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Introducing novel (ψ, ϕ)-fractional operators advances in fractional calculus

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Abstract

This study introduces novel generalized fractional derivatives known as (ψ, ϕ)-fractional derivatives of the Riemann-Liouville and Caputo types, each incorporating exponential function kernels. These new operators offer distinct advantages, including a semi-group property and a seamless extension of the Riemann-Liouville (RL-FD) and Caputo fractional derivatives (C-FD), as well as integrals (RL-FI). We explore the Laplace transform of these (ψ, ϕ)-fractional derivatives and (ψ, ϕ)-integral, leveraging them to address linear (ψ, ϕ)-fractional differential equations. Moreover, these fractional operators are general to classical fractional operators, cotangent fractional operators, and generalized proportional operators.

Keywords. (ψ, ϕ)-fractional operators, Laplace transform, linear (ψ, ϕ)-fractional differential equations.
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1. Introduction

Fractional calculus (FC) [26, 28–32] stands as a comprehensive extension of conventional differentiation and integration, encompassing non-integer orders. Over the past few decades, it has emerged as a highly potent tool for characterizing long-memory processes. Its applications span across various domains including chemistry, physics, electricity, and mechanics, as extensively covered in references such as [9, 11, 22, 25]. The literature on FC is vast, with numerous works discussing its applications and theories. For further exploration, you may refer to publications like [4, 10, 12, 15–17]. Local fractional derivatives (FDs), allowing differentiation with arbitrary orders beyond traditional FDs, play a pivotal role in advancing classical FC. In [20], the conformable derivative (CD) was introduced; however, its limitation lies in the failure to converge to the original function at $\varkappa = 1$. Addressing this, [1] explored alternative CD concepts and posed an open challenge to harness CD for generating more generalized FDs.

To address these concerns, generalized FDs and fractional integrals (FI) were proposed and analyzed in [18, 19], offering partial solutions to this problem. Additionally, Anderson in [5, 6] introduced a novel local FD that converges to the original function at $\varkappa = 1$, thus enhancing the CD. Subsequently, various authors developed new FD types accommodating exponential functions [2, 3, 8, 14, 24, 27] or the Mittag-Leffler function [7] in their operator kernels. However, these novel nonsingular kernel-type FDs suffer a drawback: their corresponding integral operators lack a semi-group property, posing challenges in solving complex fractional systems within their frameworks. Nevertheless, considerable efforts have been invested in defining general FDs and integrals involving the general Mittag-Leffler functions in their representations, as evident in papers see [13, 21, 23]. Motivated by these advancements, we introduce a novel generalized FC based on a specific case of proportional derivatives (PD) as discussed in [5].

Our novel (ψ, ϕ)-fractional operators exhibit three distinct features that set them apart:

(1) The fractional operator's kernel incorporates the exponential function.
(2) The resulting fractional integral demonstrate a semi-group property.

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(3) At order $\varkappa = 1$, these operators revert to the Riemann-Liouville fractional derivative (RL-FD), Caputo fractional derivative (C-FD), and Riemann-Liouville fractional integral (RL-FI).

This paper encompasses various sections detailing distinct aspects of the subject: Section 2 presents essential definitions for FD and FI, crucial for subsequent comparisons. In section 3, a discussion on PD and their corresponding integrals, accompanied by their Laplace transforms, forms the basis of the primary results. This section defines the (ψ, ϕ)-fractional derivatives and integrals while exploring their inherent properties. Section 4 delves into the Laplace transforms about Riemann-Liouville (ψ, ϕ)-fractional derivatives. The final section focuses on the exploration of Caputo (ψ, ϕ)-fractional derivatives, examining their linear (ψ, ϕ)-fractional equations. Additionally, this section elucidates the relationship linking RL-FD and C-FD.

2. Review of FC

In this section, we present several foundational definitions related to FD and FI, which are instrumental for the analysis and comparisons carried out in this work. Let $\vartheta \in \mathbb{C}$ be such that $\operatorname{Re}(\vartheta) > 0$, and denote $n_\vartheta = [\vartheta] + 1$. We begin by defining the left-sided RL-FI of a function x of order ϑ as follows:

$$\left({}_aJ^{\vartheta}x\right)(\Theta) = \frac{1}{\Gamma(\vartheta)}\int_a^{\Theta}(\Theta - s)^{\vartheta-1}x(s)ds. \tag{2.1}$$

The right-sided RL-FI of a function x, of order ϑ, is given by the expression:

$$\left(J_b^{\vartheta}x\right)(\Theta) = \frac{1}{\Gamma(\vartheta)}\int_{\Theta}^{b}(s - \Theta)^{\vartheta-1}x(s)ds. \tag{2.2}$$

The left-sided RL-FD of a function x, of order ϑ, is given by the expression:

$$\left({}_aD^{\vartheta}x\right)(\Theta) = \left(\frac{d}{d\Theta}\right)^{n_\vartheta}\left({}_aJ^{n_\vartheta-\vartheta}x\right)(\Theta). \tag{2.3}$$

The right-sided RL-FD of a function x, of order ϑ, is given by the expression:

$$\left(D_b^{\vartheta}x\right)(\Theta) = \left(-\frac{d}{d\Theta}\right)^{n_\vartheta}\left(J_b^{n_\vartheta-\vartheta}x\right)(\Theta). \tag{2.4}$$

The left-sided C-FD of a function x, of order ϑ, is given by the expression:

$$\left({}_a^CD^{\vartheta}x\right)(\Theta) = \left({}_aJ^{n_\vartheta-\vartheta}x^{(n_\vartheta)}\right)(\Theta). \tag{2.5}$$

The right-sided C-FD of a function x, of order ϑ, is given by the expression:

$$\left({}^CD_b^{\vartheta}x\right)(\Theta) = \left(J_b^{n_\vartheta-\vartheta}(-1)^{n_\vartheta}x^{(n_\vartheta)}\right)(\Theta). \tag{2.6}$$

The left and right generalized proportional integral (GPI) [14] are defined respectively as

$$\left({}_a^{\vartheta}\mathbf{J}^{\vartheta,\varkappa}x\right)(\Theta) = \frac{1}{\varkappa^{\vartheta}\Gamma(\vartheta)}\int_a^{\Theta}e^{\frac{\varkappa-1}{\varkappa}(\Theta-s)}\left(\Theta - s\right)^{\vartheta-1}x(s)ds, \tag{2.7}$$

and

$$\left(\mathbf{J}_b^{\vartheta,\varkappa}x\right)(\Theta) = \frac{1}{\varkappa^{\vartheta}\Gamma(\vartheta)}\int_{\Theta}^{b}e^{\frac{\varkappa-1}{\varkappa}(s-\Theta)}\left(s - \Theta\right)^{\vartheta-1}x(s)ds. \tag{2.8}$$

Consider $\varkappa \in (0, 1]$. The left- and right-sided generalized proportional derivatives of Riemann–Liouville (GPD-RL) type, as introduced in [14], are defined respectively by:

$$\left({}_a\mathbf{D}^{\vartheta,\varkappa}x\right)(\Theta) = \frac{D_{\Theta}^{n_\vartheta,\varkappa}}{\varkappa^{n_\vartheta-\vartheta}\Gamma(n_\vartheta - \vartheta)}\int_a^{\Theta}e^{\frac{\varkappa-1}{\varkappa}(\Theta-s)}(\Theta - s)^{n_\vartheta-\vartheta-1}x(s)ds, \tag{2.9}$$

and

$$\left(\mathbf{D}_b^{\vartheta,\varkappa}x\right)(\Theta) = \frac{D_{\Theta}^{n_\vartheta,\varkappa}}{\varkappa^{n_\vartheta-\vartheta}\Gamma(n_\vartheta - \vartheta)}\int_{\Theta}^{b}e^{\frac{\varkappa-1}{\varkappa}(s-\Theta)}(s - \Theta)^{n_\vartheta-\vartheta-1}x(s)ds, \tag{2.10}$$

where

$$D^{1,\varkappa}_{\Theta}x(\Theta) = (1-\varkappa)x(\Theta) + \varkappa x'(\Theta), \text{ and } D^{n_\vartheta,\varkappa}_{\Theta} = D^{1,\varkappa}D^{n_\vartheta-1,\varkappa}_{\Theta}.$$

