Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-13
DOI:10.22034/cmde.2025.61363.2639

Exactness of solution to the stochastic fractional impulsive differential equations

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Abstract

This paper investigates the averaging principle for the solutions to stochastic fractional impulsive differential equations (SFIDEs) with nonlocal conditions. The main focus lies in deriving sufficient conditions for the convergence of the averaged SFIDEs. According to certain proposals, solutions to SFIDEs can be approximated by averaged stochastic systems using mean square. Furthermore, two illustrative examples are provided to demonstrate the effectiveness of the proposed method in approximating the solutions to our model. The numerical simulations highlight the applicability and accuracy of the proposed approach in practical scenarios. This work contributes to the understanding and analysis of SFIDEs with complex conditions, paving the way for further research in the field of finance and industry.

Keywords. Fractional derivative, Impulse, Stochastic differential equation, Brownian motion, Averaging method.
2010 Mathematics Subject Classification. 26A33, 34A37, 60H10, 60J65, 34C29.

1. Introduction

Fractional Differential Equations (FDEs) provide a powerful tool for modeling and understanding complex phenomena in various fields of science and engineering. With the increasing interest in Fractional Calculus and its applications, the study of FDEs continues to be an active area of research, paving the way for innovative solutions to problems that cannot be addressed using classical calculus. Also, Stochastic Differential Equations (SDEs) contribute a significant mathematical framework for modeling systems that involve both deterministic and random components [12, 15–18]. The importance of modeling and understanding systems with inherent uncertainties and fluctuations, the study of SDEs continues to be an active and important area of research across various disciplines. Together, SFIDEs serve as crucial mathematical models for describing dynamic systems to random fluctuations and delays in their evolution. These equations find extensive applications across various fields, including physics, engineering, biology, and finance, among others [10, 11, 20]. SFIDEs exhibit intricate behaviors due to the combined effects of fractional derivatives, stochastic perturbations, and neutral delays, making their analysis and solution challenging tasks. Recently, there has been growing interest in studying SFIDEs with impulse and nonlocal conditions, as these conditions capture real-world phenomena more accurately.

In financial mathematics [1, 2, 5, 13], SFIDEs with impulse and nonlocal conditions can be used to model the dynamics of asset prices and interest rates, leading to more accurate option pricing models that account for sudden market shocks (impulses) and long-range dependencies (nonlocality). SFIDEs can be employed to develop risk management models that capture the impact of extreme market events (impulses) and systemic risks (nonlocal effects) on portfolios and financial derivatives. Next, as can be seen in [7], Khasminskii refined this technique to address a category of second-order parabolic partial differential equations. After that, the averaging principle for partial differential equations caused a great deal of work concern. Khasminskii originally examined the averaging principle for stochastic differential equations in [8, 9]. Since then, a lot of effort has gone into expanding this theory, the averaging principle is a decisive mathematical technique widely used to analyze the long-term behavior of deterministic and stochastic

Received: 25 April 2024 ; Accepted: 22 May 2025.
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differential equations, one can refer [3, 14, 19, 22, 23]. However, applying this principle to SFIDEs with impulse and nonlocal conditions requires careful consideration and adaptation to account for the fractional nature of the derivatives and the presence of stochastic perturbations. These applications demonstrate the versatility of stochastic fractional neutral differential equations in capturing the intricate dynamics of systems across various scientific and engineering disciplines.

In this paper, we aim to extend the classical averaging principle to the realm of SFIDEs with impulse and nonlocal conditions. Our goal is to establish the existence and uniqueness of solutions to such equations and investigate the convergence properties of the averaged SFIDEs. By developing novel mathematical techniques and deriving suitable conditions, we seek to provide a rigorous framework for approximating the solutions to SFIDEs with complex conditions.

Consider the following SFIDEs with impulse and nonlocal condition of the form:

$$\begin{aligned} &{}^{C}D^{q}[X(t)-\mathfrak{h}_3(t,X(t))]=\mathfrak{h}_1(t,X(t))+\mathfrak{h}_2(t,X(t))\frac{dW(t)}{dt}, \qquad t\in\mathcal{J}=[0,\alpha],\\ &\Delta X(t_k)=I_k(X(t_k^-)), \quad t=t_k, \quad k=1,2,3,...,l,\\ &X(0)+\mathfrak{h}_4(X)=X_0, \end{aligned} \tag{1.1}$$

where $q\in(\frac{1}{2},1)$, $\mathfrak{h}_1,\mathfrak{h}_3,\in C(\mathcal{J}\times\mathbb{R}^n,\ \mathbb{R}^n)$, $\mathfrak{h}_2\in C(\mathcal{J}\times\mathbb{R}^n,\ \mathbb{R}^{n\times m})$, $\mathfrak{h}_4$ is a continuous function on $\mathbb{R}^n$, $\Delta X(t_k)=X(t_k^+)-X(t_k^-)$ for $t=t_k$, $X(t_k^+)=\lim_{h\to 0^+}X(t_k+h)$, $I_k:\mathbb{R}^n\to\mathbb{R}^n$ $(k=1,2,3,...,l)$ stands for impulsive disruption of $X(t)$ at time t_k, t_k fulfills $0=t_0<t_1<\cdots<t_k<t_{k+1}=\alpha$. The left and right limits of I_k at time t_k are denoted by $I(t_k^-)$ and $I(t_k^+)$, respectively. The sudden change in state I at time t_k is represented by $\Delta X(t_k)$. And $W=\{W(t),t\geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$. The initial value X_0 is a random variable in $\mathbb{R}^n$ that is measurable with respect to $\mathcal{F}_0$ and it meets the condition $\mathbb{E}|X_0|^2<\infty$.

The structure of this document is as follows: The fundamental notations, definitions, lemmas, and theorems are all contained in section 2. In section 3, the averaging principle is discussed. An illustrated example is given in section 4 to support the developed theory. Finally, section 5 concludes the paper and outlines potential future research directions.

