkappa}W^j_m = \frac{W^j_{m-1} - 2W^j_m + W^j_{m+1}}{h^2}\ . \tag{2.22}$$

Lemma 2.4. *Suppose $\mathcal{R}^j_m$ denotes the local truncation error of (2.17)-(2.19). Then*

$$\mathcal{R}^j_m = \mathcal{O}\left(h^2, \tau^2_{j+1}, t^{-\alpha}_{j+\theta}K^{-\min\{3-\alpha, r\alpha\}}\right),\ \textit{for}\ \ 0 \le j \le K-1. \tag{2.23}$$

Proof. Observe that Eq. (2.13) is equivalent to Eq. (2.17). So, the local truncation error of Eq. (2.13) is given by

$$R^j_m = \frac{1}{h^2}[w^{j,\theta}(\varkappa_{m+1}) - 2w^{j,\theta}(\varkappa_m) + w^{j,\theta}(\varkappa_{m-1})] - HP^{j,\theta}_m, \tag{2.24}$$

which on using Eq. (2.16) yields

$$\mathcal{R}^j_m = \frac{1}{h^2}[w^{j,\theta}(\varkappa_{m+1}) - 2w^{j,\theta}(\varkappa_m) + w^{j,\theta}(\varkappa_{m-1})] - H\frac{\partial^2 w^{j,\theta}(\varkappa_m)}{\partial\varkappa^2} + \mathcal{R}^1_t.$$

Now by means of Taylor expansions, we can have the proof. □

2.2.2. **Two-dimensional discretization.** Take uniform meshes of (M_1+1) and (M_2+1) points for spatial domain $(\varkappa, y) \in \Omega = (0, Ł) \times (0, Ł)$. We define $\varkappa_m = mh_{\varkappa}$, $0 \le m \le M_1$, and $y_n = nh_y$, $0 \le n \le M_2$, where $h_{\varkappa} = \frac{Ł}{M_1}$ and $h_y = \frac{Ł}{M_2}$. Let $w(\varkappa, y, t)$ denote the exact solution of problem (1.1). For $0 \le m \le M_1$, $0 \le n \le M_2$, and $0 \le j \le K$, we shall use the notation $w(\varkappa_m, y_n, t_j) = w^j_{m,n}$. Further, we define

$$H_{\varkappa}w^j_{m,n} = \begin{cases} \frac{1}{6}w^j_{m-1,n} + \frac{4}{6}w^j_{m,n} + \frac{1}{6}w^j_{m+1,n}, & 1 \le m \le M_1-1, \\ w^j_{m,n}, & m = 0, M_1, \end{cases} \tag{2.25}$$

$$H_y w^j_{m,n} = \begin{cases} \frac{1}{6}w^j_{m,n-1} + \frac{4}{6}w^j_{m,n} + \frac{1}{6}w^j_{m,n+1}, & 1 \le n \le M_2-1, \\ w^j_{m,n}, & n = 0, M_2, \end{cases} \tag{2.26}$$

which can also be defined as

$$H_{\varkappa}w^j_{m,n} = \left(1 + \frac{h^2_{\varkappa}}{6}\delta^2_{\varkappa}\right)w^j_{m,n}, \tag{2.27}$$

$$H_y w_{m,n}^j = \left(1 + \frac{h_y^2}{6}\delta_y^2\right) w_{m,n}^j, \tag{2.28}$$

where

$$\delta_\varkappa^2 w_{m,n}^j = \frac{w_{m-1,n}^j - 2w_{m,n}^j + w_{m+1,n}^j}{h_\varkappa^2}, \tag{2.29}$$

$$\delta_y^2 w_{m,n}^j = \frac{w_{m,n-1}^j - 2w_{m,n}^j + w_{m,n+1}^j}{h_y^2}. \tag{2.30}$$

Theorem 2.5. *Consider* $f(\varkappa) \in C^4[\varkappa_{i-1}, \varkappa_{i+1}]$. *Then*

$$\frac{1}{6}\left[f''(\varkappa_{i+1}) + 4f''(\varkappa_i) + f''(\varkappa_{i-1})\right] - \frac{1}{h_\varkappa^2}\left[f(\varkappa_{i+1}) - 2f(\varkappa_i) + f(\varkappa_{i-1})\right] = \frac{h_\varkappa^2 f^{(4)}(\xi_i)}{12}, \ \xi_i \in (\varkappa_{i-1}, \varkappa_{i+1}). \tag{2.31}$$

Proof. By Taylor expansions, we have

$$f(\varkappa_{i+1}) = f(\varkappa_i) + h_\varkappa f'(\varkappa_i) + \frac{h_\varkappa^2}{2!} f''(\varkappa_i) + \frac{h_\varkappa^3}{3!} f'''(\varkappa_i) + \frac{h_\varkappa^4}{3!}\int_0^1 f^{(4)}(\varkappa_i + sh_\varkappa)(1-s)^3 ds, \tag{2.32}$$

$$f(\varkappa_{i-1}) = f(\varkappa_i) - h_\varkappa f'(\varkappa_i) + \frac{h_\varkappa^2}{2!} f''(\varkappa_i) - \frac{h_\varkappa^3}{3!} f'''(\varkappa_i) + \frac{h_\varkappa^4}{3!}\int_0^1 f^{(4)}(\varkappa_i - sh_\varkappa)(1-s)^3 ds. \tag{2.33}$$

Adding Eq. (2.32) and (2.33) we get

$$\frac{1}{h_\varkappa^2}[f(\varkappa_{i+1}) - 2f(\varkappa_i) + f(\varkappa_{i-1})] = f''(\varkappa_i) + \frac{h_\varkappa^2}{3!}\int_0^1 \left[f^{(4)}(\varkappa_i + sh_\varkappa) + f^{(4)}(\varkappa_i - sh_\varkappa)\right](1-s)^3 ds. \tag{2.34}$$

Similarly, by Taylor expansions we get

$$f''(\varkappa_{i+1}) = f''(\varkappa_i) + h_\varkappa f'''(\varkappa_i) + \frac{h_\varkappa^2}{2!}\int_0^1 f^{(4)}(\varkappa_i + sh_\varkappa)(1-s) ds, \tag{2.35}$$

$$f''(\varkappa_{i-1}) = f''(\varkappa_i) - h_\varkappa f'''(\varkappa_i) + \frac{h_\varkappa^2}{2!}\int_0^1 f^{(4)}(\varkappa_i - sh_\varkappa)(1-s) ds. \tag{2.36}$$

From the above two equations we obtain

$$\frac{1}{6}[f''(\varkappa_{i+1}) + 4f''(\varkappa_i) + f''(\varkappa_{i-1})] = f''(\varkappa_i) + \frac{h_\varkappa^2}{3!}\int_0^1 \left[f^{(4)}(\varkappa_i + sh_\varkappa) + f^{(4)}(\varkappa_i - sh_\varkappa)\right](1-s) ds. \tag{2.37}$$

Subtracting Eq. (2.34) from Eq. (2.37) and using mean value theorem of integration we obtain

$$\begin{aligned}
&\frac{1}{6}[f''(\varkappa_{i+1}) + 4f''(\varkappa_i) + f''(\varkappa_{i-1})] - \frac{1}{h_\varkappa^2}[f(\varkappa_{i+1}) - 2f(\varkappa_i) + f(\varkappa_{i-1})] \\
=&\frac{h_\varkappa^2}{3!}\int_0^1 \left[f^{(4)}(\varkappa_i + sh_\varkappa) + f^{(4)}(\varkappa_i - sh_\varkappa)\right](1-s)[1-(1-s)^2] ds \\
=&\frac{h_\varkappa^2}{3!}\left[f^{(4)}(\varkappa_i + \tilde{s}h_\varkappa) + f^{(4)}(\varkappa_i - \tilde{s}h_\varkappa)\right]\int_0^1 (1-s)[1-(1-s)^2] ds \\
=&\frac{h_\varkappa^2}{12} f^{(4)}(\xi_i), \ \tilde{s} \in (0,1), \ \xi_i \in (\varkappa_{i-1}, \varkappa_{i+1}).
\end{aligned}$$

This completes the proof. □

Eq. (2.31) can be viewed as

$$\tfrac{1}{6}(6I + h_\varkappa^2\delta_\varkappa^2) f''(\varkappa_i) - \delta_\varkappa^2 f(\varkappa_i) = (\mathcal{R}_x)_{m,n},$$

which is same as

$$\left(I + \frac{h_\varkappa^2 \delta_\varkappa^2}{6}\right) f''(\varkappa_i) = \delta_\varkappa^2 f(\varkappa_i) + (\mathcal{R}_x)_{m,n}.$$

Multiplying $H_\varkappa^{-1}$ on both side of above equation

$$f''(\varkappa_i) = H_\varkappa^{-1}\delta_\varkappa^2 f(\varkappa_i) + (\mathcal{R}_x)_{m,n}\,. \tag{2.38}$$

Utilizing result (2.38) for space derivatives in Eq. (2.5) at points $(\varkappa_m, y_n)$ we get

$$r_{j,j}^\alpha w_{m,n}^{j+1} - \sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) w_{m,n}^l - r_{j,0}^\alpha w_{m,n}^0 = (p_1^{j+\theta} H_\varkappa^{-1}\delta_\varkappa^2 + p_2^{j+\theta} H_y^{-1}\delta_y^2) w_{m,n}^{j,\theta} \tag{2.39}$$

$$- q^{j+\theta} \times w_{m,n}^{j,\theta} + F_{m,n}^{j+\theta} + \mathcal{R}_t^{j+\theta} + \mathcal{R}_\theta^{j,\theta} + \mathcal{R}_{m,n}, \tag{2.40}$$

for $1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1,\ 0 \le j \le K - 1$, where $F_{m,n}^{j+\theta} = F^{j+\theta}(\varkappa_m, y_n)$, $w_{m,n}^{j,\theta} = w^{j,\theta}(\varkappa_m, y_n)$, and $\mathcal{R}_{m,n} = (\mathcal{R}_x)_{m,n} + (\mathcal{R}_y)_{m,n}$ such that

$$(\mathcal{R}_x)_{m,n} = \mathcal{O}(h_\varkappa^2),\ (\mathcal{R}_y)_{m,n} = \mathcal{O}(h_y^2).$$

This implies

$$\mathcal{R}_{m,n} \le c(h_\varkappa^2 + h_y^2). \tag{2.41}$$

Further, Eq. (2.39) reduces to

$$H_\varkappa H_y \left[r_{j,j}^\alpha w_{m,n}^{j+1} - \sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) w_{m,n}^l - r_{j,0}^\alpha w_{m,n}^0 \right] = (p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2) w_{m,n}^{j,\theta} \tag{2.42}$$

$$- H_\varkappa H_y w_{m,n}^{j,\theta} + H_\varkappa H_y F_{m,n}^{j+\theta} + \mathcal{R}_t^{j+\theta} + \mathcal{R}_\theta^{j,\theta} + \mathcal{R}_{m,n},$$

$$1 \le m \le M_1 - 1, 1 \le n \le M_2 - 1, 0 \le j \le K - 1.$$

By rewriting the above equation, we obtain

$$\beta H_\varkappa H_y w_{m,n}^{j+1} - \theta(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2) w_{m,n}^{j+1} = (1-\theta)(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y)$$

$$\times\, w_{m,n}^j + \sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y w_{m,n}^l + r_{j,0}^\alpha H_\varkappa H_y w_{m,n}^0 + H_\varkappa H_y F_{m,n}^{j+\theta} + \mathcal{R}_t^{j+\theta} + \mathcal{R}_{m,n} + \mathcal{R}_\theta^{j,\theta},$$

$$1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1,\ 0 \le j \le K - 1, \tag{2.43}$$

where $\beta = \left(r_{j,j}^\alpha + \theta\right) > 0$.

