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On the Green function to the Poisson and the Helmholtz equations on the n-dimensional unit sphere

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Abstract

In the paper a new method is presented to obtain a closed form of the generalized Green function to the Poisson and the Helmholtz equations on the n-dimensional unit sphere.

Keywords. n-spheres, PDE, Poisson equation, Helmholtz equation, Green function.

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1. Introduction

In the recently published paper [2], written by Piotr Stefaniak and me, a formula for the Green function to the Poisson and the Helmholtz equations $\Delta^* u + au = f$ on the n-dimensional unit sphere was derived.

The method we used was based on the spherical wavelet transform derived from approximate identities, since the Laplace-Beltrami operator in the wavelet space is expressed by a multiplication with a polynomial and can be easily inverted. The solution u to the Helmholtz or Poisson equation is given as a convolution of function f with an integral kernel G which is explicitely given as a series. Using the fact that both the wavelet transform and the inverse wavelet transform are convergent, we established convergence of the series defining the Green function G, something that had been omitted in the proofs known so far. Further, for distinct dimensions and distinct parameters, we synthesized the kernel to an explicitely given function. The method we used was based a double integration of the Poisson kernel [2, Theorem 2], whose series expansion is known. It was a new idea and in this way we obtained an explicit representation of the kernel for a wider range of indices than it had been known so far.

In the present paper, the method of kernel synthesizing is further simplified. Namely, it is shown that it can be done by a single integration of the Poisson kernel (Theorem 3.1). This allows to find an explicit formula for the kernel for a much wider range of indices than in [2]. If $a = L(n + L - 1)$ and $L \in \left(1 - n, \frac{1-n}{2}\right) \cup \left(\frac{1-n}{2}, 0\right)$, G can be expressed in terms of special functions (Theorem 4.1). If n is even and L integer, the integration can be performed via Euler substitution. Further, for odd dimensions n and rational L (different from $\frac{1-n}{2}$), with a proper substitution one obtains an integral of a rational function. The last two cases are illustrated on examples.

2. Preliminaries

A square integrable function f over the n-dimensional unit sphere $\mathcal{S}^n \subseteq \mathbb{R}^{n+1}$, $n \geq 2$, with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma(x) = \frac{2\pi^{(n+1)/2}}{\Gamma\left((n+1)/2\right)},$$

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can be represented as a Fourier series in terms of the hyperspherical harmonics,

$$f = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a_l^k(f)\, Y_l^k, \tag{2.1}$$

where $\mathcal{M}_{n-1}(l)$ denotes the set of sequences $k = (k_0, k_1, \dots, k_{n-1})$ in $\mathbb{N}_0^{n-1} \times \mathbb{Z}$ such that $l \geq k_0 \geq k_1 \geq \dots \geq |k_{n-1}|$ and $a_l^k(f)$ are the Fourier coefficients of f. The hyperspherical harmonics of degree l and order k are given by

$$Y_l^k(x) = A_l^k \prod_{\tau=1}^{n-1} C_{k_{\tau-1}-k_\tau}^{\frac{n-\tau}{2}+k_\tau}(t_\tau) \sin^{k_\tau} \vartheta_\tau \cdot e^{\pm i k_{n-1} \varphi} \tag{2.2}$$

for some constants A_l^k. Here, $(\vartheta_1, \dots, \vartheta_{n-1}, \varphi)$ are the hyperspherical coordinates of $x \in \mathcal{S}^n$,

$$\begin{aligned} x_1 &= \cos \vartheta_1, \\ x_2 &= \sin \vartheta_1 \cos \vartheta_2, \\ &\dots \\ x_n &= \sin \vartheta_1 \dots \sin \vartheta_{n-2} \sin \vartheta_{n-1} \cos \varphi, \\ x_{n+1} &= \sin \vartheta_1 \dots \sin \vartheta_{n-2} \sin \vartheta_{n-1} \sin \varphi, \end{aligned}$$

and $\mathcal{C}_\kappa^K$ are the Gegenbauer polynomials of degree κ and order K.

Zonal (rotation-invariant) functions are those depending only on the first hyperspherical coordinate $\vartheta = \vartheta_1$. Unless it leads to misunderstandings, we identify them with functions of ϑ or $t = \cos \vartheta$. A zonal $\mathcal{L}^1$-function f has the following Gegenbauer expansion

$$f(t) = \sum_{l=0}^{\infty} \widehat{f}(l)\, C_l^\lambda(t), \qquad t = \cos \vartheta, \tag{2.3}$$

where $\widehat{f}(l)$ are the Gegenbauer coefficients of f and λ is related to the space dimension by

$$\lambda = \frac{n-1}{2}.$$

Consequently, for a zonal $\mathcal{L}^2$-function f one has

$$\widehat{f}(l) = A_l^0 \cdot a_l^0(f),$$

compare (2.1), (2.2), and (2.3).