2. Preliminaries

This section includes an introduction to certain fundamental terms, definitions, lemmas, and theorems.

Definition 2.1. [6] The Riemann-Liouville and Caputo derivative of order q for the function $\mathfrak{h}:\mathbb{R}^+\to\mathbb{R}$ with the lower limit zero is expressed as

(1) ${}^{L}\mathcal{D}^{q}\mathfrak{h}(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int\limits_0^t(t-\nu)^{n-q-1}\mathfrak{h}(\nu)d\nu, \quad t>0, \quad q\in(n-1,n),$

(2) ${}^{C}\mathcal{D}^{q}\mathfrak{h}(t)={}^{L}\mathcal{D}^{q}\big(\mathfrak{h}(t)-\sum\limits_{m=0}^{n-1}\frac{t^m}{m!}\mathfrak{h}^m(0)\big), \quad t>0, \quad q\in(n-1,n).$

Definition 2.2. [22] Considering that W is a typical wiener process,

$$\eta\Big[\int\limits_0^s X(t)dW(t)\Big]\leq\sqrt{\alpha}\eta\big(X(t)\big), \text{ it is given that } X\subset C\big([0,\alpha];\mathbb{R}^{n\times m}\big),$$

where

$$\int\limits_0^s X(t)dW(t)=\Big\{\int\limits_0^s x(t)dW(t): \text{ for all } x\in X, s\in[0,\alpha]\Big\}.$$

Lemma 2.3. *[16] Let $\mathbb{R}(q)>0$ and let $n=[\mathbb{R}(q)]+1$ for $q\notin\mathbb{N}_0$; $n=q$ for $q\in\mathbb{N}_0$. If $\mathfrak{h}(x)\in AC^n[a,b]$ or $\mathfrak{h}(x)\in C^n[a,b]$, then*

$$\big(\mathcal{I}_{a+}^{q}\,{}^{C}D_{a+}^{q}\mathfrak{h}\big)(x)=\mathfrak{h}(x)-\sum_{k=0}^{n-1}\frac{\mathfrak{h}^{(k)}(a)}{k!}(x-a)^k.$$

Lemma 2.4. *[4] Suppose $\vartheta \geq 0$, $q > 0$, and a function $\varphi(t) \geq 0$ which is integrable locally on $0 \leq t < \alpha$ ($\alpha \leq +\infty$), and for instance $\mathfrak{h}(t) \geq 0$ and integrable locally on $0 \leq t < \alpha$ with*

$$\mathfrak{h}(t) \leq \varphi(t) + \vartheta \int_0^t (t-s)^{q-1}\mathfrak{h}(s)ds,$$

then

$$\mathfrak{h}(t) \leq \varphi(t) + \int_0^t \sum_{n=1}^{\infty} \frac{\big(\vartheta\Gamma(q)\big)^n}{\Gamma(nq)}(t-s)^{nq-1}\varphi(s)ds, \quad 0 \leq t < \alpha,$$

where $\Gamma(\cdot)$ is the Gamma function.

Theorem 2.5. *(Cauchy-Schwarz inequality) If $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_m$ are arbitrary real numbers, we have*

$$\Big(\sum_{r=1}^{m} u_r v_r\Big)^2 \leq \sum_{r=1}^{m} u_r^2 \cdot \sum_{r=1}^{m} v_r^2.$$

Theorem 2.6 (Doob's martingale inequality)**.** *If $\mathfrak{h}_t$ is a martingale such that $t \to \mathfrak{h}_t(\omega)$ is continuous a.s., then for all $m \geq 1$, $q > 0$, and all $\lambda > 0$,*

$$\mathbb{P}\big[\sup_{0 \leq t \leq q} |\mathfrak{h}_t| \geq \lambda\big] \leq \frac{1}{\lambda^m} \cdot \mathbb{E}\big[|\mathfrak{h}_q|^m\big].$$

For the existence of solution, we employed picard-lindelof successive approximation techniques in the broadest field as a result from Equation (1.1).

$$X(t) = X_0 - \mathfrak{h}_4\big(X(t)\big) - \mathfrak{h}_3\big(0, X_0 - \mathfrak{h}_4\big(X(t)\big)\big) + \mathfrak{h}_3\big(t, X(t)\big) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\mathfrak{h}_1\big(s, X(s)\big)ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\mathfrak{h}_2\big(s, X(s)\big)dW(s) + \sum_{0<t_k<t} I_k\big(X(t_k)\big), \tag{2.1}$$

$X(t)$ is $\mathcal{F}_t$ adapted and $\mathbb{E}\big(\int_0^\alpha \big|X(t)\big|^2 dt\big) < \infty$.

In order to examine the qualitative aspects of solving the Equation (1.1), we shall impose some restrictions on the coefficient functions in this section as follows:

(H1): For all $x_1, x_2 \in \mathbb{R}^n$ and $t \in [0, \alpha]$, there arise two positive constants C_1 and C_2, so that

$|\mathfrak{h}_1(t, x_1)|^2 \vee |\mathfrak{h}_2(t, x_1)|^2 \leq C_1\big(1 + |x_1|^2\big)$,

$|\mathfrak{h}_1(t, x_1) - \mathfrak{h}_1(t, x_2)| \vee |\mathfrak{h}_2(t, x_1) - \mathfrak{h}_2(t, x_2)| \leq C_2|x_1 - x_2|$,

where $|\cdot|$ is the norm of $\mathbb{R}^n$.

(H2): $\mathfrak{h}_3(t, 0) = 0 = \mathfrak{h}_4(t, 0)$ and for all $x_1, x_2 \in \mathbb{R}^n$, there exists some constants $C_{\mathfrak{h}_3}, C_{\mathfrak{h}_4} \in (0, 1)$ such that

$|\mathfrak{h}_3(t, x_1) - \bar{\mathfrak{h}}_3(t, x_2)| \leq C_{\mathfrak{h}_3}|x_1 - x_2|$,

$|\mathfrak{h}_4(x_1) - \bar{\mathfrak{h}}_4(x_2)| \leq C_{\mathfrak{h}_4}|x_1 - x_2|$.

