Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-7
DOI:10.22034/cmde.2025.64979.2963

Ambarzumyan type theorem with local derivative

Beyhan Kemaloglu* **and Hasan Bulut**
Faculty of Science, Department of Mathematics 23119, Firat University, Elazig, Turkey.

Abstract

We show the Ambarzumyan theorem in this paper by taking into account the Sturm-Liouville problem with separable boundary conditions by local derivative. We proved that if the spectrum consists of the first eigenvalue, then the potential function can be found depending on the first eigenvalue. Also, we give some examples like periodic and anti-periodic boundary conditions. In the case of $\alpha = 1$, results in the classical case can be obtained. Although the concept of conformable fractional is debatable, we think the results will be useful for Sturm-Liouville theory.

Keywords. Ambarzumyan theorem, Sturm-Liouville problem, Conformable derivatives and integrals, Eigenvalues.
2010 Mathematics Subject Classification. 34A55, 34B24,34A08.

Uncorrected Proof

1. Introduction

The Sturm-Liouville equation is a fundamental concept in mathematical physics, particularly in the study of differential equations and boundary value problems. It has significant physical interpretations in various areas of physics, including quantum mechanics, heat transfer, vibration analysis, and fluid dynamics. The physical meaning of the Sturm-Liouville equation arises from its application to problems governed by second-order differential equations subject to certain boundary conditions. Here are a few examples:

Quantum Mechanics: In quantum mechanics, the wave function of a quantum system satisfies a time-independent Schrödinger equation, which can be cast into a Sturm-Liouville form. The solutions to this equation correspond to the allowed energy states of the quantum system, and the eigenvalues represent the possible energy levels.

Vibrations of a String: The equation can describe the transverse vibrations of a string fixed at both ends. The eigenvalues correspond to the natural frequencies of vibration, and the eigenfunctions represent the shapes of the vibrating modes. Heat Conduction: In heat transfer problems, the Sturm-Liouville equation can describe the distribution of temperature in a medium. The eigenvalues represent the rates of heat transfer, and the eigenfunctions represent the temperature profiles.

Fluid Dynamics: In certain fluid flow problems, such as the study of eigenmodes of oscillation in a fluid-filled cavity or the stability analysis of fluid flows, the Sturm-Liouville equation can arise to characterize the behaviour of the system.

On the other hand, the generalization of classical calculus is defined by fractional calculus. The Sturm-Liouville problem with local derivative is then achieved by substituting the fractional derivative for the ordinary derivative. Engineering, physics, and chemistry have produced a wide range of publications on this topic [2, 6–9, 11, 13–15, 18, 19, 22, 25, 27, 31, 32, 34]. The authors of [23] came up with a few generalizations that encompass the novel outcomes that fractional operators are examining. Additionally, they provided two distinct methods that, with the presumption of convexity, can be utilized to solve a few additional generalizations of rising functions. Authors enhanced a unique framework in [24] to investigate two new types of convex functions in Hilbert spaces that rely on any arbitrary non-negative function. Reverse Minkowski and reverse Holder inequalities via quantum Hahn, and various additional

Received: 11 December 2024 ; Accepted: 22 May 2025.
* Corresponding author. Email: beyhanozturk1980@gmail.com.

variations derived by quantum Hahn fractional integral operator affecting convex functions are presented in [26]. Furthermore, some results on local derivative Sturm-Liouville problems are presented. For example, [29] computed the eigenvalues and eigenfunctions of the Sturm-Liouville problem with local derivatives numerically using the Caputo fractional derivative. However there are many studies on this subject [3, 4, 17, 20, 28], it seems that these results are not enough in inverse Sturm-Liouville problem. In this study, Ambarzumyan theorem which is the first step of inverse Sturm-Liouville theory will be given for fractional Sturm-Liouville problem. Historically, the study of inverse Sturm-Liouville problem started by Ambarzumyan [5]. It is not difficult to see that if in Sturm-Liouville problem with Neumann conditions, then eigenvalues are then $q(x) = 0$. In addition, inverse Sturm-Liouville problem was solved in various case [10, 12, 16, 21, 30, 33, 35–37].