For constructing an ADI scheme, we add the following perturbation term

$$\frac{\theta^2}{\beta} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 \left(w_{m,n}^{j+1} - w_{m,n}^j\right) = \left(\mathcal{R}_s^{j+1}\right)_{m,n} \tag{2.44}$$

in Eq. (2.43) to get

$$\beta H_\varkappa H_y w_{m,n}^{j+1} - \theta(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2) w_{m,n}^{j+1} + \frac{\theta^2}{\beta} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 w_{m,n}^{j+1}$$

$$= (1-\theta)(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y) w_{m,n}^j + \sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y w_{m,n}^l$$

$$+ r_{j,0}^\alpha H_\varkappa H_y w_{m,n}^0 + H_\varkappa H_y F_{m,n}^{j+\theta} + \frac{\theta^2}{\beta} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 w_{m,n}^j + \mathcal{R}_t^{j+\theta} + \mathcal{R}_{m,n} + \mathcal{R}_\theta^{j,\theta} + \left(\mathcal{R}_s^{j+1}\right)_{m,n},$$

$$1 \le m \le M_1 - 1,\ 0 \le n \le M_2 - 1,\ 0 \le j \le K - 1. \tag{2.45}$$

Multiplying both sides of Eq. (2.45) by $\frac{1}{\beta}$ gives

$$H_\varkappa H_y w_{m,n}^{j+1} - \frac{\theta}{\beta}(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2) w_{m,n}^{j+1} + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 w_{m,n}^{j+1}$$
$$= \frac{(1-\theta)}{\beta}(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y) w_{m,n}^{j} + \frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y w_{m,n}^{l}$$
$$+ \frac{1}{\beta} r_{j,0}^\alpha H_\varkappa H_y w_{m,n}^0 + \frac{1}{\beta} H_\varkappa H_y F_{m,n}^{j+\theta} + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 w_{m,n}^{j} + \frac{\mathcal{R}_t^{j+\theta}}{\beta} + \frac{\mathcal{R}_{m,n}}{\beta} + \frac{\mathcal{R}_\theta^{j,\theta}}{\beta} + \frac{(\mathcal{R}_s^{j+1})_{m,n}}{\beta},$$
$$1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1,\ 0 \le j \le K-1.$$

Equivalently, we have

$$\left(H_\varkappa - \frac{\theta}{\beta} p_1^{j+\theta} \delta_\varkappa^2\right)\left(H_y - \frac{\theta}{\beta} p_2^{j+\theta} \delta_y^2\right) w_{m,n}^{j+1} = \frac{(1-\theta)}{\beta}(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y) w_{m,n}^{j}$$
$$+ \frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y w_{m,n}^{l} + \frac{1}{\beta} r_{j,0}^\alpha H_\varkappa H_y w_{m,n}^0 + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 w_{m,n}^{j} + \frac{1}{\beta} H_\varkappa H_y F_{m,n}^{j+\theta} + \widetilde{\mathcal{R}}_{m,n}^{j+1}, \quad (2.46)$$
$$1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1,\ 0 \le j \le K-1,$$

where

$$\widetilde{\mathcal{R}}_{m,n}^{j+1} = \frac{\mathcal{R}_t^{j+\theta}}{\beta} + \frac{\mathcal{R}_{m,n}}{\beta} + \frac{\mathcal{R}_\theta^{j,\theta}}{\beta} + \frac{(\mathcal{R}_s^{j+1})_{m,n}}{\beta}. \quad (2.47)$$

Removing the truncation error term from Eq. (2.46), we obtain

$$\left(H_\varkappa - \frac{\theta}{\beta} p_1^{j+\theta} \delta_\varkappa^2\right)\left(H_y - \frac{\theta}{\beta} p_2^{j+\theta} \delta_y^2\right) W_{m,n}^{j+1} = \frac{(1-\theta)}{\beta}(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y) W_{m,n}^{j}$$
$$+ \frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y W_{m,n}^{l}$$
$$+ \frac{1}{\beta} r_{j,0}^\alpha H_\varkappa H_y W_{m,n}^0 + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 W_{m,n}^{j} + \frac{1}{\beta} H_\varkappa H_y F_{m,n}^{j+\theta}, \quad (2.48)$$
$$1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1,\ 0 \le j \le K-1,$$

where $W_{m,n}^{j+1}$ is the numerical approximation of solution $w_{m,n}^{j+1}$.

Suppose

$$\left(H_y - \frac{\theta}{\beta} p_2^{j+\theta} \delta_y^2\right) W_{m,n}^{j+1} = W_{m,n}^{\star},\ 1 \le m \le M_1 - 1,\ 1 \le n \le M_2 - 1. \quad (2.49)$$

Now, we will first calculate $W_{m,n}^{\star}$ for fixed values of $n \in \{1, 2, ..., M_2 - 1\}$ as follows

$$\left(H_\varkappa - \frac{\theta}{\beta} p_1^{j+\theta} \delta_\varkappa^2\right) W_{m,n}^{\star} = \frac{(1-\theta)}{\beta}(p_1^{j+\theta} H_y \delta_\varkappa^2 + p_2^{j+\theta} H_\varkappa \delta_y^2 - H_\varkappa H_y) W_{m,n}^{j} + \frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^\alpha - r_{j,l-1}^\alpha) H_\varkappa H_y W_{m,n}^{l}$$
$$+ \frac{1}{\beta} r_{j,0}^\alpha H_\varkappa H_y W_{m,n}^0 + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \delta_\varkappa^2 \delta_y^2 W_{m,n}^{j}$$
$$+ \frac{1}{\beta} H_\varkappa H_y F_{m,n}^{j+\theta}, \qquad 1 \le m \le M_1 - 1, \quad (2.50)$$

with the boundary conditions

$$W_{0,n}^{\star} = \left(H_y - \frac{\theta}{\beta} p_2^{j+\theta} \delta_y^2\right) W_{0,n}^{j+1}, \quad (2.51)$$

$$W^{\star}_{M_1,n} = \left(H_y - \frac{\theta}{\beta} p_2^{j+\theta} \delta_y^2\right) W^{j+1}_{M_1,n}. \tag{2.52}$$

After obtaining $W^{\star}_{m,n}$, we can calculate $W^{j+1}_{m,n}$ using Eq. (2.49) for fixed values of $m \in \{1, 2, ..., M_1 - 1\}$ with boundary conditions

$$W^{j+1}_{m,0} = W^{j+1}_{m,M_2} = 0. \tag{2.53}$$

3. Theoretical Analysis

Here, we will discuss stability and convergence analysis of the developed methods for both 1D and 2D problems. We start by proving a lemma, which is valuable for examining the stability and convergence of the proposed schemes.

Lemma 3.1. *Assuming $\theta = 1 - \frac{\alpha}{2}$ and that the local mesh ratio $\mathcal{B}_j = \frac{\tau_{j+1}}{\tau_j}$ for $1 \leq j \leq K-1$ satisfies $\frac{3}{4} \leq \mathcal{B}_j \leq 62$, then*

$$\frac{1}{r^{\alpha}_{j,j}} = \mathcal{O}\left(\tau^{\alpha}_{j+1}\right), \ \textit{for } 0 \leq j \leq K-1. \tag{3.1}$$

Proof. It is trivial that for $j = 0$,

$$r^{\alpha}_{0,0} = \frac{\theta^{1-\alpha}}{\Gamma(2-\alpha)} \tau_1^{-\alpha}, \tag{3.2}$$

which simplifies to

$$\left|\frac{1}{r^{\alpha}_{0,0}}\right| \leq \frac{\Gamma(2-\alpha)}{\theta^{1-\alpha}} \tau_1^{\alpha}. \tag{3.3}$$

for $j \geq 1$,

$$r^{\alpha}_{j,j} = \frac{1}{\tau_{j+1}} \left(a^{\alpha}_{j,j} + b^{\alpha}_{j,j-1}\right) = \frac{1}{\tau^{\alpha}_{j+1}} \left[\frac{\theta^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{b^{\alpha}_{j,j-1}}{\tau^{1-\alpha}_{j+1}}\right],$$

which gives

$$r^{\alpha}_{j,j} = \frac{\theta^{1-\alpha}}{\Gamma(2-\alpha)} \tau^{-\alpha}_{j+1} \mathcal{J}, \tag{3.4}$$

where

$$\mathcal{J} = 1 + \left(1 + \frac{1}{\mathcal{B}_j}\right)^{-1} \left\{\left[\left(1 + \frac{1}{\theta \mathcal{B}_j}\right)^{2-\alpha} - 1\right] - \frac{1}{\mathcal{B}_j}\left[\left(1 + \frac{1}{\theta \mathcal{B}_j}\right)^{1-\alpha} + 1\right]\right\},$$

where $\mathcal{B}_j = \frac{\tau_{j+1}}{\tau_j}$. Furthermore, utilizing the established bounds of $\mathcal{B}_j$ for $1 \leq j \leq K-1$, where the values are confined within the range $\frac{3}{4} \leq \mathcal{B}_j \leq 62$ as established in [6], we can derive the corresponding bounds of $\mathcal{J}$ as

$$1.6597542 \leq \mathcal{J} \leq 13.215168. \tag{3.5}$$

By employing the relation (3.3) and integrating the provided bounds of $\mathcal{J}$ from (3.5) into the expression (3.4), we derive the following result:

$$\left|\frac{1}{r^{\alpha}_{j,j}}\right| \leq c \tau^{\alpha}_{i+1}, \tag{3.6}$$

where $c > 0$ is a generic constant. Hence this gives the Lemma. □

3.1. For one-dimensional problem.

3.1.1. *Stability analysis.* Suppose $\bar{W}_m^j$ be the perturbed solution of the cubic spline difference scheme (2.17)-(2.19). Let $\vartheta_m^j = W_m^j - \bar{W}_m^j,\ 1 \le m \le M_1 - 1,\ 1 \le j \le K - 1$. Then

$$\left[r_{j,j}^{\alpha} H \vartheta_m^{j+1} - \sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) H \vartheta_m^l - r_{j,0}^{\alpha} H \vartheta_m^0 \right] = p^{j+\theta} \delta_{\varkappa}^2 \vartheta_m^{j,\theta} - q^{j+\theta} H \vartheta_m^{j,\theta}, \quad 1 \le m \le M-1,\ 0 \le j \le K-1. \quad (3.7)$$

The grid function $\vartheta^j(\varkappa)$ is defined as

$$\vartheta^j(\varkappa) = \begin{cases} \vartheta_m^j, & \varkappa \in \left(\varkappa_{m-\frac{1}{2}}, \varkappa_{m+\frac{1}{2}} \right], \\ 0, & \varkappa \in \left[0, \frac{h}{2} \right],\ \varkappa \in \left(\text{Ł} - \frac{h}{2}, \text{Ł} \right], \end{cases} \quad (3.8)$$

for $1 \le j \le K$ and $1 \le m \le M_1 - 1$.