For $f, g \in \mathcal{L}^1(\mathcal{S}^n)$, g zonal, their convolution $f * g$ is defined by

$$(f * g)(x) = \frac{1}{\Sigma_n} \int_{\mathcal{S}^n} f(y)\, \tau_x g(y)\, d\sigma(y), \qquad \tau_x g(y) = g(x \cdot y), \tag{2.4}$$

and for $f \in \mathcal{L}^2(\mathcal{S}^n)$ it is equal to

$$f * g = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} \frac{\lambda}{\lambda + l}\, a_l^k(f)\, \widehat{g}(l)\, Y_l^k.$$

The Laplace-Beltrami operator Δ^* on the sphere is defined by

$$\begin{aligned} \Delta^* f(\vartheta_1, \dots, \vartheta_{n-1}, \varphi) = \sum_{k=1}^{n-1} \left(\prod_{j=1}^{k} \sin \vartheta_j \right)^{-2} (\sin \vartheta_k)^{k+2-n} \frac{\partial}{\partial \vartheta_k} \left[\sin^{n-k} \vartheta_k \frac{\partial f(\vartheta_1, \dots, \vartheta_{n-1}, \varphi)}{\partial \vartheta_k} \right] \\ + \left(\prod_{j=1}^{k} \sin \vartheta_j \right)^{-2} \frac{\partial^2 f(\vartheta_1, \dots, \vartheta_{n-1}, \varphi)}{\partial \varphi^2}. \end{aligned}$$

It is known that the hyperspherical harmonics are the eigenfunctions of Δ^*, i.e.,

$$\Delta^* Y_l^k = -l(n+l-1)Y_l^k, \tag{2.5}$$

see [3, Chapter II, Theorem 4.1]. The relation of Δ^* and the Laplace operator Δ is given by

$$\Delta f = R^{-n}\frac{\partial}{\partial R}\left(R^n \frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\Delta^* f, \tag{2.6}$$

where $R \geq 0$ is the radial distance of $x \in \mathbb{R}^n$ in the hyperspherical coordinates, see [3, Chapter II, Proposition 3.3].

The Poisson kernel for the n-dimensional sphere is given by

$$p_r(y) = \frac{1}{\Sigma_n}\sum_{l=0}^{\infty} r^l \frac{\lambda+l}{\lambda}\mathcal{C}_l^{\lambda}(t) = \frac{1}{\Sigma_n}\frac{1-r^2}{(1-2rt+r^2)^{(n+1)/2}}, \qquad r \in [0,1). \tag{2.7}$$

The following statements are the content of [2, Theorem 1] together with [2, Remark 2, point 2].

(1) Suppose that $f \in \mathcal{C}(\mathcal{S}^n)$ and $u \in \mathcal{C}^2(\mathcal{S}^n)$ satisfy

$$\Delta^* u + au = f, \tag{2.8}$$

where $a \in \mathbb{R} \setminus \{L(n+L-1),\, L \in \mathbb{N}_0\}$. Then,

$$u = f * G, \tag{2.9}$$

for

$$G = \sum_{l=0}^{\infty} \frac{1}{a - l(n+l-1)} \frac{\lambda+l}{\lambda} \mathcal{C}_l^{\lambda}. \tag{2.10}$$

(2) Suppose that $f \in \mathcal{C}(\mathcal{S}^n)$, $u \in \mathcal{C}^2(\mathcal{S}^n)$ are such that $f * \mathcal{C}_L^{\lambda} = u * \mathcal{C}_L^{\lambda} = 0$. Then,

$$u = f * G, \tag{2.11}$$

for

$$G = \sum_{l=0, l\neq L}^{\infty} \frac{1}{a - l(n+l-1)} \frac{\lambda+l}{\lambda} \mathcal{C}_l^{\lambda}. \tag{2.12}$$

3. The main theorem

In this section, a formula is given to sum up the series (2.10) or (2.12). The advantage with respect to [2, Theorem 2] is that the number of integrations is reduced to one.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, be fixed, $\lambda = \frac{n-1}{2}$, and suppose that $a = L(n+L-1)$ for some $L \in \mathbb{R} \setminus \mathbb{Z}$, $L \neq \frac{1-n}{2}$. Let $L_0 := \max\{[L], [-n-L+1]\}$, where $[x]$ stays for the biggest integer less than or equal to x. Denote by G the function*

$$G := \sum_{l=0}^{\infty} \frac{1}{a - l(n+l-1)} \frac{\lambda+l}{\lambda} \mathcal{C}_l^{\lambda}. \tag{3.1}$$