We consider boundary value problem on a finite interval $[0, \pi]$

$$-D_x^\alpha D_x^\alpha y + q(x)y = \lambda y, \tag{1.1}$$

with separable boundary conditions

$$D_x^\alpha y(0) + hy(0) = D_x^\alpha y(\pi) + Hy(\pi) = 0. \tag{1.2}$$

The Sturm-Liouville problem with local derivative is the name given to this issue, and it has been extensively studied in the literature [1, 4, 17, 20]. Here, λ is spectral parameter, $q(x) \in L_\alpha^2(0, \pi), h, H$ are real constants. Also,D_x^α is the conformable derivative of order α, $0 < \alpha \leq 1$.

Before giving the main part of the study, it is useful to give some basic conclusions about the fractional theory [2, 6–8, 13, 15, 19].

Definition 1.1. Let α be a positive number with $\alpha \in (0, 1]$. The conformable derivative of order α of f with regard to $x > 0$ is defined as a function

$$D^\alpha f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}. \tag{1.3}$$

If f is differentiable, that is $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, then $D^\alpha f(x) = x^{1-\alpha} f'(x)$.

Definition 1.2. $f : [0, \infty) \longrightarrow R$ be a given function. Then, the conformable integral of f of order α is defined

$$I_\alpha f(x) = \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt, \tag{1.4}$$

for all $x > 0$ and $0 < \alpha \leq 1$.

Theorem 1.3. *Let f and g be α-differential at x, $x > 0$. Then,*
i) $D_x^\alpha(af + bg) = aD_x^\alpha f + bD_x^\alpha g, \forall a, b \in R$
ii) $D_x^\alpha(c) = 0$, *(c is a constant)*
iii) $D_x^\alpha(\frac{f}{g}) = \frac{D_x^\alpha(f.g) - f.D_x^\alpha g}{g^2}$.

2. Main Results

In this section, the Ambarzumyan theorem proved by some authors for the classical derivative Sturm-Liouville problem will be given for the Sturm-Liouville Problem with local derivative. Especially, when different boundary conditions are taken into account, the motion of the potential function will be examined according to state of the eigenvalues . It is worth mentioning here that when we consider $\alpha = 1$, the results were given in [33].

Theorem 2.1. *For $\lambda_o = \frac{\alpha}{\pi}^\alpha \left\{H - h + \int_0^\pi q d_\alpha x)\right\}$, if the the spectrum (collection of eigenvalues) is consisted of λ_o then the potential function is $q(x) = \lambda_o$ almost everywhere on $L_\alpha^2(0, \pi)$ for the problems (1.1) and (1.2).*

Proof. Let $y_\alpha(x)$ be an eigenfunction corresponding to λ_o then

$$D_x^\alpha \left(\frac{D_x^\alpha y}{y} \right) = \frac{(D_x^\alpha D_x^\alpha y)y - D_x^\alpha y.D_x^\alpha y}{y^2},$$

Uncorrected Proof

$$\frac{D_x^\alpha D_x^\alpha y}{y} = D_x^\alpha\left(\frac{D_x^\alpha y}{y}\right) + \left(\frac{D_x^\alpha y}{y}\right)^2,$$

and taking α integration on $[0, \pi]$ and by (1.2)

$$\int_0^\pi \frac{D_x^\alpha D_x^\alpha y}{y} d_\alpha x = \left.\frac{D_x^\alpha y}{y}\right|_{x=\pi} - \left.\frac{D_x^\alpha y}{y}\right|_{x=0} + \int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x,$$

$$\int_0^\pi (q-\lambda_o) d_\alpha x = h - H + \int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x.$$

Inserting the λ_o in above we conclude that

$$\int_0^\pi \left(\frac{D_x^\alpha y}{y}\right)^2 d_\alpha x = 0.$$

and it gives us that $y = c$ (c is a constant). Let write the (1.1) equation for a constant solution then we complete the proof. □

Let us define the problems (1.1) and (1.2) with $L_\alpha(q, h, H)$ and $\tilde{L}_\alpha(\tilde{q}, h, H)$ and these problems refer to different two problems for q and $\tilde{q}$, respectively.