The left and right generalized proportional derivatives of Caputo type (GPD-C) [14] are defined respectively as

$$\left({}^{C}_{a}\mathbf{D}^{\vartheta,\varkappa}x\right)(\Theta) = \frac{1}{\varkappa^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}\int_a^{\Theta} e^{\frac{\varkappa-1}{\varkappa}(\Theta-s)}(\Theta-s)^{n_\vartheta-\vartheta-1}D^{n_\vartheta,\varkappa}_{s}x(s)ds, \tag{2.11}$$

and

$$\left({}^{C}\mathbf{D}^{\vartheta,\varkappa}_{b}x\right)(\Theta) = \frac{1}{\varkappa^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}\int_{\Theta}^{b} e^{\frac{\varkappa-1}{\varkappa}(s-\Theta)}(s-\Theta)^{n_\vartheta-\vartheta-1}D^{n_\vartheta,\varkappa}_{s}x(s)ds. \tag{2.12}$$

The left and right cotangent fractional integral (CFI) [27] are defined respectively as

$$\left({}^{\vartheta}_{a}\mathbf{J}^{\vartheta,\varkappa}x\right)(\Theta) = \frac{1}{\sin(\varkappa\frac{\pi}{2})^{\vartheta}\Gamma(\vartheta)}\int_a^{\Theta} e^{-\cot(\varkappa\frac{\pi}{2})(\Theta-s)}\left(\Theta-s\right)^{\vartheta-1}x(s)ds, \tag{2.13}$$

and

$$\left(\mathbf{J}^{\vartheta,\varkappa}_{b}x\right)(\Theta) = \frac{1}{\sin(\varkappa\frac{\pi}{2})^{\vartheta}\Gamma(\vartheta)}\int_{\Theta}^{b} e^{-\cot(\varkappa\frac{\pi}{2})(s-\Theta)}\left(s-\Theta\right)^{\vartheta-1}x(s)ds. \tag{2.14}$$

Let $\varkappa\in]0,1]$. The left and right cotangent fractional derivatives of RL type (CFD-RL) [27] are defined respectively as

$$\left({}_{a}\mathbf{D}^{\vartheta,\varkappa}x\right)(\Theta) = \frac{D^{n_\vartheta,\varkappa}_{\Theta}}{\sin(\varkappa\frac{\pi}{2})^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}\int_a^{\Theta} e^{-\cot(\varkappa\frac{\pi}{2})(\Theta-s)}(\Theta-s)^{n_\vartheta-\vartheta-1}x(s)ds, \tag{2.15}$$

and

$$\left(\mathbf{D}^{\vartheta,\varkappa}_{b}x\right)(\Theta) = \frac{D^{n_\vartheta,\varkappa}_{\Theta}}{\sin(\varkappa\frac{\pi}{2})^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}\int_{\Theta}^{b} e^{-\cot(\varkappa\frac{\pi}{2})(s-\Theta)}(s-\Theta)^{n_\vartheta-\vartheta-1}x(s)ds, \tag{2.16}$$

where $D^{1,\varkappa}_{\Theta}x(\Theta) = \cos(\varkappa\frac{\pi}{2})x(\Theta) + \sin(\varkappa\frac{\pi}{2})x'(\Theta)$. The left and right cotangent fractional derivatives of Caputo type (CFD-C) [27] are defined respectively as

$$\left({}^{C}_{a}\mathbf{D}^{\vartheta,\varkappa}x\right)(\Theta) = \int_a^{\Theta}\frac{e^{-\cot(\varkappa\frac{\pi}{2})(\Theta-s)}(\Theta-s)^{n_\vartheta-\vartheta-1}}{\sin(\varkappa\frac{\pi}{2})^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}D^{n_\vartheta,\varkappa}_{s}x(s)ds, \tag{2.17}$$

and

$$\left({}^{C}\mathbf{D}^{\vartheta,\varkappa}_{b}x\right)(\Theta) = \int_{\Theta}^{b}\frac{e^{-\cot(\varkappa\frac{\pi}{2})(s-\Theta)}(s-\Theta)^{n_\vartheta-\vartheta-1}}{\sin(\varkappa\frac{\pi}{2})^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)}D^{n_\vartheta,\varkappa}_{s}x(s)ds. \tag{2.18}$$

It's notable to mention that when $\varkappa = 1$:

- the GPI (2.7) and (2.8) reduce to RL-FI (2.1) and (2.2)
- the GPD-RL (2.9) and (2.10) become the RL-FD (2.3) and (2.4)
- the GPD-C (2.11) and (2.12) have the forms of the C-FD (2.5) and (2.6).
- the CFI (2.13) and (2.14) reduce to RL-FI (2.1) and (2.2)
- the CFD-RL (2.15) and (2.16) become the RL-FD (2.3) and (2.4)
- the CFD-C (2.17) and (2.18) have the forms of the C-FD(2.5) and (2.6).

3. The Riemann-Liouville (ψ,ϕ)-fractional derivative

In this section, we give the first part of the main results by defining the (ψ,ϕ)-fractional derivatives and (ψ,ϕ)-fractional integrals and studying their properties. From [5, 6] we have the following definition.

Definition 3.1. Let $\varkappa \in [0,1]$ and $\psi,\phi : [0,1] \to [0,+\infty[$ be continuous where

$$\lim_{\varkappa\to 0^+} \phi(\varkappa) = 1,\ \ \lim_{\varkappa\to 0^+} \psi(\varkappa) = 0,\ \lim_{\varkappa\to 1^-} \phi(\varkappa) = 0,\ \ \lim_{\varkappa\to 1^-} \psi(\varkappa) = 1,$$

and

$$\phi(\varkappa) \neq 0,\ \ \psi(\varkappa) \neq 0,\ \ \varkappa \in]0,1].$$

The FD of x of order $\varkappa$ is defined by

$$D^{\varkappa}x(\Theta) = \phi(\varkappa)x(\Theta) + \psi(\varkappa)x'(\Theta). \tag{3.1}$$

We aim to find the integral related to the FD in (3.1). Consider the following equation:

$$D^{\varkappa}y(\Theta) = \phi(\varkappa)y(\Theta) + \psi(\varkappa)y'(\Theta) = x(\Theta), \quad \Theta \geq a. \tag{3.2}$$

the solution of (3.2) is:

$$y(\Theta) = \frac{1}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} x(s)ds.$$

The fractional integral ((ψ,ϕ)-fractional integral) of FD Eq. (3.1) is defined by:

$${}_aJ^{1,\varkappa}x(\Theta) = \frac{1}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} x(s)ds, \tag{3.3}$$

where we accept that ${}_aJ^{0,\varkappa}x(\Theta) = x(\Theta)$.