3. An Averaging Principle

We begin this section with the investigation of averaging principle for SFIDEs with impulse and nonlocal conditions. Let us analyze the Equation (1.1) in its standard form:

$$X_\rho(t) = X_0 - \sqrt{\rho}\mathfrak{h}_4\big(X_\rho(t)\big) - \sqrt{\rho}\mathfrak{h}_3\big(0, X_0 - \sqrt{\rho}\mathfrak{h}_4\big(X_\rho(t)\big)\big) + \rho\mathfrak{h}_3\big(t, X_\rho(t)\big)$$

$$+\frac{\rho}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathfrak{h}_1\big(s,X_\rho(s)\big)ds+\frac{\sqrt{\rho}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathfrak{h}_2\big(s,X_\rho(s)\big)dW(s)+\rho\sum_{i=1}^{k}I_i\big(X_\rho(t_i)\big), \tag{3.1}$$

where initial value X_0, the significance of the coefficients $\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3$, and $\mathfrak{h}_4$ are the same as that of Equation (1.1).

Additionally, we designate a fixed number by ρ_0 and a positive small parameter by $\rho\in[0,\rho_0]$. Prior to proceeding with the notion of averaging, we apply a quantifiable coefficients $\bar{\mathfrak{h}}_2:\mathcal{J}\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $\bar{\mathfrak{h}}_1,\bar{\mathfrak{h}}_3:\mathcal{J}\times\mathbb{R}^n\to\mathbb{R}^n$, and $\bar{\mathfrak{h}}_4:\mathbb{R}^n\to\mathbb{R}^n$ fulfilling (H1) and (H2), therefore, the following inequalities:

(H3): For some $\alpha_1\in[0,\alpha], x\in\mathbb{R}^n$, there are bounded functions $\psi_i(\alpha_1),\ i=1,2,3,4$ that are positive, so that

$$\frac{1}{\alpha_1}\int_0^{\alpha_1}|\mathfrak{h}_1(s,x)-\bar{\mathfrak{h}}_1(x)|^2ds\le\psi_1(\alpha_1)(1+|x|^2),$$
$$\frac{1}{\alpha_1}\int_0^{\alpha_1}|\mathfrak{h}_2(s,x)-\bar{\mathfrak{h}}_2(x)|^2ds\le\psi_2(\alpha_1)(1+|x|^2),$$
$$\frac{1}{\alpha_1}\int_0^{\alpha_1}|\mathfrak{h}_3(s,x)-\bar{\mathfrak{h}}_3(x)|^2ds\le\psi_3(\alpha_1)(1+|x|^2),$$
$$\frac{1}{\alpha_1}\int_0^{\alpha_1}|\mathfrak{h}_4(x)-\bar{\mathfrak{h}}_4(x)|^2ds\le\psi_4(\alpha_1)(1+|x|^2),$$
$$\bar{I}(x)\le\frac{1}{\alpha_1}\sum_{i=1}^{k}I_i(x),$$

where $\lim_{\alpha_1\to 0}\psi_i(\alpha_1)=0,$ and $i=1,2,3,4$.

(H4): For all I_i, there arise a constant $\bar{l}$ which is positive such that for $\forall x\in\mathbb{R}^n$,

$$|I_i(x)|^2\le\bar{l}.$$

With the assistance provided above, we will demonstrate that the solution $X_\rho(t)$ converges, as $\rho\to 0$, leading to $Z_\rho(t)$ of the averaged system.

$$Z_\rho(t)=X_0-\sqrt{\rho}\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)-\rho\bar{\mathfrak{h}}_3\big(0,X_0-\sqrt{\rho}\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big)+\sqrt{\rho}\bar{\mathfrak{h}}_3\big(t,Z_\rho(t)\big)$$
$$+\frac{\rho}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bar{\mathfrak{h}}_1\big(s,Z_\rho(s)\big)ds+\frac{\sqrt{\rho}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bar{\mathfrak{h}}_2\big(s,Z_\rho(s)\big)dW(s)+\rho\int_0^t\bar{I}_i(Z_\rho(s))ds, \tag{3.2}$$

We are here to present the primary finding from our study.

Theorem 3.1. *Suggest that (H1) to (H4) are satisfied. For $\delta_1>0$ there exists $L>1$, $\rho_1\in(0,\rho_0]$ and $\xi\in(0,1)$ is such for $\rho\in(0,\rho_1]$,*

$$\mathbb{E}\Bigg(\sup_{t\rho[1,L^{\rho^{-\xi}}]}\big|X_\rho(t)-Z_\rho(t)\big|^2\Bigg)\le\delta_1.$$

Proof. For any $t\in[0,\mu]\subset[0,\alpha]$,

$$X_\rho(t)-Z_\rho(t)=\sqrt{\rho}\big[\mathfrak{h}_3\big(t,X_\rho(t)\big)-\bar{\mathfrak{h}}_3\big(Z_\rho(t)\big)\big]-\rho\big[\mathfrak{h}_4\big(X_\rho(t)\big)-\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big]$$
$$-\sqrt{\rho}\big[\mathfrak{h}_3\big(0,X_0-\rho\mathfrak{h}_4\big(X_\rho(t)\big)\big)-\bar{\mathfrak{h}}_3\big(0,X_0-\rho\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big)\big]$$

$$+\frac{\rho}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_1\big(Z_\rho(s)\big)\big]ds+\rho\sum_{i=1}^{k}\big(I_i(t_i)\big)-\rho\int_0^t \bar{I}_i(Z_\rho(s))ds$$

$$+\frac{\sqrt{\rho}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_2\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big]dW(s).$$