Theorem 2.2. *Let* $\lambda_o = \tilde{\lambda_o} + \frac{\langle \tilde{q}\tilde{y}, \tilde{y}\rangle_\alpha}{\langle \tilde{y}, \tilde{y}\rangle_\alpha}$ *. Then* $q - \tilde{q} = \lambda_o - \tilde{\lambda_o}$*. Where* $\tilde{y}$ *is an eigenfunction related to* $\tilde{\lambda_o}$ *in the problems* (1.1) *and* (1.2) *and the inner product defined* $\langle . \rangle = \int_0^\pi (.) d_\alpha x$ *in* $L_\alpha^2(0, \pi)$*.*

Proof. Consider y_o is an eigenfunction of L_α related with λ_o. Then

$$\begin{aligned}
\frac{\langle L_\alpha \tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} &= \frac{-\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x + \int_0^\pi q\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{-\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x + \int_0^\pi q\tilde{y_o}\tilde{y_o} d_{\alpha x}}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&- \frac{\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x - \int_0^\pi \tilde{q}\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&+ \frac{\int_0^\pi D_x^\alpha D_x^\alpha \tilde{y_o}\tilde{y_o} d_\alpha x - \int_0^\pi \tilde{q}\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{\int_0^\pi \left(-D_x^\alpha D_x^\alpha \tilde{y_o} + \tilde{q}\tilde{y_o}\right)\tilde{y_o} d_\alpha x + \int_0^\pi (q - \tilde{q})\tilde{y_o}\tilde{y_o} d_\alpha x}{\int_0^\pi \tilde{y_o}.\tilde{y_o} d_\alpha x} \\
&= \frac{\langle \tilde{\lambda_o}\tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} + \frac{\langle \left(\lambda_o - \tilde{\lambda_o}\right)\tilde{y_o}, \tilde{y_o}\rangle}{\langle \tilde{y_o}, \tilde{y_o}\rangle} \\
&= \lambda_o.
\end{aligned}$$

The rest of proof is ommited. □

Now, we will give some Ambarzumyan type results in various cases.

Corollary 2.3. *Consider the Sturm-Liouville Problem with local derivative* $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $D_x^\alpha y(0) = D_x^\alpha y(\pi) = 0$*. Also let* $\tilde{q}(x) = 0$*. Then,* $\tilde{\lambda_o} = 0$ *and* $\tilde{y_o} = 1$*. By Theorem 2.2,* $q = \lambda_o$ *that implies the Theorem 2.1.*

Uncorrected Proof

Corollary 2.4. *Consider the Sturm-Liouville Problem with local derivative* $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = y(\pi) = 0$. *Also let* $\widetilde{q}(x) = 0$. *Then,* $\tilde{\lambda_o} = \dfrac{\alpha}{\pi^{\alpha-1}}$ *and* $\tilde{y}_o = \sin(\dfrac{x^\alpha}{\pi^{\alpha-1}})$. *Then Theorem 2.2 gives that if* $\lambda_o = \frac{\alpha}{\pi^{\alpha-1}} + \dfrac{2\alpha}{\pi^\alpha}\int_o^\pi q(x)\sin^2\left(\frac{x^\alpha}{\pi^{\alpha-1}}\right) d_\alpha x$, *then* $q(x) = \lambda_o - \dfrac{\alpha}{\pi^{\alpha-1}}$.