Remark 3.2. Let $\varkappa \in]0,1[$. Then the $x(\Theta) = e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}$ is a nonconstant function. However, $\left(D^{\varkappa}x\right)(\Theta) = 0$.

$$\begin{aligned}
\left(D^{\varkappa}x\right)(\Theta) &= \phi(\varkappa)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta} + \psi(\varkappa)(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta})' \\
&= \phi(\varkappa)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta} + \psi(\varkappa)(-\frac{\phi(\varkappa)}{\psi(\varkappa)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}) \\
&= \phi(\varkappa)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta} - \phi(\varkappa)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}) \\
&= 0.
\end{aligned}$$

Proposition 3.3. *Let x differentiable on $]a,+\infty[$ and $\varkappa \in]0,1]$. We get*

$${}_aJ^{1,\varkappa}D^{\varkappa}x(\Theta) = x(\Theta) - e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}x(a).$$

Proof. We have

$$\begin{aligned}
{}_aJ^{1,\varkappa}D^{\varkappa}x(\Theta) &= \frac{1}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} D^{\varkappa}x(s)ds \\
&= \frac{1}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} \Big(\phi(\varkappa)x(s) + \psi(\varkappa)x'(s)\Big)ds \\
&= \frac{1}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}\phi(\varkappa)x(s)ds + \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}x'(s)ds \\
&= \frac{\phi(\varkappa)}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}x(s)ds + \left[e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}x(s)\right]_{s=a}^{s=\Theta} \\
&\quad -\frac{\phi(\varkappa)}{\psi(\varkappa)} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}x(s)ds
\end{aligned}$$

$$= \left[e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} x(s) \right]_{s=a}^{s=\Theta}$$
$$= x(\Theta) - e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} x(a).$$

□

New let $\left({}_aJ^{n,\varkappa}x\right)(\Theta) = \left({}_aJ^{1,\varkappa}\left({}_aJ^{n-1,\varkappa}x\right)(\Theta)\right)$.

Theorem 3.4. *Let $n \in \mathbb{N}^*$, we have*

$$\left({}_aJ^{n,\varkappa}x\right)(\Theta) = \frac{1}{\psi(\varkappa)^n\Gamma(n)} \int_a^\Theta e^{-\frac{\phi(\gamma)}{\psi(\gamma)}(\Theta-s)} (\Theta-s)^{n-1} x(s)ds. \tag{3.4}$$

Proof. By changing the order of integrals we have

$$\begin{aligned} \left({}_aJ^{n,\varkappa}x\right)(\Theta) &= \frac{1}{\psi(\varkappa)} \int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-\tau_1)} d\tau_1 \frac{1}{\psi(\varkappa)} \int_a^{\tau_1} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\tau_1-\tau_2)} d\tau_2 \times \dots \\ &\quad \dots \times \frac{1}{\psi(\varkappa)} \int_a^{\tau_{n-1}} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\tau_{n-1}-\tau_n)} x(\tau_n) d\tau_n \\ &= \frac{1}{\psi(\varkappa)^n\Gamma(n)} \int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} (\Theta-s)^{n-1} x(s)ds. \end{aligned} \tag{3.5}$$

□

From equation (3.4), we can express the subsequent generalized form of FI.

Definition 3.5. Let $\varkappa \in]0,1]$ and $\vartheta \in \mathbb{C}$, $Re(\vartheta) > 0$, the left and right Riemann-Liouville (ψ,ϕ)-fractional integrals of x are respectively defined as

$$({}_aJ^{\vartheta,\varkappa}x)(\Theta) = \frac{1}{\psi(\varkappa)^\vartheta\Gamma(\vartheta)} \int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} (\Theta-s)^{\vartheta-1} x(s)ds, \tag{3.6}$$

and

$$(J_b^{\vartheta,\varkappa}x)(\Theta) = \frac{1}{\psi(\varkappa)^\vartheta\Gamma(\vartheta)} \int_\Theta^b e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(s-\Theta)} (s-\Theta)^{\vartheta-1} x(s)ds. \tag{3.7}$$

Remark 3.6. We have

- If $\varkappa = 1$, then we get the left and right RL-FI (2.1) and (2.2), respectively.
- If $\psi(\varkappa) = \varkappa$ and $\phi(\varkappa) = 1 - \varkappa$, then we get the left and right GPI (2.7) and (2.8), respectively.
- If $\psi(\varkappa) = \sin(\frac{\pi}{2}\varkappa)$ and $\phi(\varkappa) = \cos(\frac{\pi}{2}\varkappa)$, then we get the left and right CFI (2.13) and (2.14), respectively.

Let $n \in \mathbb{N}$, we use the notation

$$\left(D_\Theta^{n,\varkappa}x\right)(\Theta) = (\underbrace{D_\Theta^\varkappa \ D_\Theta^\varkappa \cdots D_\Theta^\varkappa}_{\text{n times}} x)(\Theta).$$

Definition 3.7. Let $\varkappa \in]0,1]$ and $\vartheta \in \mathbb{C}$, $Re(\vartheta) \geq 0$. The left and right RL (ψ,ϕ)-fractional derivatives of x are respectively presented as follows:

$$\begin{aligned} ({}_aD^{\vartheta,\varkappa}x)(\Theta) &= D_\Theta^{n_\vartheta,\varkappa} \ {}_aJ^{n_\vartheta-\vartheta,\varkappa}x(\Theta) \\ &= D_\Theta^{n_\vartheta,\varkappa} \int_a^\Theta \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{n_\vartheta-\vartheta-1}}{\psi(\varkappa)^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)} x(s)ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} (D_b^{\vartheta,\varkappa}x)(\Theta) &= D_\Theta^{n_\vartheta,\varkappa} J_b^{n_\vartheta-\vartheta,\varkappa}x(\Theta) \\ &= D_\Theta^{n_\vartheta,\varkappa} \int_\Theta^b \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(s-\Theta)}(s-\Theta)^{n_\vartheta-\vartheta-1}}{\psi(\varkappa)^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)} x(s)ds, \end{aligned} \tag{3.9}$$

with $n_\vartheta = [Re(\vartheta)] + 1$.

Remark 3.8. We have

- If $\varkappa = 1$, then we get the left and right RL-FD (2.3) and (2.4), respectively.
- If $\psi(\varkappa) = \varkappa$ and $\phi(\varkappa) = 1 - \varkappa$, then we get the left and right GPD (2.9) and (2.10), respectively.
- If $\psi(\varkappa) = \sin(\frac{\pi}{2}\varkappa)$ and $\phi(\varkappa) = \cos(\frac{\pi}{2}\varkappa)$, then we get the left and right CFD (2.15) and (2.16), respectively.

Lemma 3.9. *Let function $y(\Theta)$, we have*

$$D^{\varkappa}\left(y(\Theta)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}\right) = \psi(\varkappa)y'(\Theta)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}.$$

Proposition 3.10. *Let $\varkappa \in]0,1]$ and $\vartheta_1, \vartheta_2 \in \mathbb{C}$ be such that* $\mathrm{Re}(\vartheta_1) \geq 0$ *and* $\mathrm{Re}(\vartheta_2) > 0$. *We have*

(1) $_aJ^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_2-1}\right) = \frac{\Gamma(\vartheta_2)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_1+\vartheta_2-1}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1+\vartheta_2)}$.

(2) $J_b^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_2-1}\right) = \frac{\Gamma(\vartheta_2)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_1+\vartheta_2-1}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1+\vartheta_2)}$.

(3) $_aD^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_2-1}\right) = \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_2-1-\vartheta_1}}{\Gamma(\vartheta_2-\vartheta_1)}$.

(4) $D_b^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_2-1}\right) = \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_2-1-\vartheta_1}}{\Gamma(\vartheta_2-\vartheta_1)}$.