Now $\vartheta^j(\varkappa)$ is expressed as a Fourier series

$$\vartheta^j(\varkappa) = \sum_{i_1=-\infty}^{\infty} \eta^j(i_1) e^{\frac{2\pi \iota i_1 \varkappa}{\text{Ł}}},\ 1 \le j \le K, \quad (3.9)$$

where

$$\eta^j(i_1) = \frac{1}{\text{Ł}} \int_0^{\text{Ł}} \vartheta^j(\varkappa) e^{\frac{-2\pi \iota i_1 \varkappa}{\text{Ł}}} d\varkappa. \quad (3.10)$$

By the definition of L_2 discrete norm and Parseval's equality, we get

$$\|\vartheta^j\|_2^2 = \sum_{m=1}^{M-1} h |\vartheta_m^j|^2 = \text{Ł} \sum_{i_1=-\infty}^{\infty} |\eta^j(i_1)|^2. \quad (3.11)$$

Suppose the solution of (3.7) has following form

$$\vartheta_m^j = \eta^j e^{\iota \theta_1 m h}, \quad (3.12)$$

where $\theta_1 = \frac{2\pi i_1}{\text{Ł}}$. Substituting Eq. (3.12) in Eq. (3.7) we get

$$\left[r_{j,j}^{\alpha} \nu_1 + \theta (p^{j+\theta} \nu_2 + q^{j+\theta} \nu_1) \right] \eta^{j+1} = \sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-l}^{\alpha}) \eta^l \nu_1 + r_{j,0}^{\alpha} \eta^0 \nu_1 - (1-\theta)(p^{j+\theta} \eta^j \nu_2 + q^{j+\theta} \eta^j \nu_1),\ 0 \le j \le K-1, \quad (3.13)$$

which gives

$$\eta^{j+1} = \frac{1}{\left[r_{j,j}^{\alpha} \nu_1 + \theta (p^{j+\theta} \nu_2 + q^{j+\theta} \nu_1) \right]} \left[\sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) \eta^l \nu_1 + r_{j,0}^{\alpha} \eta^0 \nu_1 - (1-\theta) \eta^j (p^{j+\theta} \nu_2 + q^{j+\theta} \nu_1) \right], \quad (3.14)$$

where

$$\nu_1 = \frac{1}{3} \left[2 \cos^2 \left(\frac{\theta_1 h}{2} \right) + 1 \right],\ \nu_2 = \frac{4}{h^2} \sin^2 \left(\frac{\theta_1 h}{2} \right). \quad (3.15)$$

It is easy to show that $\nu_1 \ge \frac{1}{3}$ and $\nu_2 \ge 0$.

Lemma 3.2. *Let η^j be the solution of (3.14). Then $|\eta^j| \le |\eta^0|,\ 1 \le j \le K$.*

Proof. We will prove this using mathematical induction. Put $j = 0$ in (3.14) to get

$$|\eta^1| = \frac{1}{\left[r_{0,0}^{\alpha} \nu_1 + \theta (p^{\theta} \nu_2 + q^{\theta} \nu_1) \right]} \left[r_{0,0}^{\alpha} \nu_1 + (1-\theta)(p^{\theta} \nu_2 + q^{\theta} \nu_1) \right] |\eta^0|. \quad (3.16)$$

It is easy to observe that $1 - \theta \le \theta$. Thus, using the fact $r_{0,0}^{\alpha} \ge 0$, we get

$$|\eta^1| \le |\eta^0|. \quad (3.17)$$

Suppose $|\eta^d| \leq |\eta^0|,$ for all $1 \leq d \leq j$. Putting $d = j+1$ in Eq. (3.14) and using Lemma 2.2 with these assumptions we get

$$\begin{aligned}|\eta^{j+1}| \leq &\frac{1}{\left[r^{\alpha}_{j,j}\nu_1 + \theta(p^{j+\theta}\nu_2 + q^{j+\theta}\nu_1)\right]}\left[\sum_{l=1}^{j}(r^{\alpha}_{j,l} - r^{\alpha}_{j,l-1})\nu_1 + r^{\alpha}_{j,0}\nu_1 + (1-\theta)(p^{j+\theta}\nu_2 + q^{j+\theta}\nu_1)\right]|\eta^0|\\ \leq &\frac{\nu_1(r^{\alpha}_{j,j} - r^{\alpha}_{j,0}) + \nu_1 r^{\alpha}_{j,0} + \theta(p^{j+\theta}\nu_2 + q^{j+\theta}\nu_1)}{\left[r^{\alpha}_{j,j}\nu_1 + \theta(p^{j+\theta}\nu_2 + q^{j+\theta}\nu_1)\right]}|\eta^0| = |\eta^0|.\end{aligned}$$

Thus, we have $|\eta^j| \leq |\eta^0|, \ 1 \leq j \leq K$.

This completes the proof. □

Using Eq. (3.11) and Lemma 3.2, we get

$$\|\vartheta^j\|_2^2 = Ł \sum_{i_1=-\infty}^{\infty}|\eta^j(i_1)|^2 \leq Ł \sum_{i_1=-\infty}^{\infty}|\eta^0(i_1)|^2 = \|\vartheta^0\|_2^2.$$

Thus, $\|\vartheta^j\|_2 \leq \|\vartheta^0\|_2, \ 1 \leq j \leq K$. Hence, the numerical scheme given by (2.17)-(2.19) is unconditionally stable.

3.1.2. *Convergence analysis.* In this section, we discuss convergence analysis of numerical scheme (2.17)-(2.19).

Recall that the exact solution of the considered problem (2.6)-(2.8) is $w(\varkappa_m, t_j)$ at $\varkappa = \varkappa_m$ and $t = t_j$ and W^j_m is the approximate value of $w(\varkappa_m, t_j)$. Now we define $\xi^j_m = w^j_m - W^j_m, \ 1 \leq m \leq M_1 - 1, \ 1 \leq \ j \leq K$. Since $w(\varkappa_m, t_j)$ is the exact solution, from (2.17)-(2.19), we have

$$r^{\alpha}_{j,j}Hw^{j+1}_m - \sum_{l=1}^{j}(r^{\alpha}_{j,l} - r^{\alpha}_{j,l-1})Hw^l_m - r^{\alpha}_{j,0}Hw^0_m - p^{j+\theta}\delta^2_{\varkappa}w^{j,\theta}_m + q^{j+\theta}Hw^{j,\theta}_m = HF^{j+\theta}_m + \mathcal{R}^j_m,$$

$$1 \leq m \leq M-1, \ 0 \leq j \leq K-1, \tag{3.18}$$

$$w^0_m = \phi(\varkappa_m), \ 0 \leq m \leq M, \tag{3.19}$$

$$w^j_0 = 0, \ w^j_M = 0, \ 1 \leq j \leq K. \tag{3.20}$$

Using (2.17)-(2.19), it is evident that the error equation is given by

$$(r^{\alpha}_{j,j} + \theta q^{j+\theta})H\xi^{j+1}_m - \sum_{l=1}^{j}(r^{\alpha}_{j,l} - r^{\alpha}_{j,l-1})H\xi^l_m - \theta p^{j+\theta}\delta^2_{\varkappa}\xi^{j+1}_m = (1-\theta)p^{j+\theta}\delta^2_{\varkappa}\xi^j_m - (1-\theta)q^{j+\theta}H\xi^j_m + \mathcal{R}^j_m,$$

$$1 \leq m \leq M-1, \ 0 \leq j \leq K-1, \tag{3.21}$$

$$\xi^0_m = 0, \ 0 \leq m \leq M, \tag{3.22}$$

$$\xi^j_0 = 0, \ \xi^j_M = 0, \ 1 \leq j \leq K. \tag{3.23}$$

Eq. (3.21) can be restated as

$$\left(1 + \frac{\theta q^{j+\theta}}{r^{\alpha}_{j,j}}\right)H\xi^{j+1}_m - \sum_{l=1}^{j}\frac{(r^{\alpha}_{j,l} - r^{\alpha}_{j,l-1})}{r^{\alpha}_{j,j}}H\xi^l_m - \frac{\theta p^{j+\theta}}{r^{\alpha}_{j,j}}\delta^2_{\varkappa}\xi^{j+1}_m = \frac{(1-\theta)p^{j+\theta}}{r^{\alpha}_{j,j}}\delta^2_{\varkappa}\xi^j_m - \frac{(1-\theta)q^{j+\theta}}{r^{\alpha}_{j,j}}H\xi^j_m + \hat{\mathcal{R}}^j_m, \tag{3.24}$$

$$1 \leq m \leq M-1, \ 0 \leq j \leq K-1,$$

where

$$\hat{\mathcal{R}}^j_m = \frac{\mathcal{R}^j_m}{r^{\alpha}_{j,j}}. \tag{3.25}$$

Now, we prove a lemma which will provide the bound of $\hat{\mathcal{R}}^j_m$.

Lemma 3.3. *Suppose that solution of (2.6)-(2.8) $w(x,t)$ satisfies the conditions given in (1.4) then $\hat{\mathcal{R}}_m^j$ satisfies the following bound*

$$\left|\hat{\mathcal{R}}_m^j\right| \leq c\left(h^2 + K^{-\min\{2,r\alpha\}}\right).$$

Proof. In Eq. (3.25) using Lemmas 2.4, and and a result from [43, Eq. (5.1), p.1069], we have

$$\left|\hat{\mathcal{R}}_m^j\right| \leq c_1(h^2 + K^{-2}) + c_2\tau_{j+1}^{\alpha} t_{j+\theta}^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}}. \tag{3.26}$$

Now, we will bound the last term of Eq. (3.26) as

$$\begin{aligned}\tau_{j+1}^{\alpha} t_{j+\theta}^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}} &\leq \tau_{j+1}^{\alpha}(t_j + \theta\tau_{j+1})^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}}\\ &\leq \tau_{j+1}^{\alpha}(\theta\tau_{j+1})^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}}\\ &\leq cK^{-\min\{3-\alpha,r\alpha\}}.\end{aligned} \tag{3.27}$$

Finally using Eq. (3.27) in Eq. (3.26) we get our desired theorem. □

Now, we proceed for the convergence analysis. The functions $\xi^j(\varkappa)$ and $\hat{\mathcal{R}}^j(\varkappa)$ are defined as

$$\xi^j(\varkappa) = \begin{cases}\xi_m^j, & \varkappa \in \left(\varkappa_{m-\frac{1}{2}}, \varkappa_{m+\frac{1}{2}}\right],\ m = 1, 2, \ldots, M_1 - 1,\\ 0, & \varkappa \in \left[0, \frac{h}{2}\right],\ \varkappa \in \left(\text{Ł} - \frac{h}{2}, \text{Ł}\right],\end{cases}$$

and

$$\hat{\mathcal{R}}^j(\varkappa) = \begin{cases}\hat{\mathcal{R}}_m^j, & \varkappa \in \left(\varkappa_{m-\frac{1}{2}}, \varkappa_{m+\frac{1}{2}}\right],\ m = 1, 2, \ldots, M_1 - 1,\\ 0, & \varkappa \in \left[0, \frac{h}{2}\right],\ \varkappa \in \left(\text{Ł} - \frac{h}{2}, \text{Ł}\right],\end{cases}$$

for $1 \leq j \leq K$.