Corollary 2.5. *Consider the periodic problem* $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = y(\pi)$, $D_x^\alpha y(0) = D_x^\alpha y(\pi)$. *let* $\tilde{q} = 0$. *Then,* $\tilde{\lambda_o} = 0$, $\tilde{y}_o = 1$. *Theorem 2.2 implies that* $\lambda_o = \dfrac{\alpha}{\pi^\alpha}\int_0^\pi q(x) d_\alpha x$ *then* $q = \lambda_o$ *a.e. on* $(0, \pi)$.

3. Examples

Example 3.1. Consider the boundary value problem (1.1) with Neumann conditions. Let, $\tilde{q} = 0$. Then, $\tilde{\lambda_o} = 0$ and $\tilde{y}_o = 1$. By Theorem 2.2,

$$\lambda_o = \tilde{\lambda_o} + \frac{\langle \tilde{q}\tilde{y}_0, \tilde{y}_0\rangle_\alpha}{\langle \tilde{y}_0, \tilde{y}_0\rangle_\alpha} = \frac{\alpha}{\pi^\alpha}\int_0^\pi q(x) d_\alpha x.$$

If $q(x) = x^2$, we have $\lambda_o = \frac{\alpha}{\alpha+2}\pi^2$. Then behaviour of the eigenvalue as in Figure 1.

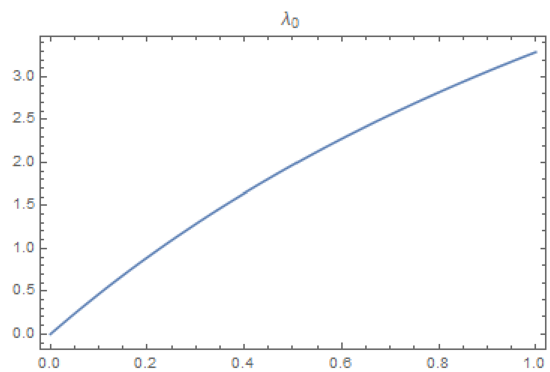

Figure 1. Behaviour of eigenvalue depending on α for $q(x) = x^2$.

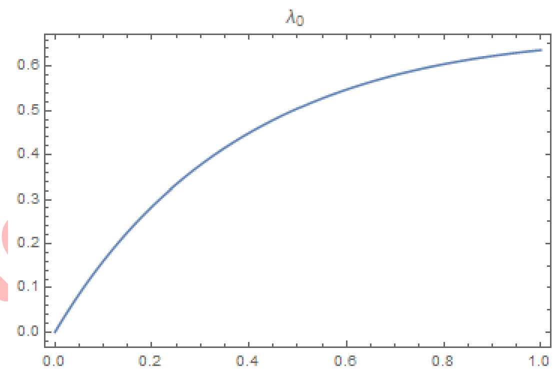

Figure 2. Behaviour of eigenvalue depending on α for $q(x) = \sin x$.

If $q(x) = \sin x$, we have $\lambda_o = \frac{\pi\alpha.PFQ[\{\frac{1}{2},\frac{\alpha}{2}\},\{\frac{3}{2},\frac{3}{2},\frac{\alpha}{2}\},-\frac{\pi^2}{4}}{\alpha+1}$. Then behaviour of the eigenvalues as in Figure 2. Where PFQ indicate the generalized Hypergeometric function.

Example 3.2. Consider the boundary value problem (1.1) with the $y(0) = D_x^\alpha y(\pi) = 0$ conditions. Let, $\tilde{q} = 0$. Then, $\tilde{\lambda_o} = 0$ and $\tilde{y}_o = 1$. By Theorem 2.2,

$$\lambda_o = -h + \frac{\alpha}{\pi^\alpha}\int_0^\pi q(x) d_\alpha x.$$

If $q(x) = x^2$ and $h = -2$, we have $\lambda_o = -2 + \frac{\alpha}{\alpha+2}\pi^2$. Then behaviour of the eigenvalues as in Figure 3.