Proof. (1) We have

$$\begin{aligned}
{}_aJ^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}\right) &= \frac{1}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}\int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta_1-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}s}(s-a)^{\vartheta_2-1}ds \\
&= \frac{1}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}\int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-s)^{\vartheta_1-1}(s-a)^{\vartheta_2-1}ds \\
&= \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}\int_a^\Theta (\Theta-s)^{\vartheta_1-1}(s-a)^{\vartheta_2-1}ds,
\end{aligned}$$

making the change of variable $y = \frac{\Theta-s}{\Theta-a}$, we get

$$\begin{aligned}
{}_aJ^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}\right) &= \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}\int_a^\Theta (\Theta-s)^{\vartheta_1-1}(s-a)^{\vartheta_2-1}ds \\
&= \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}(\Theta-a)^{\vartheta_1+\vartheta_2-1}\int_0^1 y^{\vartheta_1-1}(1-y)^{\vartheta_2-1}dy.
\end{aligned}$$

Utilizing the Beta function defined as $B(\vartheta_1,\vartheta_2) = \displaystyle\int_0^1 u^{\vartheta_1-1}(1-u)^{\vartheta_2-1}du$, and considering the identity $B(\vartheta_1,\vartheta_2) = \frac{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma(\vartheta_1+\vartheta_2)}$, thus:

$$\begin{aligned}
{}_aJ^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}\right) &= \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1)}(\Theta-a)^{\vartheta_1+\vartheta_2-1}\frac{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma(\vartheta_1+\vartheta_2)} \\
&= \frac{\Gamma(\vartheta_2)e^{-\frac{\phi(\gamma)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1+\vartheta_2)}(\Theta-a)^{\vartheta_1+\vartheta_2-1}.
\end{aligned}$$

(2) Similar of 1.

(3) Let $x(\Theta)=e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}$, using Lemma 3.9 and we have

$$\begin{aligned}
{}_aD^{\vartheta_1,\varkappa}\Big(x(\Theta)\Big) &= D_\Theta^{n_{\vartheta_1},\varkappa}\;{}_aJ^{n_{\vartheta_1}-\vartheta_1,\varkappa}\Big(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}\Big)\\
&= D_\Theta^{n_{\vartheta_1},\varkappa}\Bigg(\frac{\Gamma(\vartheta_2)e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}}{\psi(\varkappa)^{n_{\vartheta_1}-\vartheta_1}\Gamma(n_{\vartheta_1}-\vartheta_1+\vartheta_2)}(\Theta-a)^{n_{\vartheta_1}-\vartheta_1+\vartheta_2-1}\Bigg)\\
&= \frac{\psi(\varkappa)^{n_{\vartheta_1}}\Gamma(\vartheta_2)(n_{\vartheta_1}-\vartheta_1+\vartheta_2-1)(n_{\vartheta_1}-\vartheta_1+\vartheta_2-1)\cdots(\vartheta_2-\vartheta_1)}{\psi(\varkappa)^{n_{\vartheta_1}-\vartheta_1}\Gamma(n_{\vartheta_1}-\vartheta_1+\vartheta_2)}\times e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1-\vartheta_1}\\
&= \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)}{\Gamma(\vartheta_2-\vartheta_1)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1-\vartheta_1}.
\end{aligned}$$

(4) The proof follows a similar pattern to that of relation 3.

□

Lemma 3.11. *Let $\lambda\in\mathbb{R}$. Then*

$${}_aD^{\vartheta_1,\varkappa}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1})=\lambda\psi(\varkappa)^{\vartheta_1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1}),$$

where $E_{\vartheta_1,\vartheta_2}$ is the Mittag-Lefler function [22].

Proof. Let $h(\Theta)=e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1})$. We have

$$\begin{aligned}
{}_aD^{\vartheta_1,\varkappa}h(\Theta) &= {}_aD^{\vartheta_1,\varkappa}\Big(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1})\Big)\\
&= {}_aD^{\vartheta_1,\varkappa}\Big(\sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta_1k+1)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{k\vartheta_1}\Big)\\
&= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta_1k+1)}{}_aD^{\vartheta_1,\varkappa}\Big(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{k\vartheta_1}\Big)\\
&= \sum_{k=1}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta_1k+1)}\frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_1k+1)}{\Gamma(\vartheta_1k+1-\vartheta_1)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_1k-\vartheta_1}\\
&= \psi(\varkappa)^{\vartheta_1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{k=1}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta_1(k-1)+1)}(\Theta-a)^{\vartheta_1(k-1)}\\
&= \lambda\psi(\varkappa)^{\vartheta_1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{k=1}^{+\infty}\frac{\lambda^{k-1}}{\Gamma(\vartheta_1(k-1)+1)}(\Theta-a)^{\vartheta_1(k-1)}\\
&= \lambda\psi(\varkappa)^{\vartheta_1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta_1,1}(\lambda(\Theta-a)^{\vartheta_1}).
\end{aligned}$$

□

It's evident that the (ψ,ϕ)-integral operators don't exhibit the semigroup property in $\varkappa$. Nevertheless, the semigroup property of (ψ,ϕ)-integrals in ϑ remains valid.

Theorem 3.12 presents the semigroup property (ψ,ϕ)-integral Riemann-Liouville type.

Theorem 3.12. . *Let x be continuous and $\varkappa\in]0,1]$, $\mathrm{Re}(\vartheta_1)>0$, and $\mathrm{Re}(\vartheta_2)>0$. We have*

$$J^{\vartheta_1,\varkappa}\left({}_aJ^{\vartheta_2,\varkappa}x\right)(\Theta)=J^{\vartheta_2,\varkappa}\left({}_aJ^{\vartheta_1,\varkappa}x\right)(\Theta)=\left({}_aJ^{\vartheta_1+\vartheta_2,\varkappa}x\right)(\Theta),\quad\Theta\geq a. \tag{3.10}$$

Proof. We have

$$\begin{aligned} J^{\vartheta_1,\varkappa}\left({}_aJ^{\vartheta_2,\varkappa}x\right)(\Theta) &= \frac{1}{\psi(\varkappa)^{\vartheta_1+\vartheta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \\ &\times \int_a^\Theta\int_a^u e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-u)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(u-s)}(\Theta-u)^{\vartheta_1-1}(u-s)^{\vartheta_2-1}x(s)dsdu \\ &= \frac{1}{\psi(\varkappa)^{\vartheta_1+\vartheta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}\int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}x(s)\int_s^\Theta(\Theta-u)^{\vartheta_1-1}(u-s)^{\vartheta_2-1}duds, \end{aligned}$$

making the change of variable $y=\frac{u-s}{\Theta-s}$, we get

$$\begin{aligned} J^{\vartheta_1,\varkappa}\left({}_aJ^{\vartheta_2,\varkappa}x\right)(\Theta) &= \frac{1}{\psi(\varkappa)^{\vartheta_1+\vartheta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \\ &\times\int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta_1+\vartheta_2-1}x(s)ds\int_0^1(1-y)^{\vartheta_1-1}y^{\vartheta_2-1}dy \\ &= \frac{1}{\psi(\varkappa)^{\vartheta_1+\beta}\Gamma(\vartheta_1+\vartheta_2)}\int^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta_1+\vartheta_2-1}x(s)ds \\ &= \left({}_aJ^{\vartheta_1+\vartheta_2,\varkappa}x\right)(\Theta). \end{aligned}$$

□

Theorem 3.13. *Let x be integrable in each interval $[a,\Theta],\Theta>a$ and $0\le l<[\mathrm{Re}(\vartheta)]+1$. Then*

$$D^{l,\varkappa}\left({}_aJ^{\vartheta,\varkappa}x\right)(\Theta)=\left({}_aJ^{\vartheta-l,\varkappa}x\right)(\Theta). \tag{3.11}$$

Proof. Using the definition and by noting that $D_\Theta^{1,\varkappa}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}=0$, we have

$$\begin{aligned} D^{l,\varkappa}\left({}_aJ^{\vartheta,\varkappa}x\right)(\Theta) &= D^{l-1,\varkappa}\left(D^{1,\varkappa}{}_aJ^{\vartheta,\varkappa}x\right)(\Theta) \\ &= D^{l-1,\varkappa}\frac{1}{\psi(\varkappa)^{\vartheta-1}\Gamma(\vartheta-1)}\int_a^\Theta e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta-2}x(s)ds. \end{aligned}$$