Then

$$G(t) = \sum_{l=0}^{L_0} \frac{1}{a - l(n+l-1)} \cdot \frac{\lambda+l}{\lambda} \mathcal{C}_l^{\lambda}(t) \tag{3.2}$$

$$+ \frac{1}{n+2L-1}\int_0^1 \left(r^{n+L-2} - r^{-L-1}\right)\left(\Sigma_n p_r(t) - \sum_{l=0}^{L_0} r^l \frac{\lambda+l}{\lambda}\mathcal{C}_l^{\lambda}(t)\right) dr. \tag{3.3}$$

(If $L_0 < 0$, set $\sum_0^{L_0} = 0$).

Proof. Analogous to the proof of [2, Theorem 2]. Note that

$$\frac{1}{n+2L-1}\int_0^1 \left(r^{n+L-2}-r^{-L-1}\right)\cdot r^l\,dr = \frac{1}{(L-l)(n+L+l-1)}, \tag{3.4}$$

for $l > L_0$. □

Corollary 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, be fixed, $\lambda = \frac{n-1}{2}$, and suppose that $a = L(n+L-1)$ for some $L \in \mathbb{N}_0$. Further, let G denote the function*

$$G = \sum_{l=0, l\neq L}^{\infty} \frac{1}{a-l(n+l-1)}\,\frac{\lambda+l}{\lambda}\,\mathcal{C}_l^{\lambda}. \tag{3.5}$$

Then

$$G(t) = \sum_{l=0}^{L-1} \frac{1}{a-l(n+l-1)}\cdot\frac{\lambda+l}{\lambda}\,\mathcal{C}_l^{\lambda}(t) \tag{3.6}$$

$$+\frac{1}{n+2L-1}\int_0^1 \left(r^{n+L-2}-r^{-L-1}\right)\left(\Sigma_n\, p_r(t) - \sum_{l=0}^{L} r^l\,\frac{\lambda+l}{\lambda}\,\mathcal{C}_l^{\lambda}(t)\right)dr. \tag{3.7}$$

Remark 3.3. If $a = L(n+L-1)$, then also $a = L'(n+L'-1)$ for $L' = -n-L+1$. If $-\frac{(n-1)^2}{4} \leq a < 0$, both L and L' are negative (this case was not considered in [2, Remark 3, p. 28]). Choose $L = 0$ for $a = 0$ (Poisson equation) and L positive for positive a in order to apply Corollary 3.2.

4. Closed forms of the Green functions

In this section, a closed formula for the Green kernel is derived for the case $a = L(n+L-1)$,

$$L \in \left(1-n, \frac{1-n}{2}\right) \cup \left(\frac{1-n}{2}, 0\right). \tag{4.1}$$

It is expressed in terms of Appell F_1-function.

On the other hand, for some special cases, like n even and L integer (not necessarily satisfying condition (4.1)) or n odd and L rational (again not necessarily with (4.1)), integration (3.3) or (3.7), and, consequently, synthesis of the kernel G, can be performed with standard methods, which will be shown on examples.

Theorem 4.1. *Let $a = L(n+L-1)$ for $L \in (1-n, \frac{1-n}{2}) \cup (\frac{1-n}{2}, 0)$. Then*

$$\begin{aligned} G(t) = \frac{1}{n+2L-1}\cdot\Bigg[&\frac{F_1(n+L-1;\lambda+1,\lambda+1;n+L;e^{i\vartheta},e^{-i\vartheta})}{n+L-1} \\ &- \frac{F_1(n+L+1;\lambda+1,\lambda+1;n+L+2;e^{i\vartheta},e^{-i\vartheta})}{n+L+1} \\ &- \frac{F_1(-L;\lambda+1,\lambda+1;-L+1;e^{i\vartheta},e^{-i\vartheta})}{-L} + \frac{F_1(-L+2;\lambda+1,\lambda+1;-L+3;e^{i\vartheta},e^{-i\vartheta})}{-L+2}\Bigg]. \end{aligned} \tag{4.2}$$

Proof. Since $1-n < L < 0$, L_0 defined in Theorem 3.1 is less than or equal to -1 and

$$G(t) = \frac{1}{n+2L-1}\int_0^1 \left(r^{n+L-2}-r^{-L-1}\right)\frac{1-r^2}{(1-2rt+r^2)^{\lambda+1}}\,dr \tag{4.3}$$

$$= \frac{1}{n+2L-1}\int_0^1 \frac{r^{n+L-2}-r^{n+L}-r^{-L-1}+r^{-L+1}}{(1-2rt+r^2)^{\lambda+1}}\,dr. \tag{4.4}$$

All the exponents in the numerator of the integrand are greater than -1.

