Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 14, No. 3, 2026, pp. 1368-1374
DOI:10.22034/cmde.2025.64979.2963

Ambarzumyan-type theorem with local derivative

Beyhan Kemaloglu* **and Hasan Bulut**
Faculty of Science, Department of Mathematics 23119, Firat University, Elazig, Turkey.

Abstract

We show the Ambarzumyan theorem in this paper by considering the Sturm-Liouville problem with separable boundary conditions and local derivatives. We prove that if the spectrum consists of the first eigenvalue, then the potential function can be found depending on the first eigenvalue. Also, we give some examples like periodic and anti-periodic boundary conditions. In the case of $\alpha = 1$, the results of the classical case can be obtained. Although the concept of conformable fractional is debatable, we think the results will be useful for Sturm-Liouville theory.

Keywords. Ambarzumyan theorem, Sturm-Liouville problem, Conformable derivatives and integrals, Eigenvalues.
2010 Mathematics Subject Classification. 34A55, 34B24,34A08.

1. Introduction

The Sturm-Liouville equation is a fundamental concept in mathematical physics, particularly in the study of differential equations and boundary value problems. It has significant physical interpretations in various areas of physics, including quantum mechanics, heat transfer, vibration analysis, and fluid dynamics. The physical meaning of the Sturm-Liouville equation arises from its application to problems governed by second-order differential equations subject to certain boundary conditions. Here are a few examples:

Quantum mechanics: In quantum mechanics, the wave function of a quantum system satisfies a time-independent Schrödinger equation, which can be cast into a Sturm-Liouville form. The solutions to this equation correspond to the allowed energy states of the quantum system, and the eigenvalues represent the possible energy levels.

Vibrations of a string: The equation can describe the transverse vibrations of a string fixed at both ends. The eigenvalues correspond to the natural frequencies of vibration, and the eigenfunctions represent the shapes of the vibrating modes. Heat Conduction: In heat transfer problems, the Sturm-Liouville equation can describe the distribution of temperature in a medium. The eigenvalues represent the rates of heat transfer, and the eigenfunctions represent the temperature profiles.

Fluid dynamics: In certain fluid flow problems, such as the study of eigenmodes of oscillation in a fluid-filled cavity or the stability analysis of fluid flows, the Sturm-Liouville equation can arise to characterize the behaviour of the system.

On the other hand, the generalization of classical calculus is defined by fractional calculus. The Sturm-Liouville problem with local derivative is then achieved by substituting the fractional derivative for the ordinary derivative. Engineering, physics, and chemistry have produced a wide range of publications on this topic [2, 6–9, 11, 13–15, 18, 19, 22, 25, 27, 31, 32, 34]. The authors of [23] came up with a few generalizations that encompass the novel outcomes that fractional operators are examining. Additionally, they provided two distinct methods that, under the assumption of convexity, can be utilized to solve a few additional generalizations of rising functions. The authors enhanced a unique framework in [24] to investigate two new types of convex functions in Hilbert spaces that rely on any arbitrary non-negative function. Reverse Minkowski and reverse Holder inequalities via quantum Hahn, and various additional

Received: 11 December 2024; Accepted: 22 May 2025.
∗ Corresponding author. Email: beyhanozturk1980@gmail.com.

variations derived by quantum Hahn fractional integral operator affecting convex functions are presented in [26]. Furthermore, some results on local derivative Sturm-Liouville problems are presented. For example, [29] computed the eigenvalues and eigenfunctions of the Sturm-Liouville problem with local derivatives numerically using the Caputo fractional derivative. However, there are many studies on this subject [3, 4, 17, 20, 28], it seems that these results are not enough in inverse Sturm-Liouville problem. In this study, the Ambarzumyan theorem, which is the first step of inverse Sturm-Liouville theory, will be given for fractional Sturm-Liouville problem. Historically, the study of inverse Sturm-Liouville problem started by Ambarzumyan [5]. It is not difficult to see that if in Sturm-Liouville problem with Neumann conditions, then the eigenvalues are $q(x) = 0$. In addition, the inverse Sturm-Liouville problem was solved in various cases [10, 12, 16, 21, 30, 33, 35–37].

We consider a boundary value problem on a finite interval $[0, \pi]$

$$-D_x^\alpha D_x^\alpha y + q(x)y = \lambda y, \tag{1.1}$$

with separable boundary conditions

$$D_x^\alpha y(0) + hy(0) = D_x^\alpha y(\pi) + Hy(\pi) = 0. \tag{1.2}$$

The Sturm-Liouville problem with local derivative is the name given to this issue, and it has been extensively studied in the literature [1, 4, 17, 20]. Here, λ is spectral parameter, $q(x) \in L_\alpha^2(0, \pi), h, H$ are real constants. Also,D_x^α is the conformable derivative of order α, $0 < \alpha \leq 1$.