If we continue l-times in this way we reach (3.11). □

Corollary 3.14. *Let $0<\mathrm{Re}(\vartheta_2)<\mathrm{Re}(\vartheta_1)$ and $l-1<\mathrm{Re}(\vartheta_2)\le l$. Then, we have*

$${}_aD^{\vartheta_2,\varkappa}{}_aJ^{\vartheta_1,\varkappa}x(\Theta)={}_aJ^{\vartheta_1-\vartheta_2,\varkappa}x(\Theta).$$

Proof. Leveraging Theorems 3.12 and 3.13, we obtain:

$${}_aD^{\vartheta_2,\varkappa}{}_aJ^{\vartheta_1,\varkappa}x(\Theta)=D_a^{l,\varkappa}I_a^{l-\vartheta_2,\varkappa}J^{\vartheta_1,\varkappa}x(\Theta)=D_a^{l,\varkappa}J^{l-\vartheta_2+\vartheta_1,\varkappa}x(\Theta)={}_aJ^{\vartheta_1-\vartheta_2,\varkappa}x(\Theta).$$

□

Theorem 3.15. *Let x be integrable on $\Theta\ge a$ and $\mathrm{Re}[\vartheta]>0,\varkappa\in]0,1],n_\vartheta=[\mathrm{Re}(\vartheta)]+1$. We get*

$${}_aD^{\vartheta,\varkappa}{}_aJ^{\vartheta,\varkappa}x(\Theta)=x(\Theta).$$

Proof. By combining the definition and Theorem 3.12, we derive:

$${}_aD^{\vartheta,\varkappa}{}_aJ^{\vartheta,\varkappa}x(\Theta)=D^{n_\vartheta,\varkappa}{}_aJ^{n_\vartheta-\vartheta,\varkappa}{}_aJ^{\vartheta,\varkappa}x(\Theta)=D^{n_\vartheta,\varkappa}{}_aJ^{n_\vartheta,\varkappa}x(\Theta)=x(\Theta).$$

□

Lemma 3.16. *For $\vartheta>0,\varkappa\in]0,1]$ and l is positive integer we have*

$$\left({}_aJ^{\vartheta,\varkappa}D^{l,\varkappa}x\right)(\Theta)=\left(D^{l,\varkappa}{}_aJ^{\vartheta,\varkappa}x\right)(\Theta)-\sum_{k=0}^{l-1}\frac{(\Theta-a)^{\vartheta-l+k}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\Gamma(\vartheta+k-l+1)\psi(\varkappa)^{\vartheta-l+k}}\left(D^{k,\varkappa}x\right)(a). \tag{3.12}$$

If $l=1$, we have

$$\left({}_aJ^{\vartheta,\varkappa}D^{\varkappa}x\right)(\Theta)=\left(D^{\varkappa}{}_aJ^{\vartheta,\varkappa}x\right)(\Theta)-\frac{(\Theta-a)^{\vartheta-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\psi(\varkappa)^{\vartheta-1}\Gamma(\vartheta)}x(a). \tag{3.13}$$

Proof. Observing that $(D^{\varkappa}xy)(\Theta)=x(\Theta)(D^{\varkappa}y)(\Theta)+y(\Theta)(D^{\varkappa}x)(\Theta)-\phi(\varkappa)x(\Theta)y(\Theta)$ (see also Lemma 1.7 in [5]), we get

$$D^{\varkappa}\left[(\Theta-a)^{n_\vartheta-\vartheta-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}x(a)\right]=\psi(\varkappa)(n_\vartheta-\vartheta-1)(\Theta-a)^{n_\vartheta-\vartheta-2}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}x(a). \tag{3.14}$$

We establish the proof for (3.13) first. The relationship (3.12) follows from applying (3.13) inductively, utilizing (3.14). The demonstration of (3.13) relies on the observation of (3.14) and the Laplace transform. Specifically, employing (4.1) from Theorem 4.1 in the subsequent section and incorporating the identity:

$$\mathcal{L}_a\left\{D^{\varkappa}x(\Theta)\right\}(p)=(\phi(\varkappa)+\psi(\varkappa)p)X_a(p)-\psi(\varkappa)x(a),$$

where

$$\mathcal{L}_a\{x(\Theta)\}(p)=\int_a^{\infty}e^{-p(s-a)}x(s)ds,$$

we obtain

$$\mathcal{L}_a\left\{{}_aJ^{\vartheta,\varkappa}D^{\varkappa}x(\Theta)\right\}(p)=\frac{\mathcal{L}_a\left\{D^{\varkappa}x(\Theta)\right\}(p)}{(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta}}=\frac{(\phi(\varkappa)+\psi(\varkappa)p)X_a(p)-\psi(\varkappa)x(a)}{(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta}},$$

$$\mathcal{L}_a\left\{D^{\varkappa}{}_aJ^{\vartheta,\varkappa}x(\Theta)\right\}(p)=(\phi(\varkappa)+\psi(\varkappa)p)\frac{X_a(p)}{(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta}},$$

and

$$\mathcal{L}_a\left\{\frac{(\Theta-a)^{\vartheta-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\psi(\varkappa)^{\vartheta-1}\Gamma(\vartheta)}x(a)\right\}(p)=\frac{x(a)\psi(\varkappa)}{(\phi(\varkappa)+\psi(\varkappa)p)}.$$

Alternatively, relation (3.13) can be proved by integration by parts and by noting that

$$D^{\varkappa}{}_aJ^{\vartheta,\varkappa}x(\Theta)=\frac{(\vartheta-1)\psi(\varkappa)}{\psi(\varkappa)^{\vartheta}\Gamma(\vartheta)}\int_a^{\Theta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta-2}x(s)ds.$$

□

Remark 3.17. We have

- $\left({}_aD^{\vartheta,\varkappa}x\right)(\Theta)=\left({}_aJ^{-\vartheta,\varkappa}x\right)(\Theta)$.
- Using Lemma 3.16 we get

$$\begin{aligned}\left(D^{\varkappa}{}_aD^{\vartheta,\varkappa}x\right)(\Theta)&=D^{n_\vartheta,\varkappa}D^{\varkappa}{}_aJ^{n_\vartheta-\vartheta,\varkappa}x(\Theta)\\&=D^{n_\vartheta,\varkappa}\left[{}_aJ^{n_\vartheta-\vartheta,\varkappa}D^{\varkappa}-\frac{(\Theta-a)^{n_\vartheta-\vartheta-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\Gamma(n_\vartheta-\vartheta)\psi(\varkappa)^{n_\vartheta-\vartheta-1}}x(a)\right]\\&={}_aJ^{-\vartheta,\varkappa}D^{\varkappa}x(\Theta)-\frac{(\Theta-a)^{-\vartheta-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\psi(\varkappa)^{-\vartheta-1}\Gamma(-\vartheta)}x(a).\end{aligned}$$

- Lemma 3.16 is valid for any real ϑ.