Using the elementary inequality, we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2\Big)\le\ & 6\rho\mathbb{E}\sup_{0\le t\le\mu}\big[\big|\mathfrak{h}_3\big(t,X_\rho(t)\big)-\bar{\mathfrak{h}}_3\big(Z_\rho(t)\big)\big|^2+\big|\mathfrak{h}_4\big(X_\rho(t)\big)-\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big|^2\big]\\ &+6\rho\mathbb{E}\sup_{0\le t\le\mu}\big|\mathfrak{h}_3\big(0,X_0-\rho\mathfrak{h}_4\big(X_\rho(t)\big)\big)-\bar{\mathfrak{h}}_3\big(0,X_0-\rho\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big)\big|^2\\ &+\frac{6\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le\mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_1\big(Z_\rho(s)\big)\big]ds\Big|^2\\ &+\frac{6\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le\mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_2\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big]dW(s)\Big|^2\\ &+6\rho^2\mathbb{E}\sup_{0\le t\le\mu}\Big|\sum_{i=1}^{k}I_i\big(X_\rho(t_i)\big)-\int_0^t \bar{I}_i(Z_\rho(s))ds\Big|^2,\\ &=I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{3.3}$$

With the help of hypothesis (H2), one can get

$$\begin{aligned}I_1&=6\rho\mathbb{E}\sup_{0\le t\le\mu}\big[\big|\mathfrak{h}_3\big(t,X_\rho(t)\big)-\bar{\mathfrak{h}}_3\big(Z_\rho(t)\big)\big|^2+\big|\mathfrak{h}_4\big(X_\rho(t)\big)-\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big|^2\big]\\ &\le 6\rho\big[C_{\mathfrak{h}_3}\mathbb{E}\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2+C_{\mathfrak{h}_4}\mathbb{E}\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2\big]\\ &=K_1\rho,\end{aligned}\tag{3.4}$$

$$\begin{aligned}I_2&=6\rho^2\mathbb{E}\sup_{0\le t\le\mu}\big|\mathfrak{h}_3\big(0,X_0-\rho\mathfrak{h}_4\big(X_\rho(t)\big)\big)-\bar{\mathfrak{h}}_3\big(0,X_0-\rho\bar{\mathfrak{h}}_4\big(Z_\rho(t)\big)\big)\big|^2\\ &\le 6\rho C_{\mathfrak{h}_3}\big[C_{\mathfrak{h}_4}\rho\ \mathbb{E}\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2\big]\\ &=K_2\rho^2,\end{aligned}\tag{3.5}$$

where $K_1=6C_{\mathfrak{h}_3}\mathbb{E}\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2$ and $K_2=6C_{\mathfrak{h}_3}C_{\mathfrak{h}_4}\mathbb{E}\sup_{0\le t\le\mu}\big|X_\rho(t)-Z_\rho(t)\big|^2$.

$$I_3=\frac{5\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le\mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_1\big(Z_\rho(s)\big)\big]ds\Big|^2,\tag{3.6}$$

$$I_4=\frac{5\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le\mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_2\big(s,X_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big]dW(s)\Big|^2,\tag{3.7}$$

$$I_5=5\rho^2\mathbb{E}\Big(\sup_{0\le t\le\mu}\Big|\sum_{i=1}^{k}I_i\big(X_\rho(t_i)\big)-\int_0^t \bar{I}_i(Z_\rho(s))ds\Big|^2\Big).\tag{3.8}$$

Recalling the elementary inequality, we obtain from (3.6):

$$I_3 = \frac{5\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s, X_\rho(s)\big) - \bar{\mathfrak{h}}_1\big(Z_\rho(s)\big)\big]ds\Big|^2,$$

$$\begin{aligned}
I_3 \le& \frac{10\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s, X_\rho(s)\big) - \mathfrak{h}_1\big(s, Z_\rho(s)\big)\big]ds\Big|^2 \\
&+ \frac{10\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s, Z_\rho(s)\big) - \bar{\mathfrak{h}}_1\big(Z_\rho(s)\big)\big]ds\Big|^2 \\
&= I_{31} + I_{32}.
\end{aligned}$$

Using Theorem (2.5) and condition (H1), we obtain

$$I_{31} \le \frac{10\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_1\big(s, X_\rho(s)\big) - \mathfrak{h}_1\big(Z_\rho(s)\big)\big]ds\Big|^2,$$

$$\begin{aligned}
I_{31} \le& \frac{10\rho^2\mu}{(\Gamma(q))^2}\int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s}\big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)2C_2^2 ds \\
\le& \frac{20C_2^2\mu}{(\Gamma(q))^2}\rho^2\int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s}\big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)ds,
\end{aligned}$$

$$I_{31} \le K_{31}\mu\rho^2\int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s}\big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)ds, \tag{3.9}$$

where $K_{31} = \frac{20C_2^2}{(\Gamma(q))^2}$. With the help of variable upper limit integration,

$$I_{32} \le \frac{10\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t (t-s)^{q-1}d\Big[\int_0^s \big(\mathfrak{h}_1\big(\tau, Z_\rho(\tau)\big) - \bar{\mathfrak{h}}_1\big(Z_\rho(\tau)\big)\big)d\tau\Big]\Big|^2,$$

integration by parts is used,

$$I_{32} \le \frac{10(q-1)^2\rho^2}{(\Gamma(q))^2}\mathbb{E}\sup_{0\le t\le \mu}\Big|\int_0^t \Big(\int_0^s \big(\mathfrak{h}_1\big(\tau, Z_\rho(\tau)\big) - \bar{\mathfrak{h}}_1\big(Z_\rho(\tau)\big)\big)d\tau\Big)(t-s)^{q-2}ds\Big|^2.$$

Therefore in addition to the hypothesis (H3) and Theorem (2.5), we obtain

$$I_{32} \le \frac{10(q-1)^2\alpha^{2q-3}\rho^2}{(2q-3)(\Gamma(q))^2}\mathbb{E}\int_0^\mu \Big|\int_0^s \big(\mathfrak{h}_1\big(\tau, Z_\rho(\tau)\big) - \bar{\mathfrak{h}}_1\big(Z_\rho(\tau)\big)\big)d\tau\Big|^2 ds \le K_{32}\rho^2\mu^{2q}, \tag{3.10}$$

in which $K_{32} = \frac{10(q-1)^2}{(2q-3)(\Gamma(q))^2}\sup\limits_{0\le t\le \mu}\psi_1(t)^2\Big[1 + \mathbb{E}\Big(\sup\limits_{0\le t\le \mu}\big|Z_\rho(t)\big|^2\Big)\Big]$.

