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Analytical and numerical solutions of the convection-diffusion-reaction equations applying the Differential Transformation Method and the Crank-Nicolson method along with stability analysis and truncation error analysis

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Abstract

This study presents a unified approach for solving convection-diffusion-reaction equations by integrating the Differential Transformation Method (DTM) for analytical approximations with the Crank-Nicolson numerical scheme. The DTM is employed to derive an analytical solution, while the Crank-Nicolson method is used to compute the numerical solution. The results demonstrate that the analytical solution obtained via DTM is identical to the exact solution. Furthermore, the stability of the Crank-Nicolson numerical scheme is assessed using Von-Neumann stability analysis, confirming that the method is unconditionally stable. The local truncation error is determined via Taylor series expansion to establish its order of accuracy. This analysis reveals that the Crank-Nicolson scheme for the convection-diffusion-reaction equation exhibits a local truncation error of order $O(h^2 + k^2)$, ensuring a second-order accurate scheme. Numerical simulations are conducted for various parameter values to examine their impact on the solution. The simulation results demonstrate the gradual transport of the substance from high to low concentration regions, observed through the diminishing displacement of material along the x-axis. Further numerical experiments investigate the effects of different values of h and k. The results indicate a direct correlation between decreasing values of h and k and a reduction in the average error, underscoring the method's accuracy and efficiency.

Keywords. Convection-Diffusion-Reaction, Transformation Differential Method, Crank-Nicolson Method, stability analysis, truncation error analysis.

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1. Introduction

Differential equations are essential for solving problems across various disciplines, including physics, economics, chemistry, and electrical engineering. They are broadly classified into two types: ordinary differential equations (ODEs) and partial differential equations (PDEs). The Convection-Diffusion-Reaction (CDR) equation is an example of a PDE and plays a crucial role in fields such as fluid dynamics, heat transfer, chemical reaction processes, and pollutant transport [3]. Many studies in environmental mathematics utilize CDR systems of equations to model and predict pollution movement in the atmosphere, groundwater, and surface water [12].

The CDR equation encompasses three fundamental processes. The first is convection, which occurs due to the movement of material from one location to another. The second is diffusion, which describes the migration of matter from regions of high concentration to regions of low concentration. The third process is reaction, which results from decay, adsorption, and the interaction of chemicals with other components [17]. Mathematically, the Convection-Diffusion-Reaction (CDR) problem can be represented by differential equations along with the appropriate boundary

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conditions, as follows:

$$\frac{\partial u}{\partial t} = -\mathbf{V}\cdot\nabla u + \nabla\cdot D\nabla u + Q\left(u\right), \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = h(\mathbf{x},t), \quad \text{over} \quad \Gamma_u, \tag{1.2}$$

$$\nabla u\cdot\mathbf{n} = g(\mathbf{x},t), \quad \text{over} \quad \Gamma_\nabla, \tag{1.3}$$

where u is the concentration of the species studied, $\mathbf{V}$ is the velocity of fluid in the medium, D is the diffusion constant, $Q(u)$ the function that defines the reaction process, $h(\mathbf{x},t)$ is a function that defines the values of u in the Dirichlet boundary condition Γ_u, and $g(\mathbf{x},t)$ is the function that defines the normal gradient value of u along the Neumann boundary condition Γ_∇. Equation (1.1) is defined in the domain Ω, and the boundary $\partial\Omega = \Gamma_u + \Gamma_\nabla$ [11]. In this research, we propose one dimensional problem of spatial x dan temporal t as follows

$$u_t = -vu_x + Du_{xx} + f\left(u\right), \quad (x,t)\in[a,b]\times[c,d]\,. \tag{1.4}$$

Here, u denotes a function that varies with respect to x and t, where t represents the observed concentration of the substance. The variable v signifies the convection coefficient, equivalent to the fluid velocity within a medium. The term D refers to the diffusion coefficient, while $f(u)$ indicates the reaction function. The variable t serves as the time variable, and x represents spatial variables. The interval $[a,b]$ defines the spatial domain, and the interval $[c,d]$ delineates the temporal domain.

The analytical and numerical solutions of the CDR equations can be obtained using various approaches, such as the Differential Transformation Method (DTM) and the Crank-Nicolson method. The DTM is particularly advantageous for transforming complex differential equations into more manageable forms, facilitating analysis and solution derivation. Its implementation in convection-dominated situations has proven effective in simplifying the governing equations while preserving the fundamental characteristics of the original problem [9].

The DTM is an analytical method that utilizes the Taylor series to obtain solutions to differential equations in polynomial form [14]. The given differential equations and initial conditions are transformed into recursive equations, which then generate power series coefficients. This method is highly useful for solving both linear and nonlinear differential equations without requiring linearization or perturbation [4]. Furthermore, [20] discusses the solution of the Korteweg–de Vries (KdV) equation using the DTM, while [13] explores the application of the DTM to solve the Schrödinger equation.

The Crank-Nicolson method is a well-established numerical technique for solving parabolic partial differential equations. Its implicit nature enhances both stability and accuracy, making it particularly advantageous. This method achieves second-order accuracy in both time and space, making it suitable for problems involving convection and diffusion [21]. Stability analysis indicates that the Crank-Nicolson method is unconditionally stable for linear problems, which is a significant advantage when dealing with stiff equations commonly found in convection-dominated scenarios [5]. Additionally, an analysis of truncation error shows that the Crank-Nicolson method effectively balances discretization errors, providing a robust framework for numerical simulations [1].

In the context of CDR equations, various stabilization techniques have been developed to address the challenges posed by high convection rates. The Streamline Upwind/Petrov-Galerkin (SUPG) method has been integrated into the Virtual Element Method (VEM) to enhance the stability of numerical solutions in convection-dominated problems ([9], [7]). This approach improves the accuracy of numerical solutions while preserving the maximum principle, which is crucial for ensuring the physical realism of the results [6]. The effectiveness of SUPG-stabilized methods has been assessed, demonstrating their ability to maintain convergence in challenging scenarios where conventional methods may fail [19].

The CDR equation is numerically solved using the upwind forward Euler finite difference scheme, the nonstandard finite difference scheme, and the unconditionally positive finite difference scheme. The upwind forward Euler and nonstandard finite difference schemes are consistent, while the numerical stability of all three methods is conditional [18]. The CDR equation has also been solved using the finite element approach, which revealed that the numerical scheme becomes unstable when the spatial step size exceeds 0.1 [16]. Previous research has addressed the solution of convection-diffusion-reaction equations in both the temporal and spatial domains, employing explicit and implicit finite difference approaches for discretization. This research found that the implicit technique is unconditionally stable, whereas the explicit method must satisfy the von Neumann stability constraints [10]. Additionally, the Crank-Nicolson approach has been applied to solve the convection-diffusion-reaction equation under Danckwerts boundary conditions [3].