If $q(x) = \sin x$ and $h = -2$, we have $\lambda_o = -2 + \frac{\pi\alpha.PFQ[\{\frac{1}{2},\frac{\alpha}{2}\},\{\frac{3}{2},\frac{3}{2},\frac{\alpha}{2}\},-\frac{\pi^2}{4}}{\alpha+1}$. Then behaviour of the eigenvalues as in Figure 4.

Example 3.3. Consider the anti-periodic problem $-D_x^\alpha D_x^\alpha y + qy = \lambda y$, $y(0) = -y(\pi)$, $D_x^\alpha y(0) = -D_x^\alpha y(\pi)$. Let $\tilde{q} = 0$ and because of double eigenvalues $\tilde{\lambda_o} = \tilde{\lambda}_1 = \dfrac{\alpha}{\pi^{\alpha-1}}$. Then eigenfunction $\tilde{y}_o(x) = c_1\sin\left(\frac{x^\alpha}{\alpha}\right) + c_2\cos\left(\frac{x^\alpha}{\alpha}\right)$. Then, Theorem 2.2 implies that If $\lambda_o = \dfrac{\alpha}{\pi^{\alpha-1}} + \dfrac{\alpha}{\pi^\alpha(c_1^2 + c_2^2)}\int_0^\pi q(x)(c_1\sin\left(\frac{x^\alpha}{\alpha}\right) + c_2\cos\left(\frac{x^\alpha}{\alpha}\right) d_\alpha x$ for some constants c_1, c_2, Then $q = \lambda_o - \dfrac{\alpha}{\pi^{\alpha-1}}$ a.e. on $(0, \pi)$.

Uncorrected proof

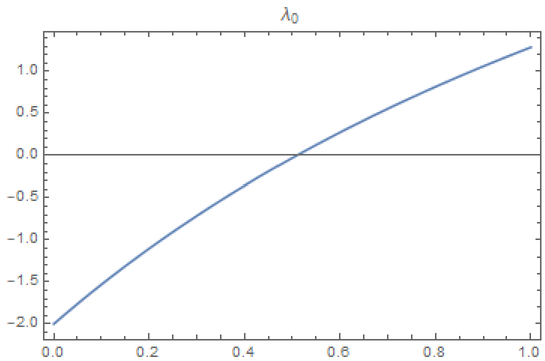

Figure 3. Behaviour of eigenvalue depending on α for $q(x) = x^2$.

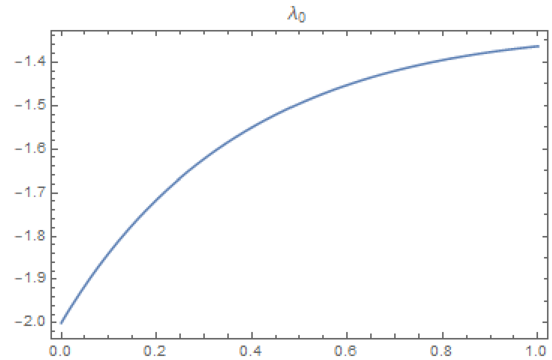

Figure 4. Behaviour of eigenvalue depending on α for $q(x) = sinx$.

Conclusion

In this paper, we proposed an effective and straightforward approach to determine the Sturm-Liouville Problem's potential function using local derivatives. For this purpose, we find the first eigenvalue of the problem and then gave the the potential function depending on the eigenvalue and some constants in the boundary conditions. Then, results will contribute to Fractional Sturm-Liouville theory. The outcomes acquired by the future plan are all cases of α. To make the results more concrete, we gave some examples and geometric interpretations showing the change of the first eigenvalue according to α in Figure 1–4. However it is possible to generalize Ambarzumyan theorem via product rule and Leibnitz rule in local derivative case, it is difficult to have non-local fractional derivative. But, author plans to study inverse problem in non-local case.