Theorem 3.18. *Let* $\mathrm{Re}(\vartheta)>0, n_\vartheta=-[-\mathrm{Re}(\vartheta)],\quad x\in L_1]a,b[$ *and* $\left({}_aJ^{\vartheta,\varkappa}x\right)(\Theta)\in AC^{n_\vartheta}[a,b]$. *Then*

$$\left({}_aJ^{\vartheta,\varkappa}{}_aD^{\vartheta,\varkappa}x\right)(\Theta)=x(\Theta)-e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{m=1}^{n_\vartheta}\left({}_aJ^{m-\vartheta,\varkappa}x\right)(a^+)\frac{(\Theta-a)^{\vartheta-m}}{\psi(\varkappa)^{\vartheta-m}\Gamma(\vartheta+1-m)}. \tag{3.15}$$

Proof. From applying (3.12) in Lemma 3.16 and using the semi-group property for Riemann-Liouville (ψ,ϕ)-integrals, we have

$$\begin{aligned}\left({}_a^{\vartheta,\varkappa}{}_aD^{\vartheta,\varkappa}x(\Theta)\right. =& {}_a^{\vartheta,\varkappa}J^{n_\vartheta,\varkappa}{}_aJ^{n_\vartheta-\vartheta,\varkappa}x(\Theta)\\ =&D^{n_\vartheta,\varkappa}{}_a^{\vartheta,\varkappa}{}_aJ^{n_\vartheta-\vartheta,\varkappa}x(\Theta)\\ &-\sum_{k=0}^{n_\vartheta-1}\frac{(\Theta-a)^{\vartheta-n_\vartheta+k}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\Gamma(\vartheta+k-n_\vartheta+1)\psi(\varkappa)^{\vartheta-n_\vartheta+k}}\left(D^{k,\varkappa}{}_aJ^{n_\vartheta-\vartheta,\varkappa}x\right)(a)\\ =&x(\Theta)-\sum_{k=0}^{n_\vartheta-1}\frac{(\Theta-a)^{\vartheta-n_\vartheta+k}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\Gamma(\vartheta+k-n_\vartheta+1)\psi(\varkappa)^{\vartheta-n_\vartheta+k}}\left({}_aJ^{n_\vartheta-\vartheta-k,\varkappa}x\right)(a^+)\\ =&x(\Theta)-e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{m=1}^{n_\vartheta}\left({}_aJ^{m-\vartheta,\varkappa}x\right)(a^+)\frac{(\Theta-a)^{\vartheta-m}}{\psi(\varkappa)^{\vartheta-m}\Gamma(\vartheta+1-m)}.\end{aligned}$$

□

4. The Laplace transform for (ψ,ϕ)-RL-FI and RL-FD

In this section, we present the Laplace transform for (ψ,ϕ)-fractional integral and the Riemann-Liouville (ψ,ϕ)-fractional derivative. We give exacte solution of linear (ψ,ϕ)-fractional differential equations of Riemann-Liouville type.

Theorem 4.1. *Let x to be exponential order, $\varkappa\in]0,1]$ and $\vartheta\in\mathbb{C}$ where $\mathrm{Re}(\vartheta)>0$. Then*

$$\mathcal{L}_a\left\{\left({}_aJ^{\vartheta,\varkappa}x\right)(\Theta)\right\}(p)=\frac{1}{(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta}}\mathcal{L}_a\{x(\Theta)\}(p). \tag{4.1}$$

Proof. We have

$$\begin{aligned}\mathcal{L}_a\left\{\left({}_aJ^{\vartheta,\varkappa}x\right)(\Theta)\right\}(p)&=\frac{1}{\psi(\varkappa)^{\vartheta}\Gamma(\vartheta)}\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}\Theta^{\vartheta-1}*x(\Theta)\right\}(p)\\ &=\frac{1}{\psi(\varkappa)^{\vartheta}\Gamma(\vartheta)}\frac{\Gamma(\vartheta)}{\left(p-\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta}}\mathcal{L}_a\{x(\Theta)\}(p)\\ &=\frac{1}{(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta}}\mathcal{L}_a\{x(\Theta)\}(p).\end{aligned}$$

□

Theorem 4.2. *Let $x\in C^{n_\vartheta-1}([a,+\infty[)$ be such that $x^{(k)}$ are of exponential order on each subinterval $[a,b]$, $k=1,2,\ldots,n_\vartheta-1$. Then,*

$$\begin{aligned}\mathcal{L}_a\left\{\left(D^{n_\vartheta,\varkappa}x\right)(\Theta)\right\}(p)=&(\phi(\varkappa)+\psi(\varkappa)p)^{n_\vartheta}\mathcal{L}_a\{x(\Theta)\}(p)\\ &-\psi(\varkappa)\sum_{k=0}^{n_\vartheta-1}(\phi(\varkappa)+\psi(\varkappa)p)^{n_\vartheta-1-k}\left(D^{k\cdot\varkappa}x\right)(a),\end{aligned} \tag{4.2}$$

where ϑ sash that $n_\vartheta=[\mathrm{Re}(\vartheta)]+1$.

Proof. Since $\mathcal{L}_a\{x'(\Theta)\}(p)=p\mathcal{L}_a\{x(\Theta)\}(p)-x(a)$, we get for $n_\vartheta=1$

$$\begin{aligned}\mathcal{L}_a\left\{\left(D^{1,\varkappa}x\right)(\Theta)\right\}(p)&=\mathcal{L}_a\left\{\phi(\varkappa)x(\Theta)+\psi(\varkappa)x'(\Theta)\right\}(p)\\ &=(\phi(\varkappa)+\psi(\varkappa)p)\mathcal{L}_a\{x(\Theta)\}(p)-\psi(\varkappa)x(a).\end{aligned} \tag{4.3}$$

Applying (4.3) leads us to derive the relation (4.2). □

Theorem 4.3. *Let* $\varkappa \in]0,1]$, $\vartheta \in \mathbb{C}$ *where* $\mathrm{Re}(\vartheta) > 0$ *and* $n_\vartheta = [\mathrm{Re}(\vartheta)] + 1$, *we have*

$$\mathcal{L}_a \left\{ \left({}_a D^{\vartheta,\varkappa} x \right)(\Theta) \right\}(p) = (\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} X_a(p) - \psi(\varkappa) \sum_{k=0}^{n_\vartheta - 1} (\phi(\varkappa) + \psi(\varkappa)p)^{n_\vartheta - k - 1} \left(J^{n_\vartheta - \vartheta - k,\varkappa} x \right)\left(a^+\right),$$

with $X_a(p) = \mathcal{L}\{x(\Theta)\}(p)$. *If* x *is continuous at* a *then*

$$\mathcal{L}_a \left\{ \left({}_a D^{\vartheta,\varkappa} x \right)(\Theta) \right\}(p) = (\phi(\varkappa) + \psi(\varkappa)s)^{\vartheta} X_a(p).$$

Proof. Through the application of Theorems 4.1 and 4.2, we obtain

$$\begin{aligned}
\mathcal{L}_a \left\{ \left({}_a D^{\vartheta,\varkappa} x \right)(\Theta) \right\}(p) &= \mathcal{L}_a \left\{ {}_a D_a^{n_\vartheta,\varkappa} J^{n_\vartheta - \vartheta,\varkappa} x)(\Theta) \right\}(p) \\
&= (\phi(\varkappa) + \psi(\varkappa)p)^{n_\vartheta} \mathcal{L}_a \left\{ {}_a J^{n_\vartheta - \vartheta,\varkappa} x)(\Theta) \right\}(p) \\
&\quad - \psi(\varkappa) \sum_{k=0}^{n_\vartheta - 1} (\phi(\varkappa) + \psi(\varkappa)p)^{n_\vartheta - 1 - k} \left(D_a^{k,\varkappa} J^{n_\vartheta - \vartheta,\varkappa} x \right)\left(a^+\right) \\
&= (\phi(\varkappa) + \psi(\varkappa)p)^{n_\vartheta} (\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta - n_\vartheta} X_a(p) \\
&\quad - \psi(\varkappa) \sum_{k=0}^{n_\vartheta - 1} (\phi(\varkappa) + \psi(\varkappa)p)^{n_\vartheta - 1 - k} \left({}_a J^{n_\vartheta - \vartheta - k,\varkappa} x \right)\left(a^+\right).
\end{aligned}$$