Moreover, analyzing truncation errors in numerical methods for CDR equations is crucial for understanding the limitations and accuracy of the obtained solutions. Research indicates that the choice of discretization and stabilization techniques significantly influences truncation errors, thereby affecting the overall effectiveness of the numerical method [8]. The use of adaptive mesh refinement in combination with stabilized approaches has been shown to improve accuracy while effectively managing truncation errors [22].

This research investigates the numerical solution of the CDR equation using the differential transformation method and the Crank-Nicolson technique under Dirichlet boundary conditions, building on previous studies. Stability analysis, accuracy order computations, and error analysis were then performed based on the numerical simulations.

This research introduces a novel approach that has not been extensively explored in the existing literature. While previous studies have examined various numerical methods for convection-diffusion equations, the specific integration of the differential transformation method with the Crank-Nicolson method, combined with a comprehensive stability and truncation error analysis, represents a significant advancement.

To solve the problems in this research, several steps were taken. The first step involved formulating the CDR equation using the Crank-Nicolson method. The variable u was approximated as the mean of the implicit and explicit schemes, with u_t discretized using the forward finite difference method and u_x and u_{xx} discretized using the central finite difference method. In the second step, a von Neumann stability analysis was performed to determine the stability requirements of the numerical scheme established in the previous step. The third step involved conducting a local truncation error analysis to determine the order of the local truncation error. The fourth step consisted of performing numerical simulations under three different conditions: varying the time intervals, modifying the spatial step size h, and adjusting the temporal step size k. Finally, conclusions were drawn based on the findings from the previous steps.

2. The Differential Transformation Method (DTM)

[2] stated that the initial concept of the differential transformation method was first introduced by Zhou in 1986 to solve linear and nonlinear problems in electrical circuits. According to [15], the differential transformation method is quite accurate for time intervals around $t = 0$. However, its accuracy decreases as the time interval increases. This method is particularly suitable for observing the behavior of variables in a model over a relatively short period. The convection-diffusion-reaction (CDR) problem in one-dimensional space x and time t is expressed as follows:

$$u_t = -vu_x + Du_{xx} + f\left(u\right), \quad \left(x,t\right) \in \left[a,b\right] \times \left[c,d\right], \tag{2.1}$$

with an initial condition

$$u\left(x, t=0\right) = p(x), \tag{2.2}$$

and the boundary conditions as follows

$$u\left(x=a,t\right) = q(t), \tag{2.3}$$

$$u\left(x=b\ ,t\right) = r(t). \tag{2.4}$$

Before we apply the TDM method, we proof the uniqueness of the solution for the CDR Equation (2.5).

Theorem 2.1 (**Uniqueness of the solution**). *Consider the initial-boundary value problem:*

$$u_t = -vu_x + Du_{xx} + f(u), \quad (x,t) \in [a,b] \times [c,d], \tag{2.5}$$

with initial condition:

$$u(x,0) = p(x), \tag{2.6}$$

and boundary conditions:

$$u(a,t) = q(t), \tag{2.7}$$

$$u(b,t) = r(t). \tag{2.8}$$

Assume that $f(u)$ is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that:

$$|f(u_1) - f(u_2)| \leq L|u_1 - u_2|, \quad \forall u_1, u_2. \tag{2.9}$$

Then, the solution to the problem is unique.

Proof. Suppose there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ satisfying the given Equation (2.5) with the same initial and boundary conditions. Define their difference:

$$v(x,t) = u_1(x,t) - u_2(x,t). \tag{2.10}$$

Then, $v(x,t)$ satisfies the homogeneous (2.5):

$$v_t + vv_x - Dv_{xx} = f(u_1) - f(u_2). \tag{2.11}$$

Using the Lipschitz condition, we obtain:

$$|f(u_1) - f(u_2)| \leq L|v|. \tag{2.12}$$

The initial and boundary conditions for v are:

$$v(x,0) = 0, \tag{2.13}$$

$$v(a,t) = 0, \tag{2.14}$$

$$v(b,t) = 0. \tag{2.15}$$

Define the energy function:

$$E(t) = \frac{1}{2}\int_a^b v^2(x,t)\,dx, \tag{2.16}$$

differentiating with respect to t, we get:

$$\frac{dE}{dt} = \int_a^b vv_t\,dx, \tag{2.17}$$

substituting $v_t = -vv_x + Dv_{xx} + f(u_1) - f(u_2)$, we obtain:

$$\frac{dE}{dt} = \int_a^b v(-vv_x + Dv_{xx} + f(u_1) - f(u_2))\,dx, \tag{2.18}$$

integration by parts and using the boundary conditions, we get:

$$\frac{dE}{dt} = -\int_a^b v^2 v_x\,dx - D\int_a^b v_x^2\,dx + \int_a^b v(f(u_1) - f(u_2))\,dx, \tag{2.19}$$

using the Lipschitz condition:

$$\int_a^b v(f(u_1) - f(u_2))\,dx \leq L\int_a^b v^2\,dx. \tag{2.20}$$

Since $v^2 v_x$ integrates to zero, we obtain:

$$\frac{dE}{dt} \leq L\int_a^b v^2\,dx. \tag{2.21}$$

Applying Grönwall's inequality:

$$E(t) \leq E(0)e^{Lt} = 0. \tag{2.22}$$

Thus, $E(t) = 0$, which implies $v(x,t) = 0$ for all (x,t), proving uniqueness. $\square$

Furthermore, we apply the two-dimensional differential transformation that is defined as

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0}, \tag{2.23}$$

with $u(x,t)$ being the original function and $U(k,h)$ being the result of the transformation. The inverse differential transform of $U(k,h)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h. \tag{2.24}$$

Based on Equations (2.23) and (2.24), it will be obtained

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^k t^h, \tag{2.25}$$

Equation (2.25) has similarities with the Maclaurin series form so that Equation (2.25) can be written as follows:

$$\begin{aligned} u(x,t) &= U(0,0)x^0t^0 + U(1,0)x^1t^0 + U(0,1)x^0t^1 + U(1,1)x^1t^1 + U(2,0)x^2t^0 \\ &\quad + U(0,2)x^0t^2 + U(1,2)x^1t^2 + ..., \end{aligned} \tag{2.26}$$

so it can be said that $U(k,h)$ are the coefficients of the power series terms. The properties of the differential transformation method can be seen in Ayaz (2004). The Convection-Diffusion-Reaction (CDR) Equation (2.5) with $f(u) = -qu$ is transformed based on the properties of differential transformations. From the CDR Equation (2.5), we have u_t, u_x, $u_x x$, and $f(u) = -qu$ terms. Furthermore, these terms are transformed by using the differential transformation method. Transformation function of u is $U(k,h)$, and the first derivative of u with respect to x is

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^k t^h\right), \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} \frac{\partial}{\partial x} x^k t^h, \\ &= \sum_{k=1}^{\infty}\sum_{h=0}^{\infty} k\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^{k-1} t^h. \end{aligned}$$