We anticipate the second term using the same approach, hence from Equation (3.7),

$$\begin{aligned} I_4 &\leq \frac{10\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\leq t\leq\mu}\left|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_2\big(s,X_\rho(s)\big)-\mathfrak{h}_2\big(s,Z_\rho(s)\big)\big]dW(s)\right|^2, \\ &+\frac{10\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\leq t\leq\mu}\left|\int_0^t (t-s)^{q-1}\big[\mathfrak{h}_2\big(s,Z_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big]dW(s)\right|^2, \\ &= I_{41}+I_{42}. \end{aligned}$$

By applying Theorem 2.6, Ito's formula and hypothesis (H1),

$$I_{41}\leq K_{41}\rho\int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\leq s_1\leq s}\big|X_\rho(s_1)-Z_\rho(s_1)\big|^2\Big)ds, \tag{3.11}$$

where $K_{41}=\frac{20C_2^2}{(\Gamma(q))^2}$. Applying Theorem 2.6 and Ito's formula again,

$$I_{42}\leq \frac{10\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\leq t\leq\mu}\int_0^t (t-s)^{2q-2}\big|\mathfrak{h}_2\big(s,Z_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big|^2ds,$$

Integrating by parts, produces

$$\begin{aligned} I_{42} &\leq \frac{10\rho}{(\Gamma(q))^2}\mathbb{E}\sup_{0\leq t\leq\mu}\int_0^t (t-s)^{2q-2}d\Big[\int_0^s \big|\mathfrak{h}_2\big(s,Z_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big|^2ds\Big], \\ &\leq \frac{10(2q-2)\rho}{(\Gamma(q))^2}\mathbb{E}\int_0^\mu\Big(\int_0^s \big|\mathfrak{h}_2\big(s,Z_\rho(s)\big)-\bar{\mathfrak{h}}_2\big(Z_\rho(s)\big)\big|^2ds\Big)(t-s)^{2q-3}ds. \end{aligned}$$

With the help of hypothesis (H3), we can draw this conclusion:

$$\begin{aligned} I_{42} &\leq \frac{10\rho(2q-2)}{(\Gamma(q))^2}\int_0^\mu \sup_{0\leq s_1\leq s}\psi_2(s_1)\Big[1+\mathbb{E}\Big(\sup_{0\leq\tau\leq s}\big|Z_\rho(\tau)\big|^2\Big)\Big](t-s)^{2q-3}ds, \\ &\leq \frac{10\rho}{(\Gamma(q))^2}\mu^{2q-1}\sup_{0\leq\tau\leq s}\psi_2(\tau)\Big[1+\mathbb{E}\Big(\sup_{0\leq\tau\leq s}\big|Z_\rho(\tau)\big|^2\Big)\Big], \\ &\leq K_{42}\mu^{2q-1}\rho, \end{aligned} \tag{3.12}$$

where $K_{42}=\frac{10}{(\Gamma(q))^2}\sup_{0\leq\tau\leq s}\psi_2(\tau)\Big[1+\mathbb{E}\Big(\sup_{0\leq\tau\leq s}\big|Z_\rho(\tau)\big|^2\Big)\Big]$.

With the help of hypothesis (H4), one can get

$$\begin{aligned} I_5 &= 10\rho^2\cdot\mathbb{E}\Big[\sup_{0\leq t\leq\mu}\Big|\sum_{i=1}^k I_i\big(X_\rho(t_i)\big)-\int_0^t \bar{I}_i\big(Z_\rho(s)\big)ds\Big|^2\Big], \\ &\leq 10\rho^2\cdot\mathbb{E}\Big[\sup_{0\leq t\leq\mu}\Big|\sum_{i=1}^k I_i\big(X_\rho(t_i)\big)\Big|^2\Big]+10\rho^2\cdot\mathbb{E}\Big[\sup_{0\leq t\leq\mu}\Big|\int_0^t \bar{I}_i\big(Z_\rho(s)\big)ds\Big|^2\Big], \end{aligned}$$

$$
\begin{aligned}
I_5 &\le 10\rho^2 k \cdot \mathbb{E}\Big[\sup_{0\le t\le \mu} \sum_{i=1}^{k} \big|I_i\big(X_\rho(t_i)\big)\big|^2\Big] + 10\rho^2 \frac{k}{\alpha_1^2}\mu \cdot \mathbb{E}\Big[\sup_{0\le t\le \mu} \int_0^t \sum_{i=1}^{k} \big|\bar{I}_i\big(Z_\rho(s)\big)ds\big|^2\Big],\\
&\le 10k^2\Big(\bar{l} + \mu^2 \bar{l}\frac{1}{\alpha_1^2}\Big)\rho^2,\\
&\le K_5\rho^2,
\end{aligned}
\tag{3.13}
$$

where $K_5 = 10k^2\Big(\bar{l} + \mu^2\bar{l}\frac{1}{\alpha_1^2}\Big)$. Now substituting the inequalities (3.4), (3.5), and from (3.9) to (3.13) into (3.3), for any $\mu \in [0,\alpha]$, we find