Now $\xi^j(\varkappa)$ and $\hat{\mathcal{R}}^j(\varkappa)$ can be expressed as a Fourier series

$$\xi^j(\varkappa) = \sum_{i_1=-\infty}^{\infty} \eta^j(i_1) e^{\frac{2\pi\iota i_1\varkappa}{\text{Ł}}},\ \hat{\mathcal{R}}^j(\varkappa) = \sum_{i_1=-\infty}^{\infty} \S^j(i_1) e^{\frac{2\pi\iota i_1\varkappa}{\text{Ł}}},$$

where

$$\begin{aligned}\eta^j(i_1) &= \frac{1}{\text{Ł}}\int_0^{\text{Ł}} \xi^j(\varkappa) e^{\frac{-2\pi\iota i_1\varkappa}{\text{Ł}}} d\varkappa,\\ \S^j(i_1) &= \frac{1}{\text{Ł}}\int_0^{\text{Ł}} \hat{\mathcal{R}}^j(\varkappa) e^{\frac{-2\pi\iota i_1\varkappa}{\text{Ł}}} d\varkappa.\end{aligned}$$

By definition of L_2 discrete norm and Parseval's equality we get

$$\|\xi^j\|_2^2 = \sum_{m=1}^{M-1} h|\xi_m^j|^2 = \text{Ł}\sum_{i_1=-\infty}^{\infty} |\eta^j(i_1)|^2, \tag{3.28}$$

$$\|\hat{\mathcal{R}}^j\|_2^2 = \sum_{m=1}^{M-1} h|\hat{\mathcal{R}}_m^j|^2 = \text{Ł}\sum_{i_1=-\infty}^{\infty} |\S^j(i_1)|^2, \tag{3.29}$$

for $1 \leq j \leq K$.

Let ξ_m^j and $\hat{\mathcal{R}}_m^j$ have following forms

$$\xi_m^j = \eta^j e^{\iota\theta_1 mh},\ \hat{\mathcal{R}}_m^j = \S^j e^{\iota\theta_1 mh}, \tag{3.30}$$

where $\theta_1 = \frac{2\pi i_1}{\mathrm{L}}$. Substituting (3.30) in (3.24) we get

$$\eta^{j+1} = \frac{1}{\left[\nu_1\left(1+\frac{\theta q^{j+\theta}}{r^{\alpha}_{j,j}}\right)+\frac{\theta p^{j+\theta}}{r^{\alpha}_{j,j}}\nu_2\right]}\left[\sum_{l=1}^{j}\frac{(r^{\alpha}_{j,l}-r^{\alpha}_{j,l-1})}{r^{\alpha}_{j,j}}\eta^{l}\nu_1 - \frac{(1-\theta)}{r^{\alpha}_{j,j}}(p^{j+\theta}\eta^{j}\nu_2+q^{j+\theta}\eta^{j}\nu_1)+\S^{j+1}\right],\ 0\le j\le K-1. \tag{3.31}$$

As we know, the series on the right side of (3.29) is convergent; therefore, for some constant $A>0$, we have

$$|\S^{j}| \equiv |\S^{j}(i_1)| \le A\tau|\S^{1}(i_1)| \equiv A\tau|\S^{1}|,\ 1\le j\le K, \tag{3.32}$$

where $\tau = \max_{1\le j\le K}\tau_j$.

Lemma 3.4. *For some constant $A>0$, it holds*

$$|\eta^{j}| \le 3A(1+\tau)^{j}|\S^{1}|,\ 1\le j\le K. \tag{3.33}$$

Proof. We will prove by mathematical induction on (3.31) and considering $\eta^0=0$. For $j=0$, we have

$$\eta^{1} = \frac{\S^{1}}{\left[\nu_1\left(1+\frac{\theta q^{\theta}}{r^{\alpha}_{0,0}}\right)+\frac{\theta p^{\theta}}{r^{\alpha}_{0,0}}\nu_2\right]}.$$

By Eq. (3.32) and using the fact $\nu_1\ge\frac{1}{3}$, we get

$$|\eta^{1}| \le 3|\S^{1}| \le 3A(1+\tau)|\S^{1}|\ . \tag{3.34}$$

Now, let us assume that

$$|\eta^{d}| \le 3A(1+\tau)^{d}|\S^{1}|\ , \tag{3.35}$$

is true for $1\le d\le j$. Now putting $d=j+1$ in Eq. (3.31) it follows that

$$\begin{aligned}
|\eta^{j+1}| &\le \frac{1}{\left[\nu_1\left(1+\frac{\theta q^{j+\theta}}{r^{\alpha}_{j,j}}\right)+\frac{\theta p^{j+\theta}}{r^{\alpha}_{j,j}}\nu_2\right]}\left[\sum_{l=1}^{j}\frac{(r^{\alpha}_{j,l}-r^{\alpha}_{j,l-1})}{r^{\alpha}_{j,j}}\nu_1\max_{1\le k\le j}|\eta^{k}| + \frac{(1-\theta)}{r^{\alpha}_{j,j}}(p^{j+\theta}\nu_2+q^{j+\theta}\nu_1)|\eta^{j}| + A\tau|\S^{1}|\right]\\
&\le \frac{1}{\left[\nu_1\left(1+\frac{\theta q^{j+\theta}}{r^{\alpha}_{j,j}}\right)+\frac{\theta p^{j+\theta}}{r^{\alpha}_{j,j}}\nu_2\right]}\left[\nu_1\left(1-\frac{r^{\alpha}_{j,0}}{r^{\alpha}_{j,j}}\right)(3A(1+\tau)^{j}|\S^{1}|)\right.\\
&\left.+\frac{(1-\theta)}{r^{\alpha}_{j,j}}(p^{j+\theta}\nu_2+q^{j+\theta}\nu_1)(3A(1+\tau)^{j}|\S^{1}|)+A\tau|\S^{1}|\right] \qquad (3.36)\\
&= \frac{1}{\left[\nu_1\left(1+\frac{\theta q^{j+\theta}}{r^{\alpha}_{j,j}}\right)+\frac{\theta p^{j+\theta}}{r^{\alpha}_{j,j}}\nu_2\right]}\left[\left(\nu_1\left(1-\frac{r^{\alpha}_{j,0}}{r^{\alpha}_{j,j}}\right)+\frac{(1-\theta)}{r^{\alpha}_{j,j}}(p^{j+\theta}\nu_2+q^{j+\theta}\nu_1)\right)\times(3A(1+\tau)^{j}|\S^{1}|)+A\tau|\S^{1}|\right].
\end{aligned}$$

Now, invoking Lemma 2.2 and using the fact that $1-\theta\le\theta$, we get

$$\begin{aligned}
|\eta^{j+1}| &\le 3A(1+\tau)^{j}|\S^{1}| + 3A(1+\tau)|\S^{1}|\\
&\le 3A\left((1+\tau)^{j}+\tau\right)|\S^{1}|\\
&\le 3A(1+\tau)^{j+1}|\S^{1}|.
\end{aligned} \tag{3.37}$$

Thus, we have the lemma. □

Theorem 3.5. *Let $w(\varkappa,t)$ be the solution of problem (2.6)-(2.8) satisfying the assumptions given in (1.4) and let $\{W^{j}_{m}, 0\le m\le M, 1\le n\le K\}$ be the solution of the discrete problem (2.17)-(2.19) on non-uniform graded mesh. Then, the following result holds*

$$\|w^{j}-W^{j}\|_2 \le c\ \left(h^2+K^{-\min\{2,r\alpha\}}\right),\ \textit{for}\ 1\le j\le K. \tag{3.38}$$

Proof. Combining Lemma 3.4 and Eq. (3.28), we have

$$\|\xi^j\|_2^2 \le \text{Ł} \sum_{i_1=-\infty}^{\infty} (3A)^2(1+\tau)^{2j}|\S^1(i_1)|^2 = (3A)^2(1+\tau)^{2j}\|\hat{\mathcal{R}}^1\|_2^2. \tag{3.39}$$

Further, from Eq. (3.29) together with Lemma 3.3, we have

$$\begin{aligned}\|\hat{\mathcal{R}}^j\|_2 &\le \sqrt{Mh}\; c\left(h^2 + K^{-\min\{2,r\alpha\}}\right)\\ &\le c\sqrt{\text{Ł}}\left(K^{-\min\{2,r\alpha\}} + h^2\right),\; 1\le j\le K.\end{aligned} \tag{3.40}$$

Using (3.40) in (3.39) we get

$$\|\xi^j\|_2^2 \le (3A)^2 e^{j\tau} c^2 \text{Ł}\left(K^{-\min\{2,r\alpha\}} + h^2\right)^2. \tag{3.41}$$

As $j\tau \le T$, we get

$$\|\xi^j\|_2 \le B\left(K^{-\min\{2,r\alpha\}} + h^2\right), \tag{3.42}$$

where $B = 3Ac\sqrt{\text{Ł}}e^T$.

Hence, we have the theorem. □

3.2. **For two-dimensional problem.**

3.2.1. *Truncation error.* Now we will estimate the value of truncation error denoted $\mathcal{R}^k_{m,n}$ described in Eq. (2.48) as

$$\widetilde{\mathcal{R}}^{j+1}_{m,n} = \frac{\mathcal{R}^{j+\theta}_t}{\beta} + \frac{\mathcal{R}_{m,n}}{\beta} + \frac{\mathcal{R}^{j,\theta}_\theta}{\beta} + \frac{\left(\mathcal{R}^{j+1}_s\right)_{m,n}}{\beta}. \tag{3.43}$$

Theorem 3.6. *Suppose that solution of (1.1)-(1.2) $w(x,y,t)$ satisfies the conditions given in (1.4) then $\widetilde{\mathcal{R}}^{j+1}_{m,n}$ satisfies the following bound*

$$\left|\widetilde{\mathcal{R}}^{j+1}_{m,n}\right| \le c\left(K^{-\min\{1+\alpha,r\alpha\}} + h_\varkappa^2 + h_y^2\right).$$

Proof. From equation (2.47) and the condition $r^\alpha_{j,j} > 0$ we deduce that

$$\begin{aligned}\left|\widetilde{\mathcal{R}}^{j+1}_{m,n}\right| &\le \left|\frac{\mathcal{R}^{j+\theta}_t}{\beta}\right| + \left|\frac{\mathcal{R}_{m,n}}{\beta}\right| + \left|\frac{\mathcal{R}^{j,\theta}_\theta}{\beta}\right| + \left|\frac{\left(\mathcal{R}^{j+1}_s\right)_{m,n}}{\beta}\right|\\ &\le \left|\frac{\mathcal{R}^{j+\theta}_t}{r^\alpha_{j,j}}\right| + \left|\frac{\mathcal{R}_{m,n}}{r^\alpha_{j,j}}\right| + \left|\frac{\mathcal{R}^{j,\theta}_\theta}{r^\alpha_{j,j}}\right| + \left|\frac{\left(\mathcal{R}^{j+1}_s\right)_{m,n}}{r^\alpha_{j,j}}\right|.\end{aligned} \tag{3.44}$$

Now we will bound each term individually in Eq. (3.44).