Suppose $l = k-1$, then we obtain

$$\frac{\partial u}{\partial x} = \sum_{l=0}^{\infty}\sum_{h=0}^{\infty} (l+1)\frac{1}{(l+1)!h!}\left[\frac{\partial^{(l+1)+h}u(x,t)}{\partial x^{l+1} \partial t^h}\right]_{x=0,t=0} x^l t^h. \tag{2.27}$$

If $l = k$, then the Equation (2.27) becomes

$$\begin{aligned} \frac{\partial u}{\partial x} &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} (k+1)\frac{1}{(k+1)!h!}\left[\frac{\partial^{(k+1)+h}u(x,t)}{\partial x^{k+1} \partial t^h}\right]_{x=0,t=0} x^k t^h, \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} (k+1)U(k+1,h)x^k t^h. \end{aligned}$$

We get the transformation function of u_x is $(k+1)U(k+1,h)$. Furthermore, we take the second derivative of u with respect to x is

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^k t^h\right),\\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0}\frac{\partial^2}{\partial x^2} x^k t^h,\\ &= \sum_{k=1}^{\infty}\sum_{h=0}^{\infty}k\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0}\frac{\partial}{\partial x} x^{k-1} t^h,\\ &= \sum_{k=2}^{\infty}\sum_{h=0}^{\infty}k(k-1)\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^{k-2} t^h.\end{aligned}$$

Suppose $l = k-2$, then we get

$$\frac{\partial^2 u}{\partial x^2} = \sum_{l=0}^{\infty}\sum_{h=0}^{\infty}(l+2)(l+1)\frac{1}{(l+2)!h!}\left[\frac{\partial^{(l+2)+h}u(x,t)}{\partial x^{l+2}\partial t^h}\right]_{x=0,t=0} x^l t^h. \tag{2.28}$$

If $l = k$, then the Equation (2.28) becomes

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(k+2)(k+1)\frac{1}{(k+2)!h!}\left[\frac{\partial^{(k+2)+h}u(x,t)}{\partial x^{k+2}\partial t^h}\right]_{x=0,t=0} x^k t^h,\\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(k+2)(k+1)U(k+2,h)x^k t^h.\end{aligned}$$

The transformation function of u_{xx} is $(k+2)(k+1)U(k+2,h)$. Then, the DTM of term u_t is similar to the DTM of u_x as follows

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0} x^k t^h\right),\\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{x=0,t=0}\frac{\partial}{\partial t} x^k t^h,\\ &= \sum_{k=0}^{\infty}\sum_{h=1}^{\infty}h\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t^h}\right]_{k=0,t=0} x^k t^{h-1}.\end{aligned}$$

Suppose $l = h-1$, then we yield

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(l+1)\frac{1}{k!(l+1)!}\left[\frac{\partial^{k+(l+1)}u(x,t)}{\partial x^k \partial t^{l+1}}\right]_{x=0,t=0} x^k t^l. \tag{2.29}$$

If $l = h$, then the Equation (2.29) becomes

$$\begin{aligned}\frac{\partial u}{\partial t} &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(h+1)\frac{1}{k!(h+1)!}\left[\frac{\partial^{k+(h+1)}u(x,t)}{\partial x^k \partial t^{h+1}}\right]_{x=0,t=0} x^k t^h,\\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(k+1)U(k,h+1)x^k t^h.\end{aligned}$$

We get the transformation function of u_t is $(h+1)U(k,h+1)$. Next, we substitute the transformation function into the Convection Diffusion-Reaction with $f(u) = -qu$, thus we get

$$U(k,h+1) = \frac{D(k+2)(k+1)U(k+2,h) - v(k+1)U(k+1,h) - qU(k,h)}{h+1}. \tag{2.30}$$

For example the initial condition of the CDR equation is $u(x,0) = p(x)$, then the initial value is tranformed into $U(k,0)$ using the definition of differential transformation as follows

$$U(k,0) = \frac{1}{k!}\left[\frac{\partial^k u(x,0)}{\partial x^k}\right]_x,$$

such that we obtain the value of $U(k,0)$ is

$$\begin{aligned}
U(1,0) &= \frac{1}{1!}\left[\frac{\partial p(x)}{\partial x}\right]_x = p'(0),\\
U(2,0) &= \frac{1}{2!}\left[\frac{\partial^2 p(x)}{\partial x^2}\right]_x = \frac{1}{2}p''(0),\\
U(3,0) &= \frac{1}{3!}\left[\frac{\partial^3 p(x)}{\partial x^3}\right]_x = \frac{1}{6}p'''(0),\\
U(4,0) &= \frac{1}{4!}\left[\frac{\partial^4 p(x)}{\partial x^4}\right]_x = \frac{1}{24}p^{(4)}(0),\\
U(5,0) &= \frac{1}{5!}\left[\frac{\partial^5 p(x)}{\partial x^5}\right]_x = \frac{1}{120}p^{(5)}(0),\\
U(6,0) &= \frac{1}{6!}\left[\frac{\partial^6 p(x)}{\partial x^6}\right]_x = \frac{1}{720}p^{(6)}(0).
\end{aligned}$$

The next step is to calculate Equation (2.30) using the value of $U(k,0)$ to get the values of $U(k,h)$, with $k = 0,1,2,\ldots$ and $h = 1,2,\ldots$. For $h = 1$ and $k = 0,1,2,\ldots$

$$\begin{aligned}
U(0,1) =& D(2)(1)U(2,0) - v(1)U(1,0) - qU(0,0)\\
=& 2D\left(\frac{1}{2}p''(0)\right) - v(p'(0)) - q(p(0)) = Dp''(0) - vp'(0) - qp(0),\\
U(1,1) =& D(3)(2)U(3,0) - v(2)U(2,0) - qU(1,0)\\
=& 6D\left(\frac{1}{6}p'''(0)\right) - 2v\left(\frac{1}{2}p''(0)\right) - q(p'(0)) = Dp'''(0) - vp''(0) - qp'(0),\\
U(2,1) =& D(4)(3)U(4,0) - v(3)U(3,0) - qU(2,0)\\
=& 12D\left(\frac{1}{24}p^{(4)}(0)\right) - 3v\left(\frac{1}{6}p'''(0)\right) - q\left(\frac{1}{2}p''(0)\right) = \frac{Dp^{(4)}(0) - vp'''(0) - qp''(0)}{2},\\
U(3,1) =& D(5)(4)U(5,0) - v(4)U(4,0) - qU(3,0)\\
=& 20D\left(\frac{1}{120}p^{(5)}(0)\right) - 4v\left(\frac{1}{24}p^{(4)}(0)\right) - q\left(\frac{1}{6}p'''(0)\right) = \frac{Dp^{(5)}(0) - vp^{(4)}(0) - qp'''(0)}{6},\\
U(4,1) =& D(6)(5)U(6,0) - v(5)U(5,0) - qU(4,0)\\
=& 30D\left(\frac{1}{720}p^{(6)}(0)\right) - 5v\left(\frac{1}{120}p^{(6)}(0)\right) - q\left(\frac{1}{24}p^{(4)}(0)\right) = \frac{Dp^{(6)}(0) - vp^{(5)}(0) - qp^{(4)}(0)}{24}.
\end{aligned}$$