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le \mu} \big|X_\rho(t) - Z_\rho(t)\big|^2\Big) \le{}& K_1\rho + K_2\rho^2 + K_{32}\rho^2\mu^{2q} + K_{42}\mu^{2q-1}\rho + K_5\rho^2\\
&+ K_{31}\rho^2 \int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s} \big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)ds\\
&+ K_{41}\rho \int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s} \big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)ds,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le \mu} \big|X_\rho(t) - Z_\rho(t)\big|^2\Big) \le{}& K_1\rho + K_2\rho^2 + K_{32}\rho^2\mu^{2q} + K_{42}\mu^{2q-1}\rho + K_5\rho^2\\
&+ \big(K_{31}\rho^2\mu + K_{41}\rho\big) \int_0^\mu (t-s)^{2q-2}\mathbb{E}\Big(\sup_{0\le s_1\le s} \big|X_\rho(s_1) - Z_\rho(s_1)\big|^2\Big)ds.
\end{aligned}
$$

Depending on the Gronwall-Bellman inequality [21], we find

$$
\mathbb{E}\Big(\sup_{0\le t\le s} \big|X_\rho(t) - Z_\rho(t)\big|^2\Big) \le \big(K_1\rho + K_2\rho^2 + K_{32}\rho^2\mu^{2q} + K_{42}\mu^{2q-2}\rho + K_5\rho^2\big) \times \sum_{j=0}^{\infty} \frac{\big((K_{31}\rho^2\mu + K_{41}\rho)\mu^{2q-1}\Gamma(2q-1)\big)^j}{\Gamma\big(j(2q-1)+1\big)}.
$$

This implies that we can select $\xi \in (0,1)$ and $L > 1$, such that for every $t \in [0, L^{\rho^{-\xi}}] \subseteq [0,\alpha]$ having

$$
\mathbb{E}\Big(\sup_{0\le t\le L^{\rho^{-\xi}}} \big|X_\rho(t) - Z_\rho(t)\big|^2\Big) \le C\rho^{1-\xi},
$$

where

$$
C = K_1\rho + K_2\rho^2 + K_{32}\rho^{1+\xi-2q\xi}L^{2q} + K_{42}L^{2q-2}\rho^{2\xi(1-q)} + K_5\rho^2 \times \sum_{j=0}^{\infty} \frac{\big((K_{31}\rho^{2(1-q\xi)}L + K_{41}\rho^{1+\xi-2q\xi})L^{2q-1}\Gamma(2q-1)\big)^j}{\Gamma\big(j(2q-1)+1\big)},
$$

is a constant. Therefore, for every number δ_1, there arise $\rho_1 \in (0,\rho_0]$ such that for each $\rho \in (0,\rho_1]$ and $t \in [0, L^{\rho^{-\xi}}]$ having

$$
\mathbb{E}\Big(\sup_{0\le t\le L^{\rho^{-\xi}}} \big|X_\rho(t) - Z_\rho(t)\big|^2\Big) \le \delta_1,
$$

completes the proof. □

4. Example

Example 4.1. Our objective is to provide an example that highlights the resultant effects of the averaging principle. Examine the following SFIDEs under impulse and nonlocal conditions:

$$\begin{aligned} {}^{C}D_t^q[X_\rho(t) - cos^2(t/2)X_\rho(t)] &= \big(sin(2t) + t^2\big)X_\rho(t) + \frac{1}{(t+1)^2}X_\rho(t)\frac{dW(t)}{dt}, \quad t \in \mathcal{J}, \\ \Delta X_\rho(t) &= \rho i^2 X_\rho(t_i^-), \quad t = t_i,\ i = 1, 2, \cdots, l, \\ X(0) &= e^{-t}X_\rho(t) + 1. \end{aligned} \tag{4.1}$$

The coefficients $\mathfrak{h}_1(s, X_\rho(s)) = \big(sin(2s) + s^2\big)X_\rho(s)$, $\mathfrak{h}_2(s, X_\rho(s)) = \frac{1}{(s+1)^2}X_\rho(s)$, $\mathfrak{h}_3(s, X_\rho(s)) = cos^2(s/2)X_\rho(s)$, $\mathfrak{h}_4(X_\rho(s)) = e^{-s}X_\rho(s)$, $I_i(s) = i^2X_\rho(s)$ satisfy the conditions (H1) - (H4). Let $\alpha_1 = 1$. Furthermore, we define

$$\bar{\mathfrak{h}}_1(X_\rho(t)) = \frac{1}{\alpha_1}\int_0^{\alpha_1} \mathfrak{h}_1(s, X_\rho(s))ds = \int_0^1 \big(sin(2s) + s^2\big)X_\rho ds = \frac{2 - 3cos(2)}{6}X_\rho,$$

$$\bar{\mathfrak{h}}_2(X_\rho(t)) = \frac{1}{\alpha_1}\int_0^{\alpha_1} \mathfrak{h}_2(s, X_\rho(s))ds = \int_0^1 \frac{1}{(s+1)^2}X_\rho ds = \frac{1}{2}X_\rho,$$

$$\bar{\mathfrak{h}}_3(X_\rho(t)) = \frac{1}{\alpha_1}\int_0^{\alpha_1} \mathfrak{h}_3(s, X_\rho(s))ds = \int_0^1 cos^2(s/2)X_\rho ds = \frac{1 + sin1}{2}X_\rho,$$

$$\bar{\mathfrak{h}}_4(X_\rho(t)) = \frac{1}{\alpha_1}\int_0^{\alpha_1} \mathfrak{h}_4(X_\rho(s))ds = \int_0^1 e^{-s}X_\rho(s)ds = \frac{e - 1}{e}X_\rho,$$

$$\bar{I}(X_\rho(t)) = \frac{1}{\alpha_1}\sum_{i=0}^{k} I_i(t) = \sum_{i=0}^{k} i^2 X_\rho = \frac{k(k+1)(2k+1)}{6}X_\rho.$$