Using Lemma 3.1 in Eq. (3.44) it gives

$$\begin{aligned}\left|\frac{\mathcal{R}^{j+\theta}_t}{r^\alpha_{j,j}}\right| &\le c\tau^\alpha_{j+1} t^{-\alpha}_{j+\theta} K^{-\min\{3-\alpha,r\alpha\}}\\ &= c\tau^\alpha_{j+1}(t_j + \theta\tau_{j+1})^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}}\\ &\le c\tau^\alpha_{j+1}(\theta\tau_{j+1})^{-\alpha} K^{-\min\{3-\alpha,r\alpha\}}\\ &\le cK^{-\min\{3-\alpha,r\alpha\}}.\end{aligned} \tag{3.45}$$

Invoking Lemma 3.1 alongside Eq. (2.41) to get

$$\left|\frac{\mathcal{R}_{m,n}}{r^\alpha_{j,j}}\right| \le c(h_\varkappa^2 + h_y^2). \tag{3.46}$$

Afterwards combining Lemma 2.3, Lemma 3.1, and a result from [43, Eq. (5.1), p.1069], we get

$$\left|\frac{\mathcal{R}_t^{j,\theta}}{r_{j,j}^{\alpha}}\right| \leq cK^{-(2+\alpha)}. \tag{3.47}$$

Now, Lemma 3.1 with Eq. (2.44) give

$$\left|\left(\mathcal{R}_s^{j+1}\right)_{m,n}\right| \leq cK^{-(1+\alpha)}. \tag{3.48}$$

Finally by combining (3.45), (3.46), (3.47), and (3.48) into (3.44) we get our desired theorem. □

3.2.2. *Stability analysis.* Suppose $\bar{W}_{m,n}^j$ be the perturbed solution of the cubic spline difference scheme (2.48). Let $\vartheta_{m,n}^j = W_{m,n}^j - \bar{W}_{m,n}^j,\ 1 \leq m \leq M_1 - 1,\ 1 \leq n \leq M_2 - 1,\ 1 \leq j \leq K$. Then

$$\begin{aligned}
&\left(H_\varkappa - \frac{\theta}{\beta}p_1^{j+\theta}\delta_\varkappa^2\right)\left(H_y - \frac{\theta}{\beta}p_2^{j+\theta}\delta_y^2\right)\vartheta_{m,n}^{j+1} = \frac{(1-\theta)}{\beta}(p_1^{j+\theta}H_y\delta_\varkappa^2 + p_2^{j+\theta}H_\varkappa\delta_y^2 - H_\varkappa H_y)\vartheta_{m,n}^j \\
&+ \frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha})H_\varkappa H_y\vartheta_{m,n}^l + \frac{1}{\beta}r_{j,0}^{\alpha}H_\varkappa H_y\vartheta_{m,n}^0 + \frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\delta_\varkappa^2\delta_y^2\vartheta_{m,n}^j, \\
&1 \leq m \leq M_1 - 1,\ 1 \leq n \leq M_2 - 1,\ 0 \leq j \leq K - 1.
\end{aligned} \tag{3.49}$$

The function $\vartheta^j(\varkappa, y)$ is defined as

$$\vartheta^j(\varkappa, y) = \begin{cases} \vartheta_{m,n}^j, & \varkappa \in \left(\varkappa_{m-\frac{1}{2}}, \varkappa_{m+\frac{1}{2}}\right],\ y \in \left(y_{n-\frac{1}{2}}, y_{n+\frac{1}{2}}\right], \\ & \quad m = 1, 2, \ldots, M_1 - 1,\ n = 1, 2, \ldots, M_2 - 1, \\ 0, & \varkappa \in \left[0, \frac{h_\varkappa}{2}\right],\ \varkappa \in \left(\text{Ł} - \frac{h_\varkappa}{2}, \text{Ł}\right], \\ & y \in \left[0, \frac{h_y}{2}\right],\ y \in \left(\text{Ł} - \frac{h_y}{2}, \text{Ł}\right], \end{cases} \tag{3.50}$$

for $1 \leq j \leq K$.

Now $\vartheta^j(\varkappa, y)$ is expressed as a Fourier series

$$\vartheta^j(\varkappa, y) = \sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}\eta^j(i_1, i_2)e^{2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}} + \frac{i_2 y}{\text{Ł}}\right)},\ 1 \leq j \leq K, \tag{3.51}$$

where

$$\eta^j(i_1, i_2) = \frac{1}{\text{Ł}^2}\int_0^{\text{Ł}}\int_0^{\text{Ł}}\vartheta^j(\varkappa, y)e^{-2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}} + \frac{i_2 y}{\text{Ł}}\right)}d\varkappa dy. \tag{3.52}$$

By L_2 discrete norm definition and Parseval's equality we get

$$\|\vartheta^j\|_2^2 = \sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}h_\varkappa h_y|\vartheta_{m,n}^j|^2 = \text{Ł}^2\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}|\eta^j(i_1, i_2)|^2. \tag{3.53}$$

Suppose the solution of (3.49) has following form

$$\vartheta_{m,n}^j = \eta^j e^{(\iota\theta_1 m h_\varkappa + \iota\theta_2 n h_y)}, \tag{3.54}$$

where $\theta_1 = \frac{2\pi i_1}{\text{Ł}}$, $\theta_2 = \frac{2\pi i_2}{\text{Ł}}$.

Substituting (3.54) in (3.49) we get

$$\begin{aligned}
\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{j+\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{j+\theta}\tilde{\nu}_4\right)\eta^{j+1} &= -\frac{(1-\theta)}{\beta}(p_1^{j+\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{j+\theta}\tilde{\nu}_1\tilde{\nu}_4 + \tilde{\nu}_1\tilde{\nu}_2)\eta^j \\
&+ \frac{\tilde{\nu}_1\tilde{\nu}_2}{\beta}\left[\sum_{l=1}^{j}(r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha})\eta^l + r_{j,0}^{\alpha}\eta^0\right] + \frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\tilde{\nu}_3\tilde{\nu}_4\eta^j, \quad 0 \leq j \leq K-1,
\end{aligned} \tag{3.55}$$

which gives

$$\eta^{j+1} = \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{j+\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{j+\theta}\tilde{\nu}_4\right)}\left[\frac{\tilde{\nu}_1\tilde{\nu}_2}{\beta}\left(\sum_{l=1}^{j}(r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha})\eta^l + r_{j,0}^{\alpha}\eta^0\right)\right.$$
$$\left. - \frac{(1-\theta)}{\beta}(p_1^{j+\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{j+\theta}\tilde{\nu}_1\tilde{\nu}_4 + \tilde{\nu}_1\tilde{\nu}_2)\eta^j + \frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\tilde{\nu}_3\tilde{\nu}_4\eta^j\right],\ 0 \le j \le K-1, \tag{3.56}$$

where

$$\tilde{\nu}_1 = \frac{1}{3}\left[2\cos^2\left(\frac{\theta_1 h_{\varkappa}}{2}\right) + 1\right],\ \tilde{\nu}_2 = \frac{1}{3}\left[2\cos^2\left(\frac{\theta_2 h_y}{2}\right) + 1\right], \tag{3.57}$$

$$\tilde{\nu}_3 = \frac{4}{h_{\varkappa}^2}\sin^2\left(\frac{\theta_1 h_{\varkappa}}{2}\right),\ \tilde{\nu}_4 = \frac{4}{h_y^2}\sin^2\left(\frac{\theta_2 h_y}{2}\right). \tag{3.58}$$

Note that $\tilde{\nu}_1, \tilde{\nu}_2 \ge \frac{1}{3}$ and $\tilde{\nu}_3, \tilde{\nu}_4 \ge 0$.

Lemma 3.7. *Let η^j be the solution of (3.56). Then,*

$$|\eta^j| \le |\eta^0|,\ 1 \le j \le K.$$

Proof. We will prove this using mathematical induction. Put $j = 0$ in (3.56) to get

$$\eta^1 = \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{\theta}\tilde{\nu}_4\right)}\left(\frac{\tilde{\nu}_1\tilde{\nu}_2}{\beta}r_{0,0}^{\alpha} - \frac{(1-\theta)}{\beta}(p_1^{\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{\theta}\tilde{\nu}_1\tilde{\nu}_4 + \tilde{\nu}_1\tilde{\nu}_2) + \frac{\theta^2}{\beta^2}p_1^{\theta}p_2^{\theta}\tilde{\nu}_3\tilde{\nu}_4\right)\eta^0. \tag{3.59}$$

As $r_{0,0}^{\alpha} \ge 0$ and $0 \le (1-\theta) \le \theta$, we have

$$|\eta^1| \le \frac{\left(\tilde{\nu}_1\tilde{\nu}_2 + \frac{\theta}{\beta}(p_1^{\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{\theta}\tilde{\nu}_1\tilde{\nu}_4) + \frac{\theta^2}{\beta^2}p_1^{\theta}p_2^{\theta}\tilde{\nu}_3\tilde{\nu}_4\right)|\eta^0|}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{\theta}\tilde{\nu}_4\right)} = |\eta^0|. \tag{3.60}$$

Now, let us assume that

$$|\eta^d| \le |\eta^0|,\ \text{for all } 1 \le d \le j. \tag{3.61}$$

Next, for $d = j+1$, from Eq. (3.56) with Lemma 2.2 and assumptions (3.61), we get

$$\left|\eta^{j+1}\right| \le \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{j+\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{j+\theta}\tilde{\nu}_4\right)}\left[\frac{\tilde{\nu}_1\tilde{\nu}_2}{\beta}\left(\sum_{l=1}^{j}(r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) + r_{j,0}^{\alpha}\right)\right.$$
$$\left. + \frac{(1-\theta)}{\beta}(p_1^{j+\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{j+\theta}\tilde{\nu}_1\tilde{\nu}_4 + \tilde{\nu}_1\tilde{\nu}_2) + \frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\tilde{\nu}_3\tilde{\nu}_4\right]|\eta^0|$$
$$\le \frac{\tilde{\nu}_1\tilde{\nu}_2 + \frac{\theta}{\beta}(p_1^{j+\theta}\tilde{\nu}_2\tilde{\nu}_3 + p_2^{j+\theta}\tilde{\nu}_1\tilde{\nu}_4) + \frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\tilde{\nu}_3\tilde{\nu}_4}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta}p_1^{j+\theta}\tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta}p_2^{j+\theta}\tilde{\nu}_4\right)}|\eta^0| = |\eta^0|.$$

This completes the proof. □

Using Eq. (3.53) and Lemma (3.7), we get

$$\|\vartheta^j\|_2^2 = Ł^2\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}|\eta^j(i_1,i_2)|^2 \le Ł^2\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}|\eta^0(i_1,i_2)|^2 = \|\vartheta^0\|_2^2.$$

Thus, $\|\vartheta^j\|_2 \le \|\vartheta^0\|_2,\ 1 \le j \le K$. This shows the unconditional stability of the scheme given by (2.48).

3.2.3. *Convergence analysis.* The convergence analysis of (2.48) is covered in this section. We have

$$\left(H_{\varkappa}-\frac{\theta}{\beta}p_1^{j+\theta}\delta_{\varkappa}^2\right)\left(H_y-\frac{\theta}{\beta}p_2^{j+\theta}\delta_y^2\right)\xi_{m,n}^{j+1}=\frac{(1-\theta)}{\beta}(p_1^{j+\theta}H_y\delta_{\varkappa}^2+p_2^{j+\theta}H_{\varkappa}\delta_y^2-H_{\varkappa}H_y)$$
$$\times\,\xi_{m,n}^{j}+\frac{1}{\beta}\sum_{l=1}^{j}(r_{j,l}^{\alpha}-r_{j,l-1}^{\alpha})H_{\varkappa}H_y\xi_{m,n}^{l}+\frac{\theta^2}{\beta^2}p_1^{j+\theta}p_2^{j+\theta}\delta_{\varkappa}^2\delta_y^2\xi_{m,n}^{j}+\widetilde{\mathcal{R}}_{m,n}^{j+1}, \tag{3.62}$$

where $\xi_{m,n}^j=w_{m,n}^j-W_{m,n}^j$, $1\le m\le M_1-1$, $1\le n\le M_2-1$, $1\le j\le K$.