Before giving the main part of the study, it is useful to give some basic conclusions about the fractional theory [2, 6–8, 13, 15, 19].

Definition 1.1. Let α be a positive number with $\alpha \in (0, 1]$. The conformable derivative of order α of f with regard to $x > 0$ is defined as a function

$$D^\alpha f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}. \tag{1.3}$$

If f is differentiable, that is $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, then $D^\alpha f(x) = x^{1-\alpha} f'(x)$.

Definition 1.2. $f : [0, \infty) \longrightarrow R$ be a given function. Then, the conformable integral of f of order α is defined

$$I_\alpha f(x) = \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt, \tag{1.4}$$

for all $x > 0$ and $0 < \alpha \leq 1$.

Theorem 1.3. *Let f and g be α-differential at x, $x > 0$. Then,*
i) $D_x^\alpha(af + bg) = aD_x^\alpha f + bD_x^\alpha g, \forall a, b \in R$
ii) $D_x^\alpha(c) = 0$, *(c is a constant)*
iii) $D_x^\alpha(\frac{f}{g}) = \frac{D_x^\alpha(f.g) - f.D_x^\alpha g}{g^2}$.

2. Main Results

In this section, the Ambarzumyan theorem, proved by some authors for the classical derivative Sturm-Liouville problem, will be given for the Sturm-Liouville Problem with local derivative. Especially, when different boundary conditions are taken into account, the motion of the potential function will be examined according to the state of the eigenvalues. It is worth mentioning here that when we consider $\alpha = 1$, the results were given in [33].

Theorem 2.1. *For $\lambda_o = \frac{\alpha}{\pi}^\alpha \left\{H - h + \int_0^\pi q d_\alpha x\right\}$, if the spectrum (collection of eigenvalues) consists of λ_o then the potential function is $q(x) = \lambda_o$ almost everywhere on $L_\alpha^2(0, \pi)$ for the problems (1.1) and (1.2).*

Proof. Let $y_\alpha(x)$ be an eigenfunction corresponding to λ_o then

$$D_x^\alpha \left(\frac{D_x^\alpha y}{y} \right) = \frac{(D_x^\alpha D_x^\alpha y)y - D_x^\alpha y.D_x^\alpha y}{y^2},$$

$$\frac{D_x^\alpha D_x^\alpha y}{y} = D_x^\alpha\left(\frac{D_x^\alpha y}{y}\right) + \left(\frac{D_x^\alpha y}{y}\right)^2,$$

and taking α integration on $[0,\pi]$ and by (1.2)

$$\int_0^\pi \frac{D_x^\alpha D_x^\alpha y}{y} d_\alpha x = \left.\frac{D_x^\alpha y}{y}\right|_{x=\pi} - \left.\frac{D_x^\alpha y}{y}\right|_{x=0} + \int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x,$$

$$\int_0^\pi (q-\lambda_o) d_\alpha x = h - H + \int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x.$$

Inserting the λ_o in above we conclude that

$$\int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x = 0.$$

and it gives us that $y = c$ (c is a constant). Let write the (1.1) equation for a constant solution, then we complete the proof. □

Let us define the problems (1.1) and (1.2) with $L_\alpha(q,h,H)$ and $\tilde{L}_\alpha(\tilde{q},h,H)$ and these problems refer to different two problems for q and $\tilde{q}$, respectively.

Theorem 2.2. *Let $\lambda_o = \tilde{\lambda_o} + \frac{\langle \tilde{q}\tilde{y},\tilde{y}\rangle_\alpha}{\langle \tilde{y},\tilde{y}\rangle_\alpha}$. Then $q - \tilde{q} = \lambda_o - \tilde{\lambda_o}$. Where $\tilde{y}$ is an eigenfunction related to $\tilde{\lambda_o}$ in the problems (1.1) and (1.2) and the inner product defined $\langle . \rangle = \int_0^\pi (.) d_\alpha x$ in $L^2_\alpha(0,\pi)$.*

Proof. Consider y_o is an eigenfunction of L_α related with λ_o. Then

$$\begin{aligned}
\frac{\langle L_\alpha \tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} &= \frac{-\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x + \int_0^\pi q\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{-\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x + \int_0^\pi q\tilde{y_o}\tilde{y_o} d_{\alpha x}}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&- \frac{\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x - \int_0^\pi \widetilde{q}\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&+ \frac{\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x - \int_0^\pi \widetilde{q}\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{\int_0^\pi \left(-D_x^\alpha D_x^\alpha \tilde{y_o} + \widetilde{q}\tilde{y_o}\right)\tilde{y_o} d_\alpha x + \int_0^\pi (q\tilde{-}q)\,\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{\langle \tilde{\lambda_o}\tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} + \frac{\langle \left(\lambda_o - \tilde{\lambda_o}\right)\tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} \\
&= \lambda_o.
\end{aligned}$$

The rest of the proof is omitted. □

Now, we will give some Ambarzumyan-type results in various cases.