Then, for $h=2$ and $k=0,1,2,...$

$$
\begin{aligned}
U(0,2) &= \frac{D(2)(1)U(2,1)-v(1)U(1,1)-qU(0,1)}{2}\\
&= \frac{1}{2}\Big(2D(\frac{1}{2}Dp^{(4)}(0)-\frac{1}{2}vp'''(0)-\frac{1}{2}qp''(0))-v(Dp'''(0)-vp''(0)-qp'(0))\\
&- q(Dp''(0)-vp'(0)+q^3p(0))\Big)\\
&= \frac{D^2p^{(4)}(0)-2vDp'''(0)-(2Dq-v^2)p''(0)+2qvp'(0)+q^3p(0)}{2},\\
U(1,2) &= \frac{D(3)(2)U(3,1)-v(2)U(2,1)-qU(1,1)}{2}\\
&= \frac{1}{2}\Big(6D(\frac{1}{6}Dp^{(5)}(0)-\frac{1}{6}vp^{(4)}(0)-\frac{1}{6}qp'''(0))-2v(\frac{1}{2}Dp^{(4)}(0)-\frac{1}{2}vp'''(0)-\frac{1}{2}qp''(0))\\
&- q(Dp'''(0)-vp''(0)+qp'(0))\Big)\\
&= \frac{D^2p^{(5)}(0)-2vDp^{(4)}(0)-(2Dq-v^2)p'''(0)+2qvp''(0)+q^2p'(0)}{2},\\
U(2,2) &= \frac{D(4)(3)U(4,1)-v(3)U(3,1)-qU(2,1)}{2}\\
&= \frac{\frac{1}{2}D^2p^{(6)}(0)-\frac{1}{2}Dvp^{(5)}(0)-\frac{1}{2}Dqp^{(4)}(0)-\frac{1}{2}vDp^{(5)}(0)+\frac{1}{2}v^2p^{(4)}(0)+\frac{1}{2}vqp'''(0)-\frac{1}{2}qDp^{(4)}(0)}{2}\\
&= \frac{\frac{1}{2}D^2p^{(6)}(0)-vDp^{(5)}(0)-(Dq-\frac{1}{2}v^2)p^{(4)}(0)+qvp'''(0)+\frac{1}{2}q^2p''(0)}{2}.
\end{aligned}
$$

By applying the same method for $h=3,4,5,...$, and $k=0,1,2,3,...$, we obtaine value of $U(k,h)$ as shown in the Table 1.

TABLE 1. $U(k,h)$ value with initial condition $u(x,0)=p(x)$.

U(k,h)	**0**	**1**	**2**	...
0	$p(0)$	$Dp''(0)-vp'(0)-qp(0)$	$\frac{D^2p^{(4)}(0)-2vDp'''(0)-(2Dq-v^2)p''(0)+2qvp'(0)+q^2p(0)}{2}$	
1	$p'(0)$	$Dp'''(0)-vp''(0)-qp'(0)$	$\frac{D^2p^{(5)}(0)-2vDp^{(4)}(0)-(2Dq-v^2)p'''(0)+2qvp''(0)+q^2p'(0)}{2}$	
2	$\frac{1}{2}p''(0)$	$\frac{1}{2}Dp^{(4)}(0)-\frac{1}{2}vp'''(0)-\frac{1}{2}qp''(0)$	$\frac{\frac{1}{2}D^2p^{(6)}(0)-vDp^{(5)}(0)-(Dq-\frac{1}{2}v^2)p^{(4)}(0)+qvp'''(0)+\frac{1}{2}q^2p''(0)}{2}$	
...				

The next step is substituting the value $U(k,h)$ into the equation

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h,$$

so that the solution is obtained from convection diffusion-reaction is as follows

$$
\begin{aligned}
u(x,t) &= p(0)+p'(0)x+(Dp''(0)-vp'(0)-qp(0))t+\frac{1}{2}p''(0)x^2+(Dp'''(0)-vp''(0)-qp'(0))xt \\
&\quad + \left(\frac{D^2p^{(4)}(0)-2vDp'''(0)-(2Dq-v^2)p''(0)+2qvp'(0)+q^2p(0)}{2}\right)+...
\end{aligned} \tag{2.31}
$$

Suppose the CDR equation is

$$
\begin{aligned}
&u_t + u_x - u_{xx} = -u, \ (x,t) \in [0,10] \times [0,10], \\
&u(x,0) = e^{-x}, \\
&u(0,t) = e^{t}, \\
&u(10,t) = e^{t-10},
\end{aligned}
\tag{2.32}
$$

so that the Table 1 becomes as shown in the Table 2.

TABLE 2. $U(k,h)$ value with initial condition $u(x,0) = e^{-x}$.

U(k,h)	**0**	**1**	**2**	...
0	1	$D+v-q$	$\frac{D^2+2vD-2Dq+v^2-2qv+q^2}{2}$	
1	-1	$-D-v+q$	$\frac{-D^2-2vD+2Dq-v^2+2qv-q^2}{2}$	
2	$\frac{1}{2}$	$\frac{1}{2}D+\frac{1}{2}v-\frac{1}{2}q$	$\frac{\frac{1}{2}D^2+vD-Dq+\frac{1}{2}v^2-qv+\frac{1}{2}q^2}{2}$	
...				

Based on the Table 2, Equation (2.31) can be written as follows

$$
\begin{aligned}
u(x,t) = {} & 1 - x + \frac{1}{2}x^2 + (D+v-q)t + (-D-v+q)xt + (\frac{1}{2}D + \frac{1}{2}v - \frac{1}{2}q)x^2 t \\
& + \left(\frac{D^2 + 2Dv - 2Dq + v^2 - 2qv + q^2}{2}\right)t^2 + \left(\frac{-D^2 - 2Dv + 2Dq - v^2 + 2qv - q^2}{2}\right)xt^2 \\
& + \left(\frac{\frac{1}{2}D^2 + Dv - Dq + \frac{1}{2}v^2 - qv + \frac{1}{2}q^2}{2}\right)x^2t^2 + \cdots
\end{aligned}
$$

Suppose

$$
A = D + v - q, \tag{2.33}
$$

then

$$
\begin{aligned}
u(x,t) &= 1 - x + \frac{1}{2}x^2 + At - Axt + \frac{Ax^2t}{2} + \frac{A^2t^2}{2} - \frac{A^2xt^2}{2} + \frac{A^2x^2t^2}{2} + ..., \\
&= \left(1 - x + \frac{x^2}{2} - ...\right) + At\left(1 - x + \frac{x^2}{2} - ...\right) + \frac{A^2t^2}{2}\left(1 - x + \frac{x^2}{2} - ...\right) + ..., \\
&= \left(1 - x + \frac{x^2}{2} - ...\right)\left[1 + At + \frac{(At)^2}{2} + ...\right].
\end{aligned}
\tag{2.34}
$$

By using the Maclaurin series, the series of function e^{-x} and e^{x} is as follows

$$
\begin{aligned}
e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ..., \\
e^{x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...,
\end{aligned}
$$

so that the equation (2.34) is obtained as follows.

$$
\begin{aligned}
u(x,t) &= e^{-x}e^{at}, \\
&= e^{at-x}.
\end{aligned}
\tag{2.35}
$$

The values of $D = v = q = 1$, based on Equation (2.33) we have the value $A = 1$, so that the Equation (2.35) can be written as

$$
u(x,t) = e^{t-x}.
$$

The solution of the Convection-Diffusion-Reaction equation is $u(x,t) = e^{t-x}$.