According to [1, 9.3(4)],

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\,du = \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)} F_1(\alpha;\beta,\beta';\gamma;x,y), \tag{4.5}$$

where F_1 is the Appell F_1-function and $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma$. Since

$$1 - 2rt + r^2 = (1 - re^{i\vartheta})(1 - re^{-i\vartheta}), \tag{4.6}$$

we obtain from (4.5)

$$\int_0^1 \frac{r^\alpha}{(1-2rt+r^2)^{\lambda+1}}\,dr = \frac{F_1(\alpha;\lambda+1,\lambda+1;\alpha+1;e^{i\vartheta},e^{-i\vartheta})}{\alpha}. \tag{4.7}$$

With this formula, (4.2) follows immediately from (4.4). □

If n is even and L integer, the integrand in (4.4), or, more general, in (3.7), is a rational function of r and $\sqrt{1-2rt+r^2}$. In this case, the function can be integrated with Euler substitution.

Example 4.2. For $n = 2$ and $L = 1$, i.e., $\lambda = \frac{1}{2}$ and $a = 2$, the integrand in (3.7) equals

$$I_{2,1} := \left(r - \frac{1}{r^2}\right)\left(\frac{1-r^2}{(1-2rt+r^2)^{3/2}} - 1 - 3rt\right), \qquad t := \cos\vartheta.$$

With the second Euler substitution $\sqrt{1-2rt+r^2} = rx + 1$, i.e., $r = \frac{2(x+t)}{1-x^2}$, one obtains

$$\begin{aligned}
I_{2,1} = &-\frac{1}{2}(7-24t^2) - 4xt + \frac{x^2}{2} - \frac{(1+t)(1-3t+3t^2)}{1-x} - \frac{3t(1+t)^2}{(1-x)^2}\\
&+ \frac{(1-t)(1+3t+3t^2)}{1+x} - \frac{3(1-t)^2 t}{(1+x)^2}\\
&+ \frac{2(7-3t-34t^2+24t^4+x+13xt-20xt^3)}{1+2xt+x^2}\\
&- \frac{8(1-t^2)(3-t-16t^2+8t^4+2x+8xt-16xt^3)}{(1+2xt+x^2)^2}\\
&+ \frac{16(1-t^2)^2(1-4t^2+x+4xt-8xt^3)}{(1+2xt+x^2)^3}
\end{aligned}$$

(note that $-1 < x < 1$ for $r \in (0,1)$ and $t \in (-1,1)$). The indefinite integral of I with respect to

$$dr = \frac{2(1+2xt+x^2)}{(1-x^2)^2}\,dx$$

equals

$$\begin{aligned}
\int I_{2,1} = &\int \frac{1-2t-2x-x^2}{(1-x^2)^4(1+2tx+x^2)^2}\cdot(1+2t+4t^2+2x+8tx+2x^2-2tx^2-2x^3+x^4)\cdot\\
&\cdot(5+18tx-7x^2+36t^2x^2+4tx^3+24t^3x^3+3x^4+12t^2x^4+2tx^5-x^6)\,dx\\
&= x - \frac{2tx}{1-x^2} - \frac{2-22t^2-4tx}{(1-x^2)^2} - \frac{8t(3t+t^3+x+3t^2x)}{(1-x^2)^3} - \frac{4(1+t-2t^2)(1-x)}{(1+2tx+x^2)} - 6t\ln(1+x) + C.
\end{aligned}$$

The integration bounds $r_1 = 0$ and $r_2 = 1$ change to $x_1 = \lim_{\epsilon\to 0}\frac{\sqrt{1-2\epsilon t+\epsilon^2}-1}{\epsilon} = -t$ and $x_2 = \sqrt{2-2t}-1$. Therefore, the definite integral in (3.7) equals

$$\int_0^1 I_{2,1}\,dr = \frac{3}{2} + (4-\ln 8)t + 3t\ln(1-t)$$

and

$$G(t) = \frac{1}{2} + \frac{1}{3}\int_0^1 I_{2,1}\, dr = 1 + \frac{4}{3}t + t\ln\frac{1-t}{2}.$$

That result coincides with that obtained in [2].

On the other hand, if n is odd and L rational, the integral in (3.7) can be transformed to an integral of a rational function.