Corollary 2.3. *Consider the Sturm-Liouville Problem with local derivative $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $D_x^\alpha y(0) = D_x^\alpha y(\pi) = 0$. Also let $\widetilde{q}(x) = 0$. Then, $\tilde{\lambda_o} = 0$ and $\tilde{y_o} = 1$. By Theorem 2.2, $q = \lambda_o$ that implies the Theorem 2.1.*

Corollary 2.4. *Consider the Sturm-Liouville Problem with local derivative* $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = y(\pi) = 0$. *Also let* $\widetilde{q}(x) = 0$. *Then,* $\tilde{\lambda_o} = \dfrac{\alpha}{\pi^{\alpha-1}}$ *and* $\tilde{y}_o = \sin(\dfrac{x^\alpha}{\pi^{\alpha-1}})$. *Then Theorem 2.2 gives that if* $\lambda_o = \frac{\alpha}{\pi^{\alpha-1}} + \dfrac{2\alpha}{\pi^\alpha} \int_o^\pi q(x) \sin^2\left(\frac{x^\alpha}{\pi^{\alpha-1}}\right) d_\alpha x$, *then* $q(x) = \lambda_o - \dfrac{\alpha}{\pi^{\alpha-1}}$.

Corollary 2.5. *Consider the periodic problem* $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = y(\pi)$, $D_x^\alpha y(0) = D_x^\alpha y(\pi)$. *let* $\tilde{q} = 0$. *Then,* $\tilde{\lambda_o} = 0$, $\tilde{y}_o = 1$. *Theorem 2.2 implies that* $\lambda_o = \dfrac{\alpha}{\pi^\alpha} \int_0^\pi q(x) d_\alpha x$ *then* $q = \lambda_o$ *a.e. on* $(0, \pi)$.

3. Examples

Example 3.1. Consider the boundary value problem (1.1) with Neumann conditions. Let, $\tilde{q} = 0$. Then, $\tilde{\lambda_o} = 0$ and $\tilde{y}_o = 1$. By Theorem 2.2,

$$\lambda_o = \tilde{\lambda_o} + \frac{\langle \tilde{q}\tilde{y}_0, \tilde{y}_0 \rangle_\alpha}{\langle \tilde{y}_0, \tilde{y}_0 \rangle_\alpha} = \frac{\alpha}{\pi^\alpha} \int_0^\pi q(x) d_\alpha x.$$

If $q(x) = x^2$, we have $\lambda_o = \frac{\alpha}{\alpha+2}\pi^2$. Then the behaviour of the eigenvalue is as in Figure 1.

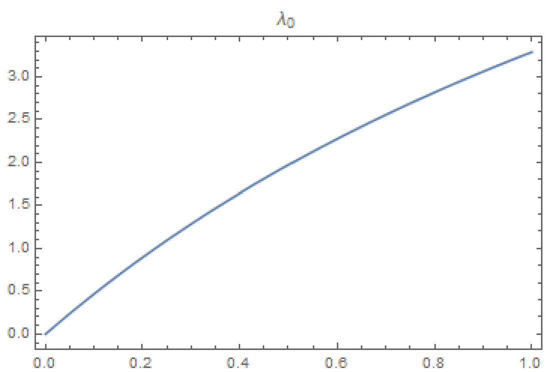

Figure 1. Behaviour of eigenvalue depending on α for $q(x) = x^2$.

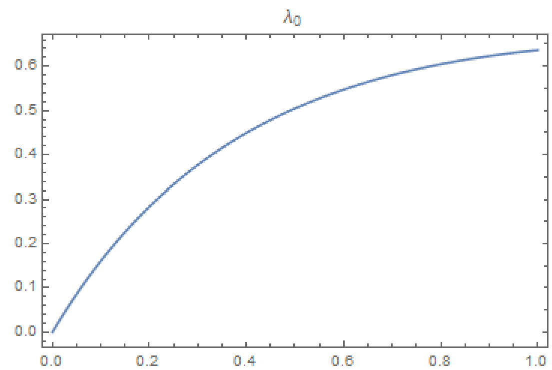

Figure 2. Behaviour of eigenvalue depending on α for $q(x) = \sin x$.

If $q(x) = \sin x$, we have $\lambda_o = \frac{\pi\alpha . PFQ[\{\frac{1}{2}, \frac{\alpha}{2}\}, \{\frac{3}{2}, \frac{3}{2}, \frac{\alpha}{2}\}, -\frac{\pi^2}{4}}{\alpha+1}$. Then behaviour of the eigenvalues as in Figure 2. Where PFQ indicates the generalized Hypergeometric function.