3. The Crank-Nicolson Numerical Scheme

In this part, the Convection-Diffusion-Reaction equation with $f(u) = -qu$ is constructed using the Crank-Nicolson scheme to obtain the following results

$$\frac{u_j^{n+1} - u_j^n}{k} + v\frac{u_{j+1}^n - u_{j-1}^n + u_{j+1}^{n+1} - u_{j-1}^{n+1}}{4h} - D\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2h^2} = -q\frac{u_j^{n+1} + u_j^n}{2}. \tag{3.1}$$

Equation (3.1) can be simplified to

$$\begin{aligned} &u_{j-1}^{n+1}\left(-\frac{kv}{4h} - \frac{Dk}{2h^2}\right) + u_j^{n+1}\left(1 + \frac{Dk}{h^2} + \frac{kq}{2}\right) + u_{j+1}^{n+1}\left(\frac{kv}{4h} - \frac{Dk}{2h^2}\right) \\ &\quad = u_{j-1}^n\left(\frac{kv}{4h} + \frac{Dk}{2h^2}\right) + u_j^n\left(1 - \frac{Dk}{h^2} - \frac{kq}{2}\right) - u_{j+1}^n\left(\frac{kv}{4h} - \frac{Dk}{2h^2}\right). \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \alpha &= \frac{kv}{4h} + \frac{Dk}{2h^2}, \\ \beta &= \frac{Dk}{h^2} + \frac{kq}{2}, \\ \gamma &= \frac{kv}{4h} - \frac{Dk}{2h^2}. \end{aligned} \tag{3.3}$$

then we have

$$-\alpha u_{j-1}^{n+1} + (1+\beta)\, u_j^{n+1} + \gamma u_{j+1}^{n+1} = \alpha u_{j-1}^n + (1-\beta)\, u_j^n - \gamma u_{j+1}^n. \tag{3.4}$$

Equation (3.4) is a numerical scheme of the Convection-Diffusion-Reaction equation using the Crank-Nicolson scheme. Then, the numerical scheme is described for $j = 1, 2, \ldots, M-1$ to obtain a system of linear equations which is expressed in matrix form as follows

$$\begin{bmatrix} (1+\beta) & \gamma & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\alpha & (1+\beta) & \gamma & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\alpha & (1+\beta) & \gamma & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\alpha & (1+\beta) & \gamma & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha & (1+\beta) & \gamma \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\alpha & (1+\beta) \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{M-3}^{n+1} \\ u_{M-2}^{n+1} \\ u_{M-1}^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha u_0^n + (1-\beta)\, u_1^n - \gamma u_2^n + \alpha u_0^{n+1} \\ \alpha u_1^n + (1-\beta)\, u_2^n - \gamma u_3^n \\ \alpha u_2^n + (1-\beta)\, u_3^n - \gamma u_4^n \\ \vdots \\ \alpha u_{M-4}^n + (1-\beta)\, u_{M-3}^n - \gamma u_{M-2}^n \\ \alpha u_{M-3}^n + (1-\beta)\, u_{M-2}^n - \gamma u_{M-1}^n \\ \alpha u_{M-2}^n + (1-\beta)\, u_{M-1}^n - \gamma u_M^n - \gamma u_M^{n+1} \end{bmatrix}. \tag{3.5}$$

4. The Von-Neumann Stability Analysis

To perform Von-Neumann stability analysis, u_j^n is first defined as follows

$$u_j^n = \left(e^{\alpha k}\right)^n e^{i\theta jh} = \lambda^n e^{i\theta jh}. \tag{4.1}$$

Next, the u_j^n value is substituted into the Crank-Nicolson scheme (3.4) to obtain the λ value as follows

$$\lambda = \frac{\alpha e^{-i\theta h} + (1-\beta) - \gamma e^{i\theta h}}{-\alpha e^{-i\theta h} + (1+\beta) + \gamma e^{i\theta h}}. \tag{4.2}$$

Then, the α, β, and γ values are substituted back to obtain the following results

$$\lambda = \frac{\left(\frac{kv}{4h} + \frac{Dk}{2h^2}\right) e^{-i\theta h} + 1 - \left(\frac{Dk}{h^2} + \frac{kq}{2}\right) + \left(\frac{kv}{4h} - \frac{Dk}{2h^2}\right) e^{i\theta h}}{-\left(\frac{kv}{4h} + \frac{Dk}{2h^2}\right) e^{-i\theta h} + 1 + \left(\frac{Dk}{h^2} + \frac{kq}{2}\right) - \left(\frac{kv}{4h} - \frac{Dk}{2h^2}\right) e^{i\theta h}}, \tag{4.3}$$

where

$$g = \frac{kv}{4h}, w = \frac{Dk}{h^2}, z = \frac{kq}{2}. \tag{4.4}$$

Furthermore, λ can be written as follows

$$\begin{aligned} \lambda &= \frac{g\left(e^{-i\theta h} + e^{i\theta h}\right) + \frac{w}{2}\left(e^{-i\theta h} - e^{i\theta h}\right) + 1 - (w+z)}{-g\left(e^{-i\theta h} + e^{i\theta h}\right) - \frac{w}{2}\left(e^{-i\theta h} - e^{i\theta h}\right) + 1 + (w+z)}, \\ &= \frac{1 - (w + z - 2g\cos\theta h) + i\ w\sin\theta h}{1 + (w + z - 2g\cos\theta h) - i\ w\sin\theta h}. \end{aligned} \tag{4.5}$$

We denote

$$\begin{aligned} X_1 &= 1 - (w + z - 2g\cos\theta h), \\ X_2 &= 1 + (w + z - 2g\cos\theta h), \\ Y &= w\sin\theta h, \end{aligned} \tag{4.6}$$

then, Equation (4.5) can be written as follows

$$\lambda = \frac{X_1 + iY}{X_2 - iY}. \tag{4.7}$$

Furthermore, we calculate the modulus of (4.6) as follows

$$|\lambda| = \frac{|X_1 + iY|}{|X_2 - iY|}, \tag{4.8}$$

$$= \frac{\sqrt{X_1^2 + Y^2}}{\sqrt{X_2^2 + Y^2}}, \tag{4.9}$$

$$= \frac{\sqrt{(1 - (w + z - 2g\cos\theta h))^2 + \ w\sin^2\theta h}}{\sqrt{(1 + (w + z - 2g\cos\theta h))^2 + \ w\sin^2\theta h}}. \tag{4.10}$$

It is clear that $|\lambda| \leq 1$, which ensures that the Crank-Nicolson scheme (3.4) for the Convection-Diffusion-Reaction equation is unconditionally stable.