In light of the discussion above, (H3) is established. After that we simplify SFIDEs with impulse and nonlocal conditions as follows:

$$\begin{aligned} Z_\rho(t) = X_0 - \sqrt{\rho}\left[\frac{1 + sin1}{2}\right]X_\rho + \sqrt{\rho}\left[\frac{e - 1}{e}\right]X_\rho + \frac{\rho}{\Gamma(q)}\int_0^t (t - s)^{q-1}\left[\frac{2 - 3cos(2)}{6}\right]X_\rho ds \\ + \frac{\sqrt{\rho}}{\Gamma(q)}\int_0^t (t - s)^{q-1}\left[\frac{1}{2}X_\rho\right]dW(s) + \rho\int_0^t \left[\frac{k(k+1)(2k+1)}{6}\right]X_\rho ds. \end{aligned}$$

Each curve on the below graph corresponds to a different values of q. The curve $X_\rho(t)$ represented in solid lines and the curve $Z_\rho(t)$ represented in dotted lines. Those are plotted at different t values. This graphical interpretation clearly illustrates how the function $X(t)$ evolves over time t for different values of q. As q increases, the rate of increase of $X(t)$ also increases, leading to steeper curves on the graph.

We are able to determine that the requirements of Theorem 3.1 are met through verification. Therefore, in terms of mean square and probability, the averaging form for the Equation (4.1)'s solution $Z_\rho(t)$ is equivalent to the existing solution $X_\rho(t)$ as $\rho \to 0$.

Averaging at Financial Market: In financial markets, prices of assets may exhibit sudden changes due to time lags in information dissemination or unprecedented reactions of traders to news. The SFIDEs can model the effect of past information on current unusual prices. The averaging principle helps reduce the complexity of the system by approximating the behavior of asset prices over a longer time horizon in stochastic uncommon asset pricing. As a result, in general, asset prices fluctuate due to a combination of factors like news, economic data, and random

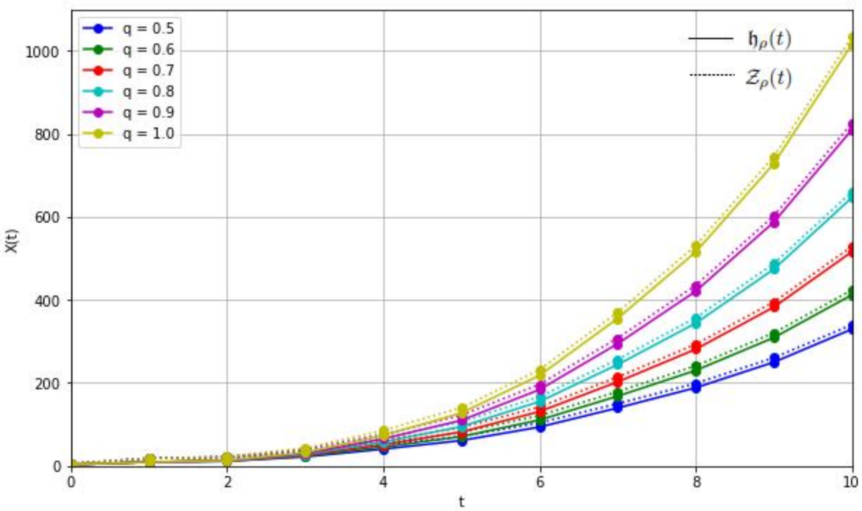

FIGURE 1. $X_\rho(t)$ and $Z_\rho(t)$ at different values of q.

market sentiment. Prices don't always react immediately to new information, leading to atypical reactions. Trades may adjust their positions based on past trends or information that takes time to propagate through the market. The model incorporates randomness (volatility), which is essential to capture the inherent unpredictability of financial markets. Short-term fluctuations, represented by random noise, create sharp ups and downs in asset prices.

By applying the averaging principle, we can smooth these fluctuations and uncover long-term price trends. This helps to focus on the fundamental trajectory of asset prices rather than getting distracted by short-term noise, which is crucial for long-term investment strategies. For example, this principle might suggest when an asset is fundamentally undervalued or overvalued, helping investors make more informed decisions.

Stock Market Model for Asset Analysis: Here, we investigate the stock market approach to asset analysis for Stochastic Dependent Impulse Analysis (SDIA). Let us consider the famous Black-Scholes asset price model with only Gaussian process

$$^{C}D^{q}X(t) = A_1\mathfrak{h}(t, X(t)) + A_2\mathfrak{h}(t, X(t))\frac{dW(t)}{dt},$$

where A_1 and A_2 are arbitrary non-negative values.

Suppose if there is a little perturbation in the asset model as a result of some embedded Technical analysis in the resulting chart; to produce a sudden jump in the stock prices and other financial asset studies which leads to the following model as an example of the asset analysis.

Example 4.2. Consider the following SDIA model:

$$\begin{cases} ^{C}D^{q}[X(t) - A_3\mathfrak{h}_3(t, X(t))] = A_1\mathfrak{h}_1(t, X(t)) + A_2\mathfrak{h}_2(t, X(t))\frac{dW(t)}{dt}, & t \in \mathcal{J} = [0, \alpha], \\ \Delta X(t_k) = I_k(A_5 X(t_k^-)), \quad t = t_k, & k = 1, 2, 3, ..., l, \\ X(0) + A_4\mathfrak{h}_4(X) = X_0, \end{cases} \tag{4.2}$$

where A_1, A_2, A_3, A_4, and A_5 are arbitrary non-negative values.

- $X(t)$ - Asset price depicts the stock price over time and the worth of a portfolio amid uncertain market conditions.