□

Theorem 4.4. *The solution of*

$$\begin{cases} {}_a D^{\vartheta,\varkappa} x(\Theta) = \lambda x(\Theta) + y(\Theta), \quad 0 < \vartheta, \varkappa \leq 1, \\ \left({}_a J^{1-\vartheta,\varkappa} x \right)\left(a^+\right) = x_a. \end{cases} \tag{4.4}$$

is:

$$\begin{aligned}
x(\Theta) =& e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta - a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta} \lambda (\Theta - a)^{\vartheta} \right) x_a \\
&+ \psi(\varkappa)^{-\vartheta} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta - s)} (\Theta - s)^{\vartheta - 1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta} \lambda (\Theta - s)^{\vartheta} \right) y(s) ds.
\end{aligned} \tag{4.5}$$

Proof. Applying the operator $\mathcal{L}_a$ to Equation (4.4) and utilizing Theorem 4.3 with $n_\vartheta = 1$, we obtain:

$$\begin{aligned}
&(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} \mathcal{L}_a\{x(\Theta)\}(p) - \psi(\varkappa) \left({}_a J^{1-\vartheta,\varkappa} x \right)\left(a^+\right) \\
&= \lambda \mathcal{L}_a\{x(\Theta)\}(p) + \mathcal{L}_a\{y(\Theta)\}(p).
\end{aligned}$$

Which is equivalent to

$$\left[(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} - \lambda \right] \mathcal{L}_a\{x(\Theta)\}(p) = \psi(\varkappa) x_a + \mathcal{L}_a\{y(\Theta)\}(p).$$

Thus,

$$\mathcal{L}_a\{x(\Theta)\}(p) = \left[(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} - \lambda \right]^{-1} \psi(\varkappa) x_a + \left[(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} - \lambda \right]^{-1} \mathcal{L}_a\{y(\Theta)\}(p). \tag{4.6}$$

Equation (4.6) can be written as:

$$\begin{aligned}
\mathcal{L}_a\{x(\Theta)\}(p) = & \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)} \right)^{\vartheta} - \psi(\varkappa)^{-\vartheta} \lambda \right]^{-1} \psi(\varkappa)^{1-\vartheta} x_a \\
& + \psi(\varkappa)^{-\vartheta} \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)} \right)^{\vartheta} - \psi(\varkappa)^{-\vartheta} \lambda \right]^{-1} \mathcal{L}_a\{y(\Theta)\}(p),
\end{aligned} \tag{4.7}$$

using

$$\mathcal{L}_a\{e^{-c(\Theta - a)} x(\Theta)\}(p) = \mathcal{L}_a\{x(\Theta)\}(p + c),$$

as follows:

$$\mathcal{L}_a\left\{(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p), \quad (4.8)$$

and

$$\mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p). \quad (4.9)$$

In our case, using the facts that (see 1.9.13 in [22])

$$\mathcal{L}_a\left\{(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)=\frac{1}{p^{\vartheta}-\lambda}, \quad (4.10)$$

and

$$\mathcal{L}_a\left\{E_{\vartheta}\left(\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)=\frac{p^{\vartheta-1}}{p^{\vartheta}-\lambda}, \quad (4.11)$$

as follows:

$$\begin{aligned}&\mathcal{L}_a\{\ (\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)\\&=\left[(p+\frac{\phi(\varkappa)}{\psi(\varkappa)})^{\vartheta}-\psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}.\end{aligned} \quad (4.12)$$

And,

$$\mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)=\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta-1}\left[\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta}-\psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}. \quad (4.13)$$

Hence, by using Equations (4.9) and (4.13), we have:

$$\begin{aligned}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta-1}\left[\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta}-\psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}&=\mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)\\&=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p).\end{aligned} \quad (4.14)$$

Further, by using Eqs. (4.8) and (4.12), we get

$$\begin{aligned}\left[\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta}-\psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}&=\mathcal{L}_a\left\{(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)\\&=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p).\end{aligned} \quad (4.15)$$

We may get the following result by combining Eqs. (4.14) and (4.15) in Eq. (4.7), we get

$$\begin{aligned}\mathcal{L}_a[x(\Theta)](p)=&\ \mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)x_a\\&+\psi(\varkappa)^{-\vartheta}\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)\mathcal{L}_a[y(\Theta)](p).\end{aligned} \quad (4.16)$$

Hence, by using convolution formula in Eq. (4.16), we have

$$\begin{aligned}\mathcal{L}_a[x(\Theta)](p)=&\ \mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)x_a\\&+\psi(\varkappa)^{-\vartheta}\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)*y(\Theta)\right\}(p).\end{aligned} \quad (4.17)$$

Applying the Laplace inverse of Eq. (4.17), we get:

$$x(\Theta) = e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right) x_a + \psi(\varkappa)^{-\vartheta}\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right) * y(\Theta)\right\}.$$

Therefore,

$$x(\Theta) = e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right) x_a + \psi(\varkappa)^{-\vartheta}\int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-s)^{\vartheta}\right) y(s)ds. \tag{4.18}$$

□

5. The Caputo (ψ, ϕ)-fractional derivative

In this section, we present the Caputo (ψ, ϕ)-fractional derivatives, Caputo (ψ, ϕ)-fractional derivatives of some special functions, the Laplace transform for Caputo (ψ, ϕ)-fractional derivatives and relation that links the Caputo and Riemann-Liouville (ψ, ϕ)-fractional derivatives. Finly we give exacte solution of linear (ψ, ϕ)-fractional differential equations of Caputo type.

Definition 5.1. Let $\varkappa \in]0,1]$ and $\vartheta \in \mathbb{C}$, $Re(\vartheta) \geq 0$, define the left Caputo (ψ, ϕ)-fractional derivative starting at a by:

$$\begin{aligned}({}_a^C D^{\vartheta,\varkappa} x)(\Theta) &= {}_a J^{n_\vartheta-\vartheta,\varkappa}(D_\Theta^{n_\vartheta,\varkappa} x(\Theta)) \\ &= \int_a^{\Theta} \frac{e^{-\frac{\phi(\gamma)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{n_\vartheta-\vartheta-1}}{\psi(\varkappa)^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)} D_s^{n,\varkappa} x(s)ds,\end{aligned} \tag{5.1}$$

and the right Caputo (ψ, ϕ)-fractional derivative ending at b is defined by:

$$\begin{aligned}({}^C D_b^{\vartheta,\varkappa} x)(\Theta) &= J_b^{n_\vartheta-\vartheta,\varkappa}(D_\Theta^{n_\vartheta,\varkappa} x(\Theta)) \\ &= \int_\Theta^{b} \frac{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(s-\Theta)}(s-\Theta)^{n_\vartheta-\vartheta-1}}{\psi(\varkappa)^{n_\vartheta-\vartheta}\Gamma(n_\vartheta-\vartheta)} D_s^{n_\vartheta,\varkappa} x(s)ds,\end{aligned} \tag{5.2}$$

where $n_\vartheta = [Re(\vartheta)] + 1$.

Remark 5.2.

- If $\varkappa = 1$, then we get the left and right RL-FD (2.5) and (2.6), respectively.
- If $\psi(\varkappa) = \varkappa$ and $\phi(\varkappa) = 1 - \varkappa$, then we get the left and right GPD (2.11) and (2.12), respectively.
- If $\psi(\varkappa) = \sin(\varkappa\frac{\pi}{2})$ and $\phi(\varkappa) = \cos(\varkappa\frac{\pi}{2})$, then we get the left and right CFD-C (2.17) and (2.18), respectively.

Proposition 5.3. *Let $\varkappa \in]0,1]$, $\vartheta_1, \vartheta_2 \in \mathbb{C}$ be such that $\mathrm{Re}(\vartheta_1) \geq 0$ and $\mathrm{Re}(\vartheta_2) > 0$. We get*

(1) ${}_a^C D^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_2-1}\right) = \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)}{\Gamma(\vartheta_2-\vartheta_1)} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta_2-1-\vartheta_1}$.

(2) ${}^C D_b^{\vartheta_1,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_2-1}\right) = \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)}{\Gamma(\beta-\vartheta_1)} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^{\vartheta_2-1-\vartheta_1}$.