5. Local Truncation Error Analysis

The local truncation error analysis is performed using a Taylor series expansion in the Crank-Nicolson scheme for the Convection-Diffusion-Reaction equation. Equation (3.1) can be rewritten in the following form:

$$FDE = \frac{u_j^{n+1} - u_j^n}{k} + \frac{v}{2}\left(\frac{\delta_x u_j^{n+1} + \delta_x u_j^n}{2h}\right) - \frac{D}{2}\left(\frac{\delta_x^2 u_j^{n+1} + \delta_x^2 u_j^n}{h^2}\right) + \frac{q}{2}\left(u_j^{n+1} + u_j^n\right), \tag{5.1}$$

with FDE is called a finite difference equation, and the Taylor series are defined as follow

$$u_j^{n+1} = \left[u + ku_t + \frac{1}{2!}k^2 u_{tt} + \frac{1}{3!}k^3 u_{ttt} + \frac{1}{4!}k^4 u_{tttt} + \dots\right]_j^n, \tag{5.2}$$

$$u_j^{n-1} = \left[u - ku_t + \frac{1}{2!}k^2u_{tt} - \frac{1}{3!}k^3u_{ttt} + \frac{1}{4!}k^4u_{tttt} + \dots\right]_j^n, \tag{5.3}$$

$$\delta_x u_j^n = \left[2hu_x + \frac{1}{3}h^3u_{xxx} + \frac{1}{60}h^5u_{xxxxx} + \dots\right]_j^n, \tag{5.4}$$

$$\delta_x^2 u_j^n = \left[h^2u_{xx} + \frac{1}{12}h^4u_{xxxx} + \frac{1}{360}h^6u_{xxxxxx} + \dots\right]_j^n. \tag{5.5}$$

The Crank-Nicolson scheme is a numerical method that is extended by considering the value $t = n + \frac{1}{2}$, allowing Equations (5.2) to (5.5) to be transformed accordingly.

$$u_j^{n+1} = \left[u + \frac{1}{2}ku_t + \frac{1}{2!}\left(\frac{1}{2}k\right)^2u_{tt} + \frac{1}{3!}\left(\frac{1}{2}k\right)^3u_{ttt} + \frac{1}{4!}\left(\frac{1}{2}k\right)^4u_{tttt} + \dots\right]_j^{n+\frac{1}{2}}, \tag{5.6}$$

$$u_j^{n} = \left[u - \frac{1}{2}ku_t + \frac{1}{2!}\left(\frac{1}{2}k\right)^2u_{tt} - \frac{1}{3!}\left(\frac{1}{2}k\right)^3u_{ttt} + \frac{1}{4!}\left(\frac{1}{2}k\right)^4u_{tttt} + \dots\right]_j^{n+\frac{1}{2}}, \tag{5.7}$$

$$\begin{aligned}\delta_x u_j^{n+1} = \Bigg[&\left(2hu_x + \frac{1}{3}h^3u_{xxx} + \frac{1}{60}h^5u_{xxxxx} + \dots\right)\\ &+ \frac{1}{2}k\left(2hu_{xt} + \frac{1}{3}h^3u_{xxxt} + \frac{1}{60}h^5u_{xxxxxt} + \dots\right)\\ &+\frac{1}{2!}\left(\frac{1}{2}k\right)^2\left(2hu_{xtt} + \frac{1}{3}h^3u_{xxxtt} + \frac{1}{60}h^5u_{xxxxxtt} + \dots\right) + \dots\Bigg]_j^{n+\frac{1}{2}},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\delta_x u_j^{n} = \Bigg[&\left(2hu_x + \frac{1}{3}h^3u_{xxx} + \frac{1}{60}h^5u_{xxxxx} + \dots\right)\\ &- \frac{1}{2}k\left(2hu_{xt} + \frac{1}{3}h^3u_{xxxt} + \frac{1}{60}h^5u_{xxxxxt} + \dots\right)\\ &+\frac{1}{2!}\left(\frac{1}{2}k\right)^2\left(2hu_{xtt} + \frac{1}{3}h^3u_{xxxtt} + \frac{1}{60}h^5u_{xxxxxtt} + \dots\right) + \dots\Bigg]_j^{n+\frac{1}{2}},\end{aligned} \tag{5.9}$$

$$\begin{aligned}\delta_x u_j^{n+1} = \Bigg[&\left(h^2u_{xx} + \frac{1}{12}h^4u_{xxxx} + \frac{1}{360}h^6u_{xxxxxx} + \dots\right)\\ &+ \frac{1}{2}k\left(h^2u_{xxt} + \frac{1}{12}h^4u_{xxxxt} + \frac{1}{360}h^6u_{xxxxxxt} + \dots\right)\\ &+\frac{1}{2!}\left(\frac{1}{2}k\right)^2\left(h^2u_{xxtt} + \frac{1}{12}h^4u_{xxxxtt} + \frac{1}{360}h^6u_{xxxxxxtt} + \dots\right) + \dots\Bigg]_j^{n+\frac{1}{2}},\end{aligned} \tag{5.10}$$

$$\delta_x u_j^n = \Bigg[\left(h^2 u_{xx} + \frac{1}{12}h^4 u_{xxxx} + \frac{1}{360}h^6 u_{xxxxxx} + \dots\right) - \frac{1}{2}k\left(h^2 u_{xxt} + \frac{1}{12}h^4 u_{xxxxt} + \frac{1}{360}h^6 u_{xxxxxxt} + \dots\right) + \frac{1}{2!}\left(\frac{1}{2}k\right)^2\left(h^2 u_{xxtt} + \frac{1}{12}h^4 u_{xxxxtt} + \frac{1}{360}h^6 u_{xxxxxxtt} + \dots\right) + \dots\Bigg]_j^{n+\frac{1}{2}}. \tag{5.11}$$

Then, the Convection-Diffusion-Reaction equation can be written as follows

$$PDE = u_t + vu_x - Du_{xx} + qu, \tag{5.12}$$

with PDE is called a partial differential equation. Equation (5.1) is expanded by Taylor series in each term with $t = n + \frac{1}{2}$ to obtain

$$FDE = u_t + vu_x - Du_{xx} + qu + q\frac{1}{8}k^2 u_{tt} + \frac{1}{24}k^2 u_{ttt} + v\frac{1}{8}k^2 u_{xtt} + v\frac{1}{6}h^2 u_{xxx} - D\frac{1}{8}k^2 u_{xxtt} - D\frac{1}{12}h^2 u_{xxxx} + \frac{1}{192}k^3 u_{tttt} + q\frac{1}{384}k^4 u_{tttt} + v\frac{1}{48}k^2h^2 u_{xxxtt} + v\frac{1}{120}h^4 u_{xxxxx} - D\frac{1}{360}h^4 u_{xxxxxx} - D\frac{1}{96}k^2h^2 u_{xxxxtt} + \dots \tag{5.13}$$