Declarations

The authors declare that they have no known competing financial interests or personal relationships in this paper. Funding Not applicable. Availability of data and materials All the data related to this study is presented in the paper.

Acknowledgment

The authors are grateful to the anonymous reviewers that improved the completion and presentation of this work through their valuable corrections and suggestions.

References

[1] A. Mohammed and T. Abdeljavad, *Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems*, Complexity, *2017*(1) (2017), Article ID 3720471, 7 pages.
[2] T. Abdeljawad, *On conformable fractional calculus,* J. Comput Appl Math, *279* (2015), 57-66.
[3] I. Adalar and A. S. Ozkan, *Inverse problems for a Fractional Sturm-Liouville operators,* J. Inverse Ill-posed Probl, *28*(6) (2020), 775-782.
[4] B. Allahverdiyev, T. Hüseyin, and Y. Yüksel, *Fractional Sturm-Liouville equation*, Math Methods Appl Sci, *42*(10) (2019), 3508-3526.
[5] V. A. Ambarzumyan, *ber eine frage der eigenwerttheorie*, Z. Physics,*53* (1929), 690-695.
[6] A. Atangana, B. Dumitru, and A. Alsaedi, *New properties of conformable derivative*, Open Math., *13*(1) (2015), 889-898.
[7] D. Avci, B. I. Eroglu, and N. Ozdemir, *The Dirichlet problem of a conformable advection diffusion equation*, Thermal Sci., *21* (2017), 9-18.

Uncorrected Proof

[8] D. Balaneu, G. Z. Burhanettin, and J. A. Machado, *New trends in nanotechnology and fractional calculus applications,* New York (NY), Springer, *10* (2010).

[9] N. Benkhettou, S. Hassani, and D. F. M. Torres, *A Fractional calculus on arbitrary time scales*, J. King Saud Univ Sci., *28*(1) (2016), 93-98.

[10] H. H. Chern, C. K. Law, and H. J. Wang, *Extensions of Ambarzumyan's theorem to general boundary conditions*, J. Math. Anal. Appl., *263*(2) (2001), 333-342.

[11] G. Chuan-Yun, F. X. Zheng, and B. Shiri, *Mittag-Leffler stability analysis of tempered fractional neural networks with short memory and variable order*, Fractals,*29*(8) (2021), 2140029 .

[12] B. Kemaloglu, *Ambarzumyan theorem by zeros of eigenfunction. International Journal of Mathematics and Computer in Engineering*, *1*(2) (2023), 211-216.

[13] R. Khalil, M. Y. Al-Horani, and A. Sababheh M. *A new definition of fractional derivative*, J. Comput Appl Math., *264* (2014), 65-70.

[14] Z. A. Khan,S. Rashid, R. Ashraf, D. Baleanu, and C. Yu-Ming, *Generalized trapezium-type inequalities in the settings of fractal sets for functions having generalized convexity property*, Advanced in Difference Equation, *2020* (2020), 1-24.

[15] A. A. Kilbas, H. M. Srivastava, and J. Trujillo, *Theory and applications of fractional differential equations*, New York, North-Holland, (Math Studies), 2006.

[16] A. A. Kirac, *Ambarzumyan's theorem for the quasi-periodic boundary conditions*, Analysis and Mathematical Physics, *6* (2016), 297-300.

[17] M. Klimek and O. P. Agrawal, *Fractional Sturm-Liouville problem*, Comput Math Appl.,*66* (2013), 795-812.

[18] J. T. Machado and V. F. Mainardi, *Recent history of fractional calculus*, Communications in nonlinear science and numerical simulation, *16*(3) (2011), 1140-1153

[19] K. S. Miller, *An introduction to fractional calculus and fractional differential equations,* New York (NY):J. Wiley and Sons, 1993.

[20] H. Mortazaasl and A. J. Akbarfam, *Trace formula and inverse nodal problem for a Fractional Sturm-Liouville problem*, Inverse Probl Sci Eng., *28*(4) (2020), 524-555.