Example 4.3. Suppose, $n = 3$ and $L = \frac{1}{2}$, i.e., $\lambda = 1$ and $a = \frac{5}{4}$. In this case, the integral in (3.7) equals

$$I_{3,1/2} := \int_0^1 \left(r^{3/2} - r^{-3/2}\right)\left(\frac{1-r^2}{(1-2rt+r^2)^2} - 1\right) dr, \qquad t := \cos\vartheta,$$

i.e.,

$$I_{3,1/2} = \int_0^1 \frac{-4tr^{-1/2} + (3+4t^2)r^{1/2} - 4tr^{3/2} + (1+4t)r^{5/2} - (3+4t^2)r^{7/2} + 4tr^{9/2} - r^{11/2}}{(1-2rt+r^2)^2}\, dr.$$

With substitution $\rho = r^{1/2}$ one obtains

$$\begin{aligned} I_{3,1/2} &= 2\int_0^1 \frac{\left(1-\rho^6\right)\left(-4t + (3+4t^2)\rho^2 - 4t\rho^4 + \rho^6\right)}{(1-2t\rho^2+\rho^4)^2}\, d\rho \\ &= \int_0^1 \left[-2 - 2\rho^4 + \frac{1-4t}{1-2c\rho+\rho^2} + \frac{(1+t-2t^2)\,\rho}{2c\,(1-2c\rho+\rho^2)^2} + \frac{1-4t}{1+2c\rho+\rho^2} - \frac{(1+t-2t^2)\,\rho}{2c\,(1+2c\rho+\rho^2)^2}\right] d\rho, \end{aligned}$$

where

$$c = \cos\frac{\vartheta}{2} = \sqrt{\frac{1+t}{2}}.$$

Now,

$$\begin{aligned} \mathcal{I}_1(\rho) &:= \int \left[\frac{1}{1-2c\rho+\rho^2} + \frac{1}{1+2c\rho+\rho^2}\right] d\rho \\ &= \frac{1}{\sqrt{1-c^2}}\left[\arctan\frac{\rho-c}{\sqrt{1-c^2}} + \arctan\frac{\rho+c}{\sqrt{1-c^2}}\right] + C = \frac{1}{\sqrt{1-c^2}}\arctan\frac{2\sqrt{1-c^2}\,\rho}{1-\rho^2} + C \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_2(\rho) &:= \int \left[\frac{\rho}{(1-2c\rho+\rho^2)^2} - \frac{\rho}{(1+2c\rho+\rho^2)^2}\right] d\rho \\ &= \frac{-1}{2(1-c^2)}\left[\frac{1-c\rho}{1-2c\rho+\rho^2} - \frac{c\arctan\frac{\rho-c}{\sqrt{1-c^2}}}{\sqrt{1-c^2}} - \frac{1+c\rho}{1+2c\rho+\rho^2} - \frac{c\arctan\frac{c+\rho}{\sqrt{1-c^2}}}{\sqrt{1-c^2}}\right] + C \\ &= \frac{1}{2(1-c^2)}\left[\frac{-2c\rho\,(1-\rho^2)}{1-2t\rho^2+\rho^4} + \frac{c}{\sqrt{1-c^2}}\arctan\frac{2\sqrt{1-c^2}\,\rho}{1-\rho^2}\right] + C. \end{aligned}$$

Thus,

$$\begin{aligned} I_{3,1/2} &= -\frac{12}{5} + (1-4t)\cdot\left[\lim_{\rho\to 1}\mathcal{I}_1(\rho) - \mathcal{I}_1(0)\right] + \frac{1+t-2t^2}{2c}\cdot\left[\lim_{\rho\to 1}\mathcal{I}_1(\rho) - \mathcal{I}_1(0)\right] \\ &= -\frac{12}{5} + (1-4t)\cdot\frac{1}{\sqrt{1-c^2}}\cdot\frac{\pi}{2} + \frac{1+t-2t^2}{2c}\cdot\frac{1}{2(1-c^2)}\cdot\frac{c}{\sqrt{1-c^2}}\cdot\frac{\pi}{2} \\ &= -\frac{12}{5} + \frac{(3-6t)\pi}{2\sqrt{2-2t}} \end{aligned}$$

and

$$G(t) = \frac{4}{5} + \frac{1}{3}I_{3,1/2} = \frac{(1-2t)\pi}{2\sqrt{2-2t}},$$

which is the same result as in [2]. (Note that since

$$\frac{1+t}{\sqrt{1-t}} + \sqrt{1-t} = \frac{2}{\sqrt{1-t}},$$

all the expressions in [2, Table 3] can be simplified).

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