The local truncation error is obtained by calculating the difference between the FDE in Equation (5.13) and the PDE in Equation (5.12) to obtain the following results

$$\tau_j^{n+\frac{1}{2}} = q\frac{1}{8}k^2 u_{tt} + \frac{1}{24}k^2 u_{ttt} + v\frac{1}{8}k^2 u_{xtt} + v\frac{1}{6}h^2 u_{xxx} - D\frac{1}{8}k^2 u_{xxtt} - D\frac{1}{12}h^2 u_{xxxx} + \frac{1}{192}k^3 u_{tttt} + q\frac{1}{384}k^4 u_{tttt} + v\frac{1}{48}k^2h^2 u_{xxxtt} + v\frac{1}{120}h^4 u_{xxxxx} - D\frac{1}{360}h^4 u_{xxxxxx} - D\frac{1}{96}k^2h^2 u_{xxxxtt} + \dots \tag{5.14}$$

Specifically, the local truncation error order is determined by identifying the lowest degree of each term involving h and k. By expanding the finite difference approximations and performing a Taylor series analysis, we have shown that the leading error terms are of order h^2 and k^2. Thus, the local error order of the Crank-Nicholson scheme for the convection-diffusion-reaction equation is given by $O(h^2 + k^2)$.

6. Numerical Simulation

The Convection-Diffusion-Reaction equation to be simulated is

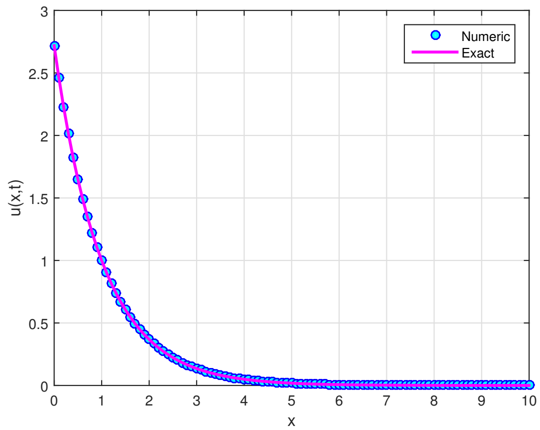
$$\begin{aligned} &u_t + u_x - u_{xx} = -u, \ (x,t) \in [0,10] \times [0,10], \\ &u(x,0) = e^{-x}, \\ &u(0,t) = e^{t}, \\ &u(10,t) = e^{t-10}, \end{aligned} \tag{6.1}$$

with exact solution

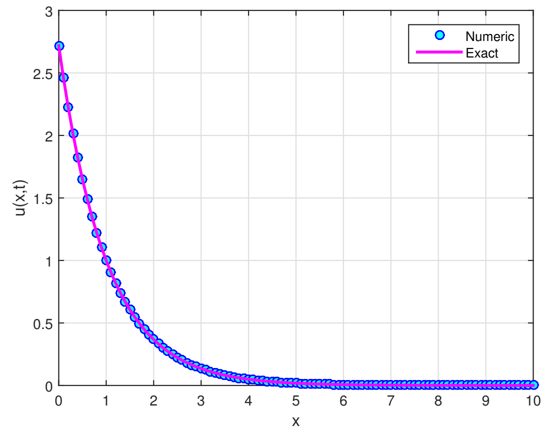
$$u(x,t) = e^{t-x}. \tag{6.2}$$

The numerical simulation to observe the effect of parameters h and k. The effect of values will be observed by simulating constant values h and varying values k. The parameter values used are $h = 0.1$ with $k = 0.01$, $k = 0.1$, and $k = 0.5$ at time $t = 10$. Figure 1 shows numerical solution and exact solution of the Convection-Diffussion-Reaction equation with $h = 0.1$ and $k = 0.01$ at time $t = 1$.

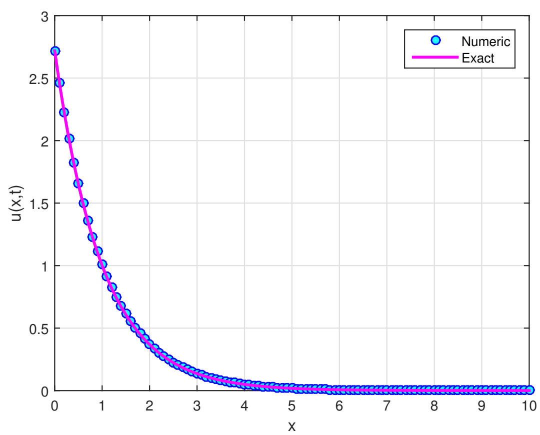
Next, we observe the effect of h values by simulating constant k values and varying h values. The parameter values are $k = 0.1$ with $h = 0.05$, $h = 0.1$, and $h = 0.2$. Numerical simulation results are presented in Figure 2. The average



(a) $h = 0.1$ and $k = 0.01$.



(b) $h = 0.1$ and $k = 0.1$.



(c) $h = 0.1$ and $k = 0.5$.

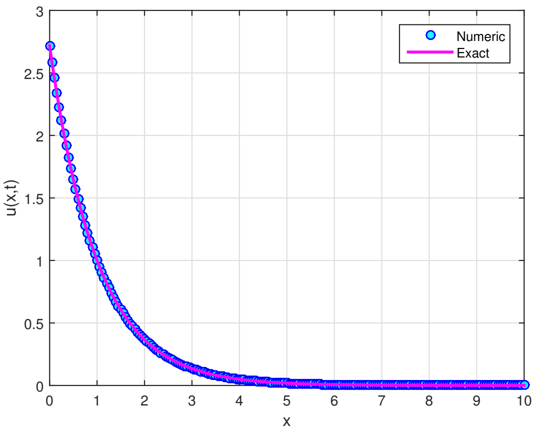
FIGURE 1. Numerical solution and exact solution of the CDF with $h = 0.1$ and $k = 0.01$, $k = 0.1$ and $k = 0.5$ at time $t = 1$

error value in each time the first and second numerical simulations is presented in Tables 3 and 4. Table 3 demonstrates

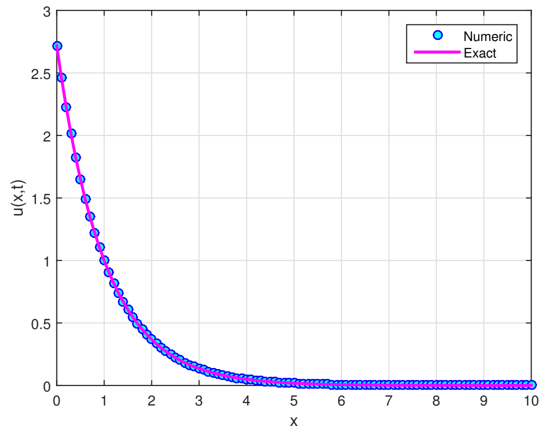
TABLE 3. Average error values for different k ($k = 0.01$, $k = 0.01$, and $k = 0.5$), at step size $h = 0.1$.