[21] S. Mosazadeh, *A new approach to uniqueness for inverse Sturm-Liouville problems on finite intervals*, Turkish Journal of Mathematics, *41*(5) (2017), 1224-1234.

[22] M. D. Ortigueira and J. A. T. Machado, *What is a fractional derivative?*, J. Comput Phys., *29*(3) (2015), 4-13.

[23] S. Rashid, D. Baleanu, and Y. M. Chu, *Some new extensions for fractional integral operator having exponential in the kernel and their applications in physical systems*, Open Physics, *18*(1) (2020), 478-491.

[24] S. Rashid, H. Kalsoom, Z. Hammouch , R. Ashraf , D. Baleanu, and Y. M. Chu, *New Multi-Parametrized Estimates Having pth-Order Differentiability in Fractional Calculus for Predominating h-Convex Functions in Hilbert Space*, Symmetry, *12*(2) (2020), 222.

[25] S. Rashid, H. Kalsoom, Z. Hammouch , D. Baleanu, and Y. M. Chu. *New generalizations in the sense of the weighted non-singular fractional integral operator*, Fractals, *28*(08) (2020), 2040003.

[26] S. Rashid, Z. Hammouch, R. Ashraf , D. Baleanu, and K. S. Nisar, *New quantum estimates in the setting of fractional calculus theory*, Advanced in Difference Equations, *2020*(1) (2020), 383.

[27] S. Rashid, D. Iscan, D. Baleanu, and Y. M. Chu, *Generation of new fractional inequalities via n polynomials s-type convexity with applications*, Advanced in Difference Equations, *2020* (2020), 1-20.

[28] M. Rivero M, J. Trujillo, and M. P. Velasco, *A fractional approach to the Sturm-Liouville problem*, Centr Eur J Phys., *11* (2013), 1246-1254.

[29] M. K. Sadabad, A. J. Akbarfam, and B. Shiri, *A numerical study of eigenvalues and eigenfunctions of fractional Sturm-Liouville problems via Laplace transform*, Indian Journal of Pure and Applied Mathematics, *51*(3) (2020), 857-868.

[30] M. Shahriari, *An Inverse Three Spectra Problem for Parameter-Dependent and Jumps Conformable Sturm Liouville Operators*, Bull. Malays. Math. Sci. Soc., *47*(1) (2024), 25.

[31] B. Shiri, G. C. Whu, and D. Baleanu, *Collocation methods for terminal value problems of tempered fractional differential equations*, Applied Numerical Mathematics, *156* (2020), 385-395.

Uncorrected Proof

[32] Y. Wang, J. Zhou, and Y. Li, *Fractional sobolev spaces on time scales via Fractional calculus and their application to a fractional differential equation on time scales*, Adv Math Phys., *2016*(1) (2016), 9636491.

[33] C. F. Yang and X. P. Yang, *Ambarzumyan's theorem with eigenparameter in the boundary conditions*, Acta Math. Sci. Ser. B Engl. Ed., *31*(4) (2011), 1561-1568.

[34] G. S. Yang, B. H. Kong, and G. C. Wu, *Intermediate value problems for fractional differential equations*, Computational and Applied Mathematics, *40*(6) (2021), 1-20.

[35] E. Yilmaz and H. Koyunbakan, *Some Ambarzumyan type theorems for Bessel operator ona finite interval*, Differ Equ Dyn Syst., *27* (2019), 553-559.

[36] V. A. Yurko, *On Ambarzumyan-type theorems*, Appl. Math. Lett., *26*(4) (2013), 506-509.

[37] R. Zhang and C. F. Yang, *Ambarzumyan-type theorem for the impulsive Sturm-Liouville operator*, Journal of Inverse and Ill-posed Problems,*29*(1) (2021), 21-25.

Uncorrected Proof