- ${}^{C}D^{q}[X(t) - A_3\mathfrak{h}_3(t, X(t))]$ - Fractional derivative (Memory effect) model simulates long-term memory effects in financial markets and interprets effects such as market inertia, in which previous prices influence the present.
- $A_1\mathfrak{h}_1(t, X(t))$ - Drift term (Trend Component) represents the foretold return or growth rate, which can be used to forecast interest rates, economic growth, or systemic market patterns.
- $A_2\mathfrak{h}_2(t, X(t))$ - Stochastic Volatility (Market Noise) represents the volatility co-efficient, which indicates how much unpredictability affects the price.
- $\Delta X(t_k) = I_k(A_5 X(t_k^-))$ - Jump process (Market Shocks) models price fluctuations, such as crashes, news shocks, or earnings releases, and manages the impact of jumps, which may represent major trades or economic events. I_k simulates how external shocks, like as political events, interest rate changes, or financial crises, affect prices.
- $X(0) + A_4\mathfrak{h}_4(X) = X_0$ - Initial condition (Market Constraints) represents the value of the asset or initial market state and the function $A_4\mathfrak{h}_4(X)$ could be a market friction or transaction cost affecting the initial setup.

According to Theorem 3.1, the solution of the Equation (4.2) is given as:

$$X_\rho(t) = \begin{cases} X_0 - \sqrt{\rho}A_4\mathfrak{h}_4(X_\rho(t)) - \sqrt{\rho}A_3\mathfrak{h}_3(0, X_0 - \sqrt{\rho}A_4\mathfrak{h}_4(X_\rho(t))) + \rho A_3\mathfrak{h}_3(t, X_\rho(t)) \\ + \frac{\rho}{\Gamma(q)} \int_0^t (t-s)^{q-1} A_1\mathfrak{h}_1(s, X_\rho(s))ds + \frac{\sqrt{\rho}}{\Gamma(q)} \int_0^t (t-s)^{q-1} A_2\mathfrak{h}_2(s, X_\rho(s))dW(s) + \rho \sum_{i=1}^{k} I_i(A_5 X_\rho(t_i)). \end{cases} \tag{4.3}$$

Take $q = 0.8, X_0 = 1, A_1 = 0.5, A_2 = 1.0, A_3 = 0.7, A_4 = 0.8$, and $A_5 = 0.2$. It is clear that the system (4.2) satisfies the hypotheses (H1) and (H2).

Furthermore, from the hypotheses (H3), (H4), and by Theorem 3.1, we obtain $Z_\rho(t)$. The solutions $X_\rho(t)$ and $Z_\rho(t)$ clearly show the weiner process as well as the jump in the system and also show entry points for potential investors in asset.

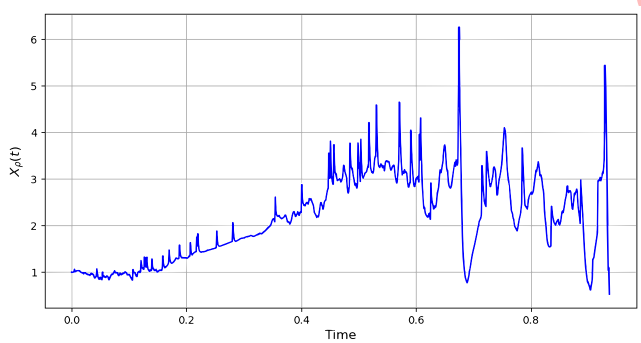

Figure 2. SDIA Model: $X_\rho(t)$

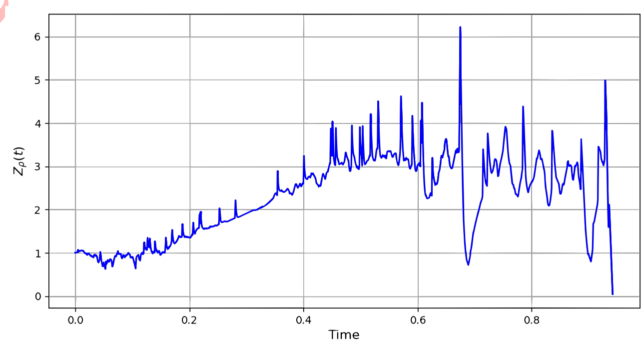

Figure 3. SDIA Model: $Z_\rho(t)$

The above graphs Figures 2 and 3 that simulate the graphical interpretation of the asset price model which clearly describes how these investment functions $X_\rho(t)$ and $Z_\rho(t)$ evolve over time t for different values of q which we can understand the sensitivity of the system to these parameters.

5. Conclusion

We have extensively explored the averaging principle for the solutions to SFIDEs equipped with impulse and non-local conditions. Our primary objective was to establish sufficient conditions under which the averaged SFIDEs. To achieve this, we innovatively extended classical averaging techniques to the fractional domain, duly considering both

TABLE 1. The table below has been constructed for the asset price and time interval $t = 2^{-6}$.

t	$X_\rho(t)$	$Z_\rho(t)$	Error	t	$X_\rho(t)$	$Z_\rho(t)$	Error
0.0	0.219069	0.219501	0.000432	0.6	0.539095	0.540446	0.001351
0.1	0.154460	0.155522	0.001062	0.7	0.157273	0.157804	0.000531
0.2	0.577279	0.578562	0.001283	0.8	0.977007	0.978253	0.001246
0.3	0.271409	0.272024	0.000615	0.9	0.941121	0.941856	0.000735
0.4	0.677542	0.678056	0.000514	1.0	0.648318	0.648903	0.000585
0.5	0.073082	0.073437	0.000355	-	-	-	-

Table 1 displays the comparison between the original solution and the averaged solution. This example is confined to theoretical and computational modeling within financial mathematics.

the neutral delay and the inherent stochasticity of the system. This research significantly contributes to the advancement of the understanding and analysis of SFIDEs under intricate conditions. The derived results not only enhance the theoretical framework but also hold promising implications for practical applications in the field of finance and industry. This work serves as a foundational step, laying the groundwork for further exploration and research in this challenging and important area of applied sciences and financial mathematics.

Funding: No external funding is received for this research.
Conflicts of Interest: There are no conflicts of interest declared by the authors.

Acknowledgement

We would like to sincerely thank the reviewers for their tremendous assistance and valuable contributions in offering comments and questions during the course of the study.

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