Example 3.2. Consider the boundary value problem (1.1) with the $y(0) = D_x^\alpha y(\pi) = 0$ conditions. Let, $\tilde{q} = 0$. Then, $\tilde{\lambda_o} = 0$ and $\tilde{y}_o = 1$. By Theorem 2.2,

$$\lambda_o = -h + \frac{\alpha}{\pi^\alpha} \int_0^\pi q(x) d_\alpha x.$$

If $q(x) = x^2$ and $h = -2$, we have $\lambda_o = -2 + \frac{\alpha}{\alpha+2}\pi^2$. Then the behaviour of the eigenvalues is as in Figure 3.

If $q(x) = \sin x$ and $h = -2$, we have $\lambda_o = -2 + \frac{\pi\alpha . PFQ[\{\frac{1}{2}, \frac{\alpha}{2}\}, \{\frac{3}{2}, \frac{3}{2}, \frac{\alpha}{2}\}, -\frac{\pi^2}{4}}{\alpha+1}$. Then the behaviour of the eigenvalues is as in Figure 4.

Example 3.3. Consider the anti-periodic problem $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = -y(\pi)$, $D_x^\alpha y(0) = -D_x^\alpha y(\pi)$. Let $\tilde{q} = 0$ and because of double eigenvalues $\tilde{\lambda_o} = \tilde{\lambda}_1 = \dfrac{\alpha}{\pi^{\alpha-1}}$. Then eigenfunction $\tilde{y}_o(x) = c_1 \sin\left(\frac{x^\alpha}{\alpha}\right) + c_2 \cos\left(\frac{x^\alpha}{\alpha}\right)$. Then, Theorem 2.2 implies that If $\lambda_o = \dfrac{\alpha}{\pi^{\alpha-1}} + \dfrac{\alpha}{\pi^\alpha(c_1^2 + c_2^2)} \int_0^\pi q(x)(c_1 \sin\left(\frac{x^\alpha}{\alpha}\right) + c_2 \cos\left(\frac{x^\alpha}{\alpha}\right) d_\alpha x$ for some constants c_1, c_2, Then $q = \lambda_o - \dfrac{\alpha}{\pi^{\alpha-1}}$ a.e. on $(0, \pi)$.

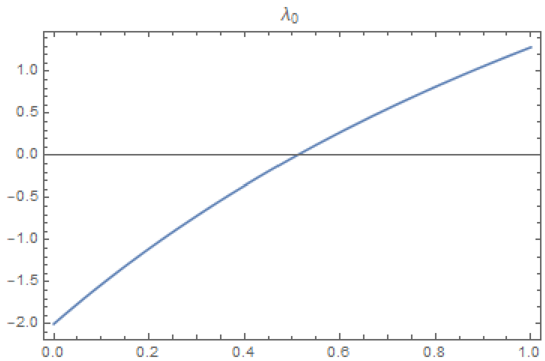

Figure 3. Behaviour of eigenvalue depending on α for $q(x) = x^2$.

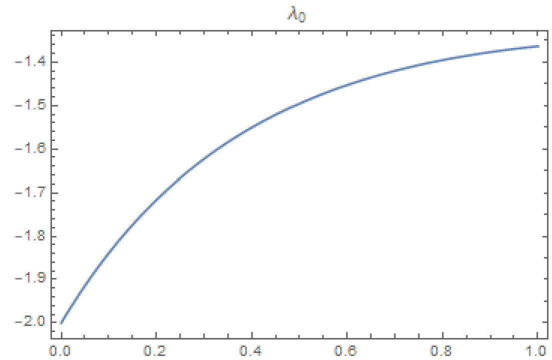

Figure 4. Behaviour of eigenvalue depending on α for $q(x) = sinx$.

4. Conclusion

In this paper, we proposed an effective and straightforward approach to determine the Sturm-Liouville Problem's potential function using local derivatives. For this purpose, we find the first eigenvalue of the problem and then give the potential function depending on the eigenvalue and some constants in the boundary conditions. Then, the results will contribute to Fractional Sturm-Liouville theory. The outcomes acquired by the future plan are all cases of α. To make the results more concrete, we gave some examples and geometric interpretations showing the change of the first eigenvalue according to α in Figure 1–4. However, it is possible to generalize Ambarzumyan's theorem via the product rule and Leibnitz's rule in the local derivative case; it is difficult to have a non-local fractional derivative. But the author plans to study the inverse problem in the non-local case.

Declarations

The authors declare that they have no known competing financial interests or personal relationships in this paper. Funding Not applicable. Availability of data and materials. All the data related to this study are presented in the paper.

Acknowledgment

The authors are grateful to the anonymous reviewers who improved the completion and presentation of this work through their valuable corrections and suggestions.

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