x	$k = 0.01$	$k = 0.1$	$k = 0.5$
2	0.49983	0.688331	5.530431
4	0.135396	0.18651	1.508347
6	0.027506	0.037901	0.30853
8	0.004952	0.006825	0.055919
10	0	0	0

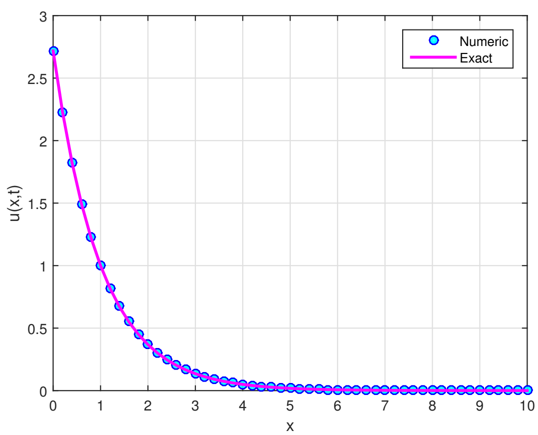
that for a fixed x, the average error increases with larger k values, indicating greater numerical discrepancies as k grows. Conversely, for a constant k, the error progressively decreases as x increases, suggesting an improvement in



(a) $k = 0.1$ and $h = 0.05$.



(b) $k = 0.1$ and $h = 0.1$.



(c) $k = 0.1$ and $h = 0.2$.

FIGURE 2. Numerical solution and exact solution of the CDF with $k = 0.1$ and $h = 0.05$, $k = 0.1$ and $h = 0.2$ at time $t = 1$

numerical stability over time. Notably, at $x = 10$, the error is zero across all k values, signifying the convergence of the numerical method. This pattern implies that while higher k values initially contribute to greater inaccuracies, the method exhibits enhanced precision as x advances. Table 4 reveals that as k increases for a fixed x, the average error

TABLE 4. Average error values for different k ($k = 0.01$, $k = 0.01$, and $k = 0.5$), at step size $h = 0.2$.

x	$k = 0.01$	$k = 0.1$	$k = 0.5$
2	1.987093	2.229276	7.315085
4	0.539609	0.605542	1.999983
6	0.109897	0.123359	0.410103
8	0.019832	0.022268	0.074513
10	0	0	0

also increases, indicating that larger k values introduce greater numerical discrepancies. Conversely, when k remains constant, the error gradually decreases as x grows, suggesting enhanced numerical stability over time. Notably, at $x = 10$, the error is zero across all k values, signifying that the numerical method successfully converges. This trend suggests that while larger k values initially produce higher errors, the method's accuracy improves as x progresses.

7. Discussion

The Convection-Diffusion-Reaction equations have been formulated analytically using the Differential Transformation Method (DTM) and numerically using the Crank-Nicolson method. The results are identical to the exact solution. The numerical scheme, analyzed using Von Neumann stability analysis, demonstrates that the Crank-Nicolson method for the Convection-Diffusion-Reaction problem is unconditionally stable. Subsequently, a local truncation error analysis is conducted via Taylor series expansion, yielding a local truncation error of order $O(h^2+k^2)$. Numerical simulations with various treatments indicate that these treatments influence the flow of chemicals. Figures 3 and Table 4 illustrate the impact of the values of h and k in the second and third simulations. According to these simulations, a reduction in h and k values correlates with a decrease in the average relative error.

The error obtained using the Crank-Nicolson approach is smaller than that obtained from the Finite Element Method (FEM), as investigated by [16]. This suggests that the Crank-Nicolson approach exhibits superior accuracy compared to FEM. Furthermore, the Crank-Nicolson technique demonstrates enhanced stability compared to the methods used in previous studies by [18] and [10].

8. Conclusion

This study develops a unified approach to solving convection-diffusion-reaction equations by integrating the Differential Transformation Method (DTM) for analytical approximations with the Crank-Nicolson numerical scheme. Through stability and truncation error analysis, this work confirms the robustness of the method and its ability to accurately model convection, diffusion, and reaction dynamics. A comprehensive numerical investigation highlights the influence of different parameters on solution behavior, revealing that reducing the discretization parameters h and k significantly improves accuracy. The results suggest that optimizing these parameters leads to a more precise representation of physical processes. The findings provide valuable insights into the interplay between convection, diffusion, and reaction mechanisms. To the best of our knowledge, this is the first attempt to systematically compare these methods in this context, offering both computational efficiency and analytical depth. Future research could explore extending this approach to multi-dimensional cases or incorporating adaptive mesh refinement strategies to further enhance accuracy and computational performance.

Acknowledgment

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Appendix

The Matlab script of Figure 1 and Figure 2 can be written as follows

```
clear all;
clc;
tic
fprintf('THE CRANK-NICOLSON of CDR');
v=1;
D=1;
q=1;
h=0.2;
k=0.1;
```

```
x=0:h:10;
t=0:k:1;
a=(k*v/(4*h))+(D*k/(2*(h^2)));
b=(D*k/(h^2))+(k*(q/2));
c=(k*v/(4*h))-(D*k/(2*(h^2)));
M=length(x);
N=length(t);
u=zeros(M,N);
u(:,1)=exp(-x);
u(1,:)=exp(t);
u(M,:)=exp(t-10);

% Matrix A
A=zeros(M-2,M-2);
A(1,1)=1+b;
A(1,2)=c;
A(M-2,M-3)=-a;
A(M-2,M-2)=1+b;
for i=2:M-3
A(i,i-1)=-a;
A(i,i)=1+b;
A(i,i+1)=c;
end

% Matrix B
B=zeros(M-2,1);
for n=1:N-1
B(1,1)=a*(u(1,n)+u(1,n+1))+(1-b)*u(2,n)-c*u(3,n);
B(M-2,1)=a*u(M-2,n)+(1-b)*u(M-1,n)-c*(u(M,n)+u(M,n+1));
for i=2:M-3
B(i,1)=a*u(i,n)+(1-b)*u(i+1,n)-c*u(i+2,n);
end
u(2:M-1,n+1)=inv(A)*B;
end

% Exact Solution
for j=1:M
for n=1:N
ueks(j,n)=exp(t(n)-x(j));
error(j,n)=abs(u(j,n)-ueks(j,n));
end
end

% Plot exact solution, numerical solution, and error
for n=1:N
if t(n)==1
```

```
figure()
plot(x,u(:,n),'o','linewidth',1,'color','b','markerfacecolor','c');
hold on;
plot(x,ueks(:,n),'linewidth',2,'color','m');
hold on;
xlabel('x');
ylabel('u(x,t)');
legend('Numeric','Exact');
grid on;
axis([0 10 0 3]);
end
end
toc
```

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