Now we define the functions $\widetilde{\mathcal{R}}^j(\varkappa,y)$ and $\xi^j(\varkappa,y)$, as

$$\widetilde{\mathcal{R}}^j(\varkappa,y)=\begin{cases}\widetilde{\mathcal{R}}_{m,n}^j, & \varkappa\in\left(\varkappa_{m-\frac{1}{2}},\varkappa_{m+\frac{1}{2}}\right],\ y\in\left(y_{n-\frac{1}{2}},y_{n+\frac{1}{2}}\right],\\ & \quad m=1,2,\dots,M_1-1,\ n=1,2,\dots,M_2-1,\\ 0, & \varkappa\in\left[0,\frac{h_{\varkappa}}{2}\right]\ ,\varkappa\in\left(\text{Ł}-\frac{h_{\varkappa}}{2},\text{Ł}\right],\\ & y\in\left[0,\frac{h_y}{2}\right],\ y\in\left(\text{Ł}-\frac{h_y}{2},\text{Ł}\right],\end{cases}$$

and

$$\xi^j(\varkappa,y)=\begin{cases}\xi_{m,n}^j, & \varkappa\in\left(\varkappa_{m-\frac{1}{2}},\varkappa_{m+\frac{1}{2}}\right],\ y\in\left(y_{n-\frac{1}{2}},y_{n+\frac{1}{2}}\right],\\ & \quad m=1,2,\dots,M_1-1,\ n=1,2,\dots,M_2-1,\\ 0, & \varkappa\in\left[0,\frac{h_{\varkappa}}{2}\right]\ ,\varkappa\in\left(\text{Ł}-\frac{h_{\varkappa}}{2},\text{Ł}\right],\\ & y\in\left[0,\frac{h_y}{2}\right],\ y\in\left(\text{Ł}-\frac{h_y}{2},\text{Ł}\right],\end{cases}$$

for $1\le j\le K$.

Further, $\xi^j(\varkappa,y)$ and $\widetilde{\mathcal{R}}^j(\varkappa,y)$ can be stated as a Fourier series

$$\xi^j(\varkappa,y)=\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}\eta^j(i_1,i_2)e^{2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}}+\frac{i_2y}{\text{Ł}}\right)},$$
$$\widetilde{\mathcal{R}}^j(\varkappa,y)=\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}\S^j(i_1,i_2)e^{2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}}+\frac{i_2y}{\text{Ł}}\right)},$$

where

$$\eta^j(i_1,i_2)=\frac{1}{\text{Ł}^2}\int_0^{\text{Ł}}\int_0^{\text{Ł}}\xi^j(\varkappa,y)e^{-2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}}+\frac{i_2y}{\text{Ł}}\right)}d\varkappa dy,$$
$$\S^j(i_1,i_2)=\frac{1}{\text{Ł}^2}\int_0^{\text{Ł}}\int_0^{\text{Ł}}\widetilde{\mathcal{R}}^j(\varkappa,y)e^{-2\pi\iota\left(\frac{i_1\varkappa}{\text{Ł}}+\frac{i_2y}{\text{Ł}}\right)}d\varkappa dy.$$

By definition of L_2 discrete norm and Parseval's equality we get

$$\|\xi^j\|_2^2=\sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}h_{\varkappa}h_y|\xi_{m,n}^j|^2=\text{Ł}^2\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}|\eta^j(i_1,i_2)|^2, \tag{3.63}$$
$$\|\widetilde{\mathcal{R}}^j\|_2^2=\sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}h_{\varkappa}h_y|\widetilde{\mathcal{R}}_{m,n}^j|^2=\text{Ł}^2\sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty}|\S^j(i_1,i_2)|^2, \tag{3.64}$$

for $1\le j\le K$.

Suppose $\xi_{m,n}^j$ and $\widetilde{\mathcal{R}}_{m,n}^j$ have following form

$$\xi_{m,n}^j=\eta^je^{(\iota\theta_1mh_{\varkappa}+\iota\theta_2nh_y)},\ \widetilde{\mathcal{R}}_{m,n}^j=\S^je^{(\iota\theta_1mh_{\varkappa}+\iota\theta_2nh_y)}, \tag{3.65}$$

where $\theta_1 = \frac{2\pi i_1}{\text{L}}$, $\theta_2 = \frac{2\pi i_2}{\text{L}}$. Substituting (3.65) in (3.62) and using $\eta^0 = 0$, we get

$$\eta^{j+1} = \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta} p_1^{j+\theta} \tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta} p_2^{j+\theta} \tilde{\nu}_4\right)} \left(\frac{\tilde{\nu}_1 \tilde{\nu}_2}{\beta} \sum_{l=1}^{j} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) \eta^l - \frac{(1-\theta)}{\beta} \right.$$
$$\left. \times (p_1^{j+\theta} \tilde{\nu}_2 \tilde{\nu}_3 + p_2^{j+\theta} \tilde{\nu}_1 \tilde{\nu}_4 + \tilde{\nu}_1 \tilde{\nu}_2) \eta^j + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \tilde{\nu}_3 \tilde{\nu}_4 \eta^j + \S^{j+1} \right),\ 0 \le j \le K-1. \tag{3.66}$$

As we know, the series on the right side of (3.64) is convergent; therefore, for some constant $A > 0$, we have

$$|\S^j| \equiv |\S^j(i_1, i_2)| \le A\tau |\S^1(i_1, i_2)| \equiv A\tau |\S^1|,\ 1 \le j \le K. \tag{3.67}$$

Lemma 3.8. *For some constant $A > 0$, it holds*

$$|\eta^j| \le 9A(1+\tau)^j |\S^1|,\ 1 \le j \le K. \tag{3.68}$$

Proof. We will prove by mathematical induction on (3.66). For $j = 0$ and taking $\eta^0 = 0$, we have

$$\eta^1 = \frac{\S^1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta} p_1^{\theta} \tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta} p_2^{\theta} \tilde{\nu}_4\right)}\ .$$

By Eq. (3.67) and using the fact $\tilde{\nu}_1, \tilde{\nu}_2 \ge \frac{1}{3}$ we get

$$|\eta^1| \le 9A\tau |\S^1| \le 9A(1+\tau)|\S^1|\ . \tag{3.69}$$

Now, let us assume that

$$|\eta^d| \le 9A(1+\tau)^d |\S^1|, \tag{3.70}$$

is true for $1 \le d \le j$.

Next, for $d = j+1$, from Eq. (3.66) with Lemma 2.2, Eq. (3.67) and assumptions (3.70), we get

$$|\eta^{j+1}| \le \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta} p_1^{j+\theta} \tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta} p_2^{j+\theta} \tilde{\nu}_4\right)} \left(\frac{\tilde{\nu}_1 \tilde{\nu}_2}{\beta} \sum_{l=1}^{K-1} (r_{j,l}^{\alpha} - r_{j,l-1}^{\alpha}) \max_{1 \le k \le j} |\eta^k| \right.$$
$$\left. + \frac{(1-\theta)}{\beta} (p_1^{j+\theta} \tilde{\nu}_2 \tilde{\nu}_3 + p_2^{j+\theta} \tilde{\nu}_1 \tilde{\nu}_4 + \tilde{\nu}_1 \tilde{\nu}_2) |\eta^j| + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \tilde{\nu}_3 \tilde{\nu}_4 |\eta^j| + A\tau |\S^1| \right)$$
$$\le \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta} p_1^{j+\theta} \tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta} p_2^{j+\theta} \tilde{\nu}_4\right)} \left(\frac{\tilde{\nu}_1 \tilde{\nu}_2}{\beta} \left(r_{j,j}^{\alpha} - r_{j,0}^{\alpha}\right) + \frac{(1-\theta)}{\beta} (p_1^{j+\theta} \tilde{\nu}_2 \tilde{\nu}_3 + p_2^{j+\theta} \right.$$
$$\left. \times \tilde{\nu}_1 \tilde{\nu}_4 + \tilde{\nu}_1 \tilde{\nu}_2) + \frac{\theta^2 \tau^{2\alpha}}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \tilde{\nu}_3 \tilde{\nu}_4 + 9A(1+\tau)^j |\S^1| + A\tau |\S^1| \right)$$
$$\le \frac{1}{\left(\tilde{\nu}_1 + \frac{\theta}{\beta} p_1^{j+\theta} \tilde{\nu}_3\right)\left(\tilde{\nu}_2 + \frac{\theta}{\beta} p_2^{j+\theta} \tilde{\nu}_4\right)} \left(\frac{\tilde{\nu}_1 \tilde{\nu}_2}{\beta} \left((r_{j,j}^{\alpha} - r_{j,0}^{\alpha}) + \theta \tau^{\alpha}\right) + \frac{\theta}{\beta} (p_1^{j+\theta} \tilde{\nu}_2 \tilde{\nu}_3 + p_2^{j+\theta} \right.$$
$$\left. \times \tilde{\nu}_1 \tilde{\nu}_4) + \frac{\theta^2}{\beta^2} p_1^{j+\theta} p_2^{j+\theta} \tilde{\nu}_3 \tilde{\nu}_4 + 9A(1+\tau)^j |\S^1| + A\tau |\S^1| \right). \tag{3.71}$$

Again, invoking Lemma 2.2 will lead to

$$\begin{aligned} |\eta^{j+1}| &\le\ 9A(1+\tau)^j |\S^1| + 9A\tau |\S^1| \\ &\le 9A\left((1+\tau)^j + \tau\right) |\S^1| \\ &\le 9A(1+\tau)^{j+1} |\S^1|. \end{aligned} \tag{3.72}$$

Thus, we have the Lemma. □

Theorem 3.9. *Assume that the problem (1.1)-(1.2) has a solution $w(\varkappa, y, t)$, which meets the assumptions provided in (1.4) and let $\{W^j_{m,n} | 0 \le m \le M_1, 0 \le n \le M_2, 1 \le j \le K\}$ be the solution of cubic spline difference scheme (2.48). Then, we have*

$$\|w^j - W^j\|_2 \le c\left(K^{-\min\{1+\alpha, r\alpha\}} + h^2_\varkappa + h^2_y\right),\ 1 \le j \le K. \tag{3.73}$$

Proof. Incorporating Lemma 3.8, Eq. (3.63), and Eq. (3.64), we get

$$\|\xi^j\|_2^2 \le \text{Ł}^2 \sum_{i_1=-\infty}^{\infty}\sum_{i_2=-\infty}^{\infty} (9A)^2(1+\tau)^{2j}|\S^1(i_1,i_2)|^2 = (9A)^2(1+\tau)^{2j}\|\widetilde{\mathcal{R}}^1\|_2^2. \tag{3.74}$$

Now, utilizing Theorem (3.6) and Eq. (3.64), we have

$$\begin{aligned}\|\widetilde{\mathcal{R}}^j\|_2 &\le \sqrt{M_1 h_\varkappa}\sqrt{M_2 h_y}\left(c\left(K^{-\min\{r\alpha,1+\alpha\}} + h^2_\varkappa + h^2_y\right)\right)\\ &\le c\text{Ł}\left(K^{-\min\{r\alpha,1+\alpha\}} + h^2_\varkappa + h^2_y\right),\ 1 \le j \le K.\end{aligned} \tag{3.75}$$

Using (3.75) in (3.74) we get

$$\|\xi^j\|_2^2 \le (9A)^2 e^{2j\tau}(c\text{Ł})^2\left(K^{-\min\{r\alpha,1+\alpha\}} + h^2_\varkappa + h^2_y\right)^2.$$

As $j\tau \le T$, we obtain

$$\|\xi^j\|_2 \le B_1\left(K^{-\min\{r\alpha,1+\alpha\}} + h^2_\varkappa + h^2_y\right),$$

where $B_1 = 9Ac\text{Ł}e^T$.

Hence, we have the theorem. □

4. Numerical results

In this section, we present numerical results to demonstrate the accuracy and efficiency of the proposed method with the help of two examples. We calculate the order of convergence for the given examples using L_∞ and L_2 errors. We have compared the temporal order of convergence of the proposed method in one-dimension with the method described in [41] using L_∞ and L_2 errors:

$$L_2(h,\tau) = \max_{1\le j\le K}\left[h\sum_{m=1}^{M-1}\left(W(\varkappa_m,t_j) - w(\varkappa_m,t_j)\right)^2\right]^{\frac{1}{2}},$$
$$L_\infty(h,\tau) = \max_{1\le j\le K}\ \max_{1\le m\le M-1}\left|W(\varkappa_m,t_j) - w(\varkappa_m,t_j)\right|,$$

where $W(\varkappa_m,t_j)$ and $w(\varkappa_m,t_j)$ are the approximate and exact solutions at the point $(\varkappa_m,t_j)$ respectively.

The spatial order of convergence can be computed using the following formula

$$Co\ in\ |.|_l = \frac{\log(L_l(2h,\tau)) - \log(L_l(h,\tau))}{\log(2)},$$

where $l = 2, \infty$.

Similarly, the temporal order of convergence can be computed using the following formula

$$Co\ in\ |.|_l = \frac{\log(L_l(h,2\tau)) - \log(L_l(h,\tau))}{\log(2)},$$

where $l = 2, \infty$.

Further, the two-dimensional L_2 and L_∞ errors are defined as follows

$$L_2(h_\varkappa, h_y, \tau) = \max_{1\le j\le K}\left[h_\varkappa h_y \sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}\left(W(\varkappa_m, y_n, t_j) - w(\varkappa_m, y_n, t_j)\right)^2\right]^{\frac{1}{2}},$$
$$L_\infty(h_\varkappa, h_y, \tau) = \max_{1\le j\le K}\max_{\substack{1\le m\le M_1-1\\ 1\le n\le M_2-1}}\left|W(\varkappa_m, y_n, t_j) - w(\varkappa_m, y_n, t_j)\right|,$$

where $W(\varkappa_m, y_n, t_j)$ and $w(\varkappa_m, y_n, t_j)$ are the approximate and exact solutions respectively at the point $(\varkappa_m, y_n, t_j)$.

Moreover, the spatial order of convergence can be computed using the following formula

$$Co\ in\ |.|_l = \frac{\log(L_l(2h_\varkappa, 2h_y, \tau)) - \log(L_l(h\varkappa, h_y, \tau))}{\log(2)}.$$

Similarly, the temporal order of convergence can be computed using the following formula

$$Co\ in\ |.|_l = \frac{\log(L_l(h_\varkappa, h_y, 2\tau)) - \log(L_l(h\varkappa, h_y, \tau))}{\log(2)},$$

where $l = 2, \infty$.

Table 1. L_2-error and corresponding order of convergence at $M_1 = M_2 = 1000$ for Example 4.1.

		Uniform mesh			Non-uniform mesh		
α	K	L_2-error	Co in $\|.\|_2$	$CPU\ time(sec)$	L_2-error	Co in $\|.\|_2$	$CPU\ time(sec)$
0.3	80	7.2772e-03		2.12	5.6018e-05		2.13
	160	8.1528e-03		6.14	1.4468e-05	1.9531	6.55
	320	8.4037e-03		22.95	3.5102e-06	2.0432	22.95
	640	8.1318e-03		84.58	7.6010e-07	2.2073	84.98
0.5	80	8.0002e-03		2.16	4.1073e-05		2.13
	160	6.5787e-03	0.2822	6.45	1.0445e-05	1.9754	6.55
	320	5.1537e-03	0.3522	22.86	2.6269e-06	1.9913	22.95
	640	3.9108e-03	0.3981	83.63	6.5771e-07	1.9978	84.98
0.7	80	3.4938e-03		2.13	3.2372e-05		2.10
	160	2.3109e-03	0.5963	6.18	8.3571e-06	1.9534	6.56
	320	1.4859e-03	0.6371	22.76	2.1124e-06	1.9841	22.55
	640	9.3939e-04	0.6615	85.65	5.2987e-07	1.9952	84.14
0.9	80	6.4991e-04		2.15	1.6194e-05		2.11
	160	3.5951e-04	0.8542	6.64	4.5043e-06	1.8461	6.25
	320	1.9595e-04	0.8755	23.14	1.2150e-06	1.8903	22.20
	640	1.0596e-04	0.8869	85.84	2.9056e-07	2.0641	84.58

Example 4.1. [3] Consider the following test problem

$$\partial_t^\alpha w(\varkappa, t) = \frac{\partial^2 w(\varkappa, t)}{\partial \varkappa^2} - (1 - \sin(2t))w(\varkappa, t) + F(\varkappa, t),\ \ \varkappa \in (0, \pi),\ \ t \in (0, 1],$$

with initial and boundary conditions

$$w(\varkappa, 0) = 0,\ \ \varkappa \in [0, 1],$$
$$w(0, t) = 0,\ \ w(\pi, t) = 0,\ \ t \in (0, 1].$$

The source term is $F(\varkappa, t) = [t^{\alpha}(2 - \sin(2t)) + \Gamma(1+\alpha)]\sin(\varkappa)$ and the exact solution is $w(\varkappa, t) = t^{\alpha}\sin(\varkappa)$.

For a 1D problem, the convergence order in time is given by $K^{-\min\{2, r\alpha\}}$. If we set $r = 1$, the convergence order simplifies to α. When we set $r = \frac{1}{\alpha}$, the convergence order becomes 1. Alternatively, if we decide on $r \geq \frac{2}{\alpha}$, we attain the optimal convergence order 2. Therefore, in solving Example 4.1, we used $r = \frac{2}{\alpha}$.

Table 2. L_∞-error and corresponding order of convergence at $M_1 = M_2 = 1000$ for Example 4.1.

		Uniform mesh			Non-uniform mesh		
α	K	L_∞-error	*Co* in $\lvert.\rvert_\infty$	*CPU time*(*sec*)	L_∞-error	*Co* in $\lvert.\rvert_\infty$	*CPU time*(*sec*)
0.3	80	5.8064e-03		2.12	4.4696e-05		2.13
	160	6.5050e-03		6.14	1.1544e-05	1.9531	6.55
	320	6.7052e-03		22.95	2.8007e-06	2.0432	22.95
	640	6.4882e-03		84.58	6.0647e-07	2.2073	84.98
0.5	80	6.3833e-03		2.16	3.2772e-05		2.13
	160	5.2490e-03	0.2822	6.45	8.3335e-06	1.9754	6.41
	320	4.1121e-03	0.3521	22.86	2.0959e-06	1.9913	22.52
	640	3.1204e-03	0.3981	83.63	5.2477e-07	1.9978	84.74
0.7	80	2.7876e-03		2.13	2.5829e-05		2.10
	160	1.8438e-03	0.5963	6.18	6.6679e-06	1.9537	6.56
	320	1.1856e-03	0.6371	22.76	1.6855e-06	1.9841	22.55
	640	7.4952e-04	0.6616	85.65	4.2277e-07	1.9952	84.14
0.9	80	6.4991e-04		2.15	1.2921e-05		2.11
	160	3.5951e-04	0.8542	6.64	3.5939e-06	1.8461	6.25
	320	1.9595e-04	0.8755	23.14	9.6945e-07	1.8903	22.20
	640	1.0596e-04	0.8869	85.84	2.2816e-07	2.0871	84.58

Numerical results for 4.1 are presented in Tables 1, 2, and 3. Table 1 represents L_2-error and the corresponding temporal convergence order for different α values, comparing results on uniform and non-uniform graded meshes. Non-uniform graded meshes validate the theoretical findings, showing convergence order K^{-2}. Table 2 displays the L_∞-error and the corresponding order of convergence for various fractional orders α. The table indicates that on a non-uniform graded mesh, the temporal convergence order is numerically computed as K^{-2}, while on a uniform mesh, it decreases due to the singularity in the derivative. Tables 3 demonstrates L_∞ and L_2-error for $K = 500$, varying spatial mesh spacing h with $\alpha = 0.2, 0.4, 0.6, 0.8$ respectively. Decreasing mesh spacing h results in decreased errors, and the spatial convergence order is observed to be two, aligning with theoretical expectations. The cubic spline difference scheme consistently produces more accurate results.

Figure 1 illustrates surface plots of absolute errors on both uniform and non-uniform graded meshes for Example 4.1 with $M = N = 70$ and $\alpha = 0.5$. The graph shows that the error increases towards the initial time on a uniform mesh, whereas it reduces in the case of a non-uniform mesh due to mesh grading near $t = 0$. Figure 2 compares the exact and numerical solutions of Example 4.1 at different time levels when $\alpha = 0.6$ and $M = N = 80$, revealing a good match between the two.

Example 4.2. Consider the following test problem

$$\partial_t^{\alpha} w(\varkappa, y, t) = \frac{\partial^2 w(\varkappa, y, t)}{\partial \varkappa^2} + \frac{\partial^2 w(\varkappa, y, t)}{\partial y^2} - w(\varkappa, y, t) + F(\varkappa, y, t),$$
$$(\varkappa, y) \in (0,1) \times (0,1), \ t \in (0,1],$$

with initial and boundary conditions

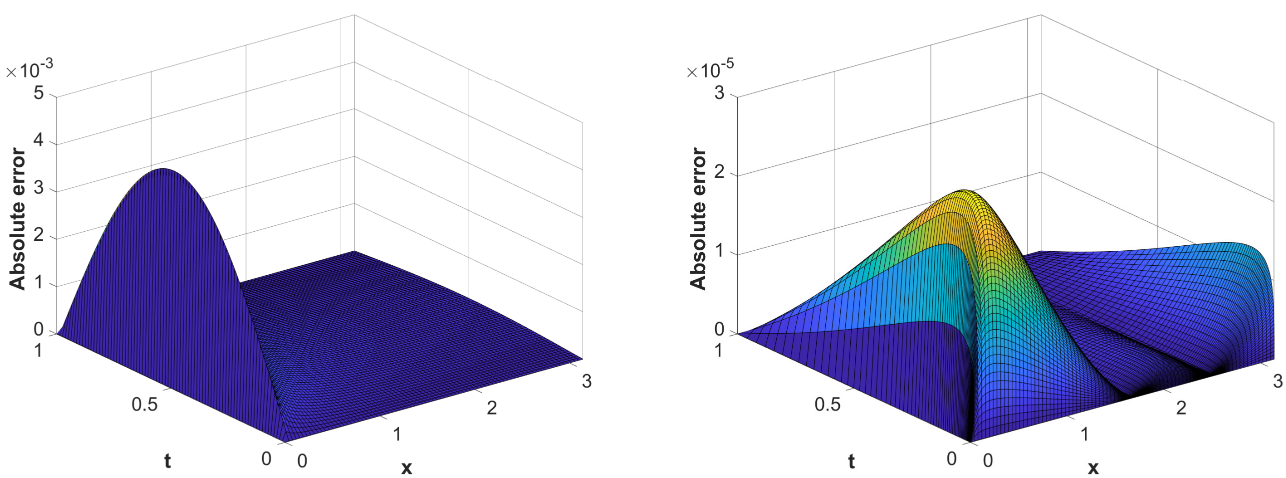
$$w(\varkappa, y, 0) = 0, \ (\varkappa, y) \in [0,1] \times [0,1],$$
$$w(\varkappa, y, t) = t^{\alpha} \exp(\varkappa + y), \ (\varkappa, y) \in \partial\Omega \ , \ t \in (0,1].$$

The source term is $F(\varkappa, y, t) = [\Gamma(1+\alpha) - t^{\alpha}] \exp(\varkappa + y)$ and the exact solution is $w(\varkappa, t) = t^{\alpha} \exp(\varkappa + y)$.

We have solved this example by selecting $r = \frac{1+\alpha}{\alpha}$ (optimal grading parameter). Tables 4 ,5 and 6 present the numerical outcomes for Example 4.2. Table 4 represents L_2-error and the corresponding temporal convergence order for different α values, comparing results on uniform and non-uniform graded meshes. Non-uniform graded meshes validate the theoretical findings, showing convergence order $K^{-\{1+\alpha\}}$. Table 5 displays the L_∞-error and the corresponding

TABLE 3. L_2 and L_∞ errors and corresponding spatial order of convergence for Example 4.1 with $K = 500$.

α	$M_1 = M_2$	L_2-error	Co in $\vert.\vert_2$	L_∞-error	Co in $\vert.\vert_\infty$	$CPU\ time(sec)$
0.4	2^2	3.1181e-02		2.4878e-02		46.46
	2^3	7.8276e-03	1.9940	6.2455e-03	1.9940	46.72
	2^4	1.9574e-03	1.9997	1.5617e-03	1.9997	46.74
	2^5	4.8842e-04	2.0027	3.8970e-04	2.0027	46.25
0.6	2^2	2.9662e-02		2.3667e-02		46.83
	2^3	7.4338e-03	1.9965	5.9313e-03	1.9965	46.63
	2^4	1.8584e-03	2.0001	1.4848e-03	2.0001	49.26
	2^5	4.6407e-04	2.0017	3.7025e-04	2.0017	46.76
0.8	2^2	2.7015e-02		2.1555e-02		47.62
	2^3	6.7535e-03	2.0009	5.3885e-03	2.0008	47.33
	2^4	1.6876e-03	2.0006	1.3465e-03	2.0006	46.76
	2^5	4.2168e-04	2.0007	3.3645e-04	2.0007	46.60



(a) On uniform mesh

(a) On non-uniform graded mesh

FIGURE 1. Surface plots of absolute error of Example 4.1 with $M = N = 70$ and $\alpha = 0.5$.

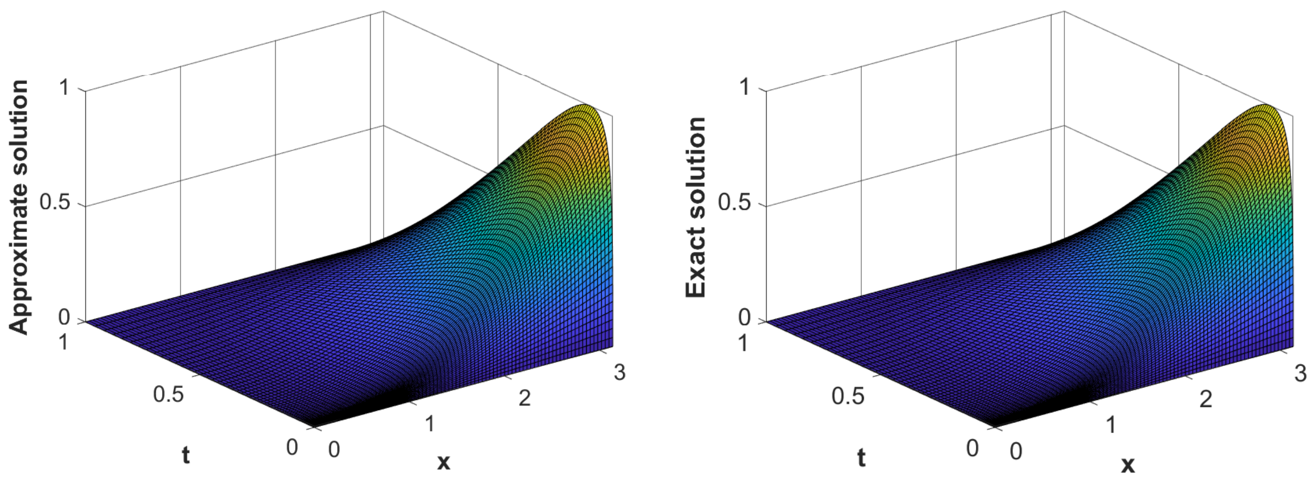


Figure 2. Surface plots of solutions of Example 4.1 with $M = N = 80$ and $\alpha = 0.6$.

order of convergence for various fractional orders α. The table indicates that on a non-uniform graded mesh, the temporal convergence order is numerically computed as $K^{-\{1+\alpha\}}$, while on a uniform mesh, it decreases due to the singularity in the derivative.

Tables 6 demonstrate L_∞ and L_2 errors for $K = \left\lceil M_1^{\frac{2}{1+\alpha}} \right\rceil$, varying spatial mesh spacing h with $\alpha = 0.4, 0.6, 0.8$ respectively. Decreasing mesh spacing h results in decreased errors, and the spatial convergence order is observed to be two, aligning with theoretical expectations. The cubic spline difference scheme consistently produces more accurate results.

5. Conclusions

In this article, we have proposed numerical methods for solving one and two-dimensional time-fractional reaction-diffusion equations defined in the Caputo sense, where the time-fractional derivative is discretized using L2-1_θ formula on a non-uniform graded mesh and the spatial discretization is done with the cubic spline difference scheme on a uniform mesh. Stability and convergence analysis are given for the numerical methods obtained for one and two dimensional time-fractional reaction-diffusion equations. For both one and two-dimensional problems, the theoretical analysis is demonstrated using the Fourier method. The proposed methods are shown to be convergent with an order of convergence of $\mathcal{O}(K^{-\min\{2,r\alpha\}}, h_x^2)$ in 1D and $\mathcal{O}(K^{-\min\{1+\alpha,r\alpha\}}, h_x^2, h_y^2)$ in 2D. Numerical outcomes indicate that the obtained results agree with the schemes theoretical findings. It is important to highlight that the method presented not only tackled the issue of weak singularity but also showcased its effectiveness in solving the time-fractional reaction-diffusion equation. However, for non-smooth solutions, the L2-1_θ approximation is fundamentally limited to a maximum temporal accuracy of two. To address this limitation, we aim to develop higher-order temporal discretization techniques in future research. Additionally, in nonlinear, higher-dimensional time-fractional problems, the computational expense remains significant due to the interplay of nonlinearity and dimensional complexity. Therefore, we also seek to explore innovative strategies to enhance computational efficiency while preserving accuracy.

TABLE 4. L_2-error and corresponding order of convergence at $M_1 = M_2 = 5$ for Example 4.2.

		Uniform mesh			Non-uniform mesh		
α	K	L_2-error	Co in $\|.\|_2$	$CPU\ time(sec)$	L_2-error	Co in $\|.\|_2$	$CPU\ time(sec)$
0.3	80	9.7986e-03		1.51	7.1228e-04		2.67
	160	9.0177e-03	0.1980	15.36	3.0486e-04	1.2243	5.29
	320	8.3885e-03	0.1043	20.82	1.2676e-04	1.2666	20.00
	640	7.8524e-03	0.0952	82.08	5.2109e-05	1.2824	78.73
0.5	80	8.6188e-03		1.54	3.4309e-04		3.42
	160	7.5939e-03	0.1826	5.25	1.2812e-04	1.4211	5.07
	320	6.5166e-03	0.2207	19.92	4.7006e-05	1.4465	19.54
	640	5.4242e-03	0.2647	78.77	1.6863e-05	1.4789	79.50
0.7	80	4.9519e-03		1.55	1.4162e-04		1.23
	160	3.6373e-03	0.4450	5.45	4.7481e-05	1.5766	5.20
	320	2.5492e-03	0.5128	21.08	1.5143e-05	1.6486	19.54
	640	1.7226e-03	0.5654	85.56	4.7292e-06	1.6790	78.54
0.9	80	1.1405e-03		1.56	3.6307e-05		1.49
	160	6.7897e-04	0.7482	5.48	1.1411e-05	1.6698	5.43
	320	3.8916e-04	0.8029	20.55	3.4209e-06	1.7379	20.51
	640	2.1730e-04	0.8406	83.13	9.4209e-07	1.8604	84.14

TABLE 5. L_∞-error and corresponding order of convergence at $M_1 = M_2 = 5$ for Example 4.2.

		Uniform mesh			Non-uniform mesh		
α	K	L_∞-error	Co in $\|.\|_\infty$	$CPU\ time(sec)$	L_∞-error	Co in $\|.\|_\infty$	$CPU\ time(sec)$
0.3	80	1.7887e-02		1.51	1.4907e-03		2.67
	160	1.6442e-02	0.1215	5.29	6.7715e-04	1.1384	15.36
	320	1.5277e-02	0.1065	20.00	2.8930e-04	1.2269	20.08
	640	1.4266e-02	0.09837	78.73	1.2031-04	1.2657	82.08
0.5	80	1.5571e-02		3.42	7.6472e-04		1.54
	160	1.3604e-02	0.1948	5.07	2.8455e-04	1.4263	5.25
	320	1.1526e-02	0.2391	19.54	1.0709e-04	1.4097	19.92
	640	9.4185e-03	0.2913	79.50	3.8953e-05	1.4591	78.77
0.7	80	8.6079e-03		1.23	3.0737e-04		1.55
	160	6.0967e-03	0.4976	5.20	1.0524e-04	1.6486	5.45
	320	4.3586e-03	0.4841	19.54	3.4705e-05	1.6004	21.08
	640	3.2151e-03	0.4390	78.54	1.0936e-05	1.6661	85.56
0.9	80	1.9822e-03		1.49	7.4266e-05		1.56
	160	1.3114e-03	0.5959	5.43	2.4327e-05	1.6101	5.48
	320	8.1010e-04	0.6949	20.51	7.5037e-06	1.6969	20.55
	640	4.7449e-04	0.7717	84.14	2.1038e-06	1.8343	83.13

TABLE 6. Spatial Error and order of convergence for Example 4.2 at $K = \left\lceil M_1^{\frac{2}{1+\alpha}} \right\rceil$.

α	$M_1 = M_2$	L_2-error	Co in $\|.\|_2$	L_∞-error	Co in $\|.\|_\infty$	$CPU\ time(sec)$
0.4	2^5	2.6191e-04		5.1984e-04		6.23
	2^6	7.0018e-05	1.9032	1.4741e-04	1.8182	55.47
	2^7	1.8024e-05	1.9578	3.9334e-05	1.9061	596.67
	2^8	4.3354e-06	2.0556	9.7765e-06	2.0082	2460
0.6	2^5	2.5584e-04		4.9055e-04		4.75
	2^6	7.2341e-05	1.8223	1.4693e-04	1.7392	27
	2^7	1.9337e-05	1.9034	4.1264e-05	1.8322	247
	2^8	4.9836e-06	1.9561	1.0987e-05	1.9090	2184
0.8	2^5	1.8770e-04		3.6022e-04		1.09
	2^6	5.7778e-05	1.6998	1.1415e-04	1.6579	4.679
	2^7	1.6323e-05	1.8231	3.3479e-05	1.7696	39.23
	2^8	4.3997e-06	1.8919	9.3492e-06	1.8403	368

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing is not applicable to this paper because no datasets were created or examined during the current study.

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