Let $n_{\vartheta_1} = [Re(\vartheta_1)] + 1$, for $k = 0, 1, \ldots, n_{\vartheta_1} - 1$, we have

$$ {}_a^C D^{\vartheta_1,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^k = 0 \quad \text{and} \quad {}^C D_b^{\vartheta_1,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}(b-\Theta)^k = 0.$$

In particular, ${}_a^C D^{\vartheta_1,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta} = 0$ *and* ${}^C D_b^{\vartheta_1,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)} = 0$.

Proof. Let $n_{\vartheta_1} = [Re(\vartheta_1)] + 1$, from Proposition 3.10, we get

$$\begin{aligned}
{}^{C}_{a}D^{\vartheta_1,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1} &= {}_{a}J^{n_{\vartheta_1}-\vartheta_1,\varkappa} D^{n_{\vartheta_1},\varkappa}\left[e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1}\right]\\
&= {}_{a}J^{n_{\vartheta_1}-\vartheta_1,\varkappa}\\
&\times\left[\psi(\varkappa)^{n_{\vartheta_1}}(\vartheta_2-1)(\vartheta_2-2)\dots(\vartheta_2-1-n_{\vartheta_1})(\Theta-a)^{\vartheta_2-n_{\vartheta_1}-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}\right]\\
&= \frac{\psi(\varkappa)^{n_{\vartheta_1}}(\vartheta_2-1)(\vartheta_2-2)\dots(\vartheta_2-1-n_{\vartheta_1})\Gamma(\vartheta_2-n)}{\Gamma(\vartheta_2-\vartheta_1)\psi(\varkappa)^{n-\vartheta_1}}\\
&\times(\Theta-a)^{\vartheta_2-\vartheta_1-1}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}\\
&= \frac{\psi(\varkappa)^{\vartheta_1}\Gamma(\vartheta_2)}{\Gamma(\vartheta_2-\vartheta_1)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}(\Theta-a)^{\vartheta_2-1-\vartheta_1}.
\end{aligned}$$

The proof of the second relationship follows a similar approach. □

Lemma 5.4. *Let $\lambda \in \mathbb{R}$. We have*

$$ {}^{C}_{a}D^{\vartheta,\varkappa} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1}) = \lambda\psi(\varkappa)^{\vartheta_1} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1}).$$

Proof. Let $h(\Theta) = e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1})$, we have

$$\begin{aligned}
{}^{C}_{a}D^{\vartheta,\varkappa}h(\Theta) &= {}^{C}_{a}D^{\vartheta,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta_1,\vartheta_2}(\lambda(\Theta-a)^{\vartheta_1})\right)\\
&= {}^{C}_{a}D^{\vartheta,\varkappa}\left(\sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta k+1)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{k\vartheta}\right)\\
&= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta k+1)}\,{}^{C}_{a}D^{\vartheta,\varkappa}\left(e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{k\vartheta}\right)\\
&= \sum_{k=1}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta k+1)}\frac{\psi(\varkappa)^{\vartheta}\Gamma(\vartheta k+1)}{\Gamma(\vartheta k+1-\vartheta)}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta k-\vartheta}\\
&= \psi(\varkappa)^{\vartheta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{k=1}^{+\infty}\frac{\lambda^k}{\Gamma(\vartheta(k-1)+1)}(\Theta-a)^{\vartheta(k-1)}\\
&= \lambda\psi(\varkappa)^{\vartheta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{k=1}^{+\infty}\frac{\lambda^{k-1}}{\Gamma(\vartheta(k-1)+1)}(\Theta-a)^{\vartheta(k-1)}\\
&= \lambda\psi(\varkappa)^{\vartheta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}(\lambda(\Theta-a)^{\vartheta}).
\end{aligned}$$

□

Theorem 5.5. *Let $\varkappa \in]0,1]$ and $n_\vartheta = [\mathrm{Re}(\vartheta)] + 1$, then*

$$ {}_{a}J^{\vartheta,\varkappa}\left({}^{C}_{a}D^{\vartheta,\varkappa}x\right)(\Theta) = x(\Theta) - \sum_{k=0}^{n-1}\frac{\left(D^{k,\varkappa}x\right)(a)}{\psi(\varkappa)^k k!}(\Theta-a)^k e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}. \tag{5.3}$$

Proof. From the Theorem 3.18 where $\vartheta = n_\vartheta$, we get

$$
\begin{aligned}
{}_aJ^{\vartheta,\varkappa}\left({}_a^CD^{\vartheta,\varkappa}x\right)(\Theta) &= {}_aJ^{\vartheta,\varkappa}\left({}_aJ^{n_\vartheta-\vartheta,\varkappa}D^{n_\vartheta,\varkappa}x\right)(\Theta) = \left({}_aJ^{n_\vartheta,\varkappa}D^{n_\vartheta,\varkappa}x\right)(\Theta)\\
&= x(\Theta) - e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\sum_{j=1}^{n_\vartheta}\frac{\left({}_aJ^{j-n_\vartheta,\varkappa}x\right)(a^+)(\Theta-a)^{n_\vartheta-j}}{\psi(\varkappa)^{n_\vartheta-j}\Gamma(n_\vartheta-j+1)}\\
&= x(\Theta) - \sum_{k=0}^{n_\vartheta-1}\frac{\left(D^{k,\varkappa}x\right)(a)}{\psi(\varkappa)^k k!}(\Theta-a)^k e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}.
\end{aligned}
$$

□

Theorem 5.6. *Let* $\varkappa \in]0,1]$ *and* $\vartheta \in \mathbb{C}$ *with* $\mathrm{Re}(\vartheta) > 0$ $n_\vartheta = [\mathrm{Re}(\vartheta)] + 1$. *If* $X_a(p) = \mathcal{L}\{x(\Theta)\}(p)$, *then*

$$
\begin{aligned}
\mathcal{L}_a\left\{\left({}_a^CD^{\vartheta,\varkappa}x\right)(\Theta)\right\}(p) &= (\phi(\varkappa)+\psi(\varkappa)p)^\vartheta X_a(p) - \psi(\varkappa)\sum_{k=0}^{n_\vartheta-1}(\phi(\varkappa)\\
&+\psi(\varkappa)p)^{\vartheta-1-k}\left(D^{k,\varkappa}x\right)(a).
\end{aligned} \tag{5.4}
$$

Proof. From the Theorems 4.1 and 4.2, we get

$$
\begin{aligned}
\mathcal{L}_a\left\{\left({}_a^CD^{\vartheta,\varkappa}x\right)(\Theta)\right\}(p) &= \mathcal{L}_a\left\{\left({}_aJ^{n_\vartheta-\vartheta,\varkappa}D^{n_\vartheta,\varkappa}x\right)(\Theta)\right\}(p)\\
&= (\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta-n_\vartheta}\mathcal{L}_a\left\{\left(D^{\vartheta,\varkappa}x\right)(\Theta)\right\}(p)\\
&= (\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta-n_\vartheta}\\
&\times\Bigg[(\phi(\varkappa)+\psi(\varkappa)p)^{n_\vartheta}X_a(p) - \psi(\varkappa)\\
&\sum_{k=0}^{n_\vartheta-1}(\phi(\varkappa)+\psi(\varkappa)p)^{n_\vartheta-1-k}\left(D^{k,\varkappa}x\right)(a)\Bigg]\\
&= (\phi(\varkappa)+\psi(\varkappa)p)^\vartheta X_a(p) - \psi(\varkappa)\\
&\times\sum_{k=0}^{n_\vartheta-1}(\phi(\varkappa)+\psi(\varkappa)p)^{\vartheta-1-k}\left(D^{k,\varkappa}x\right)(a).
\end{aligned}
$$

□

Utilizing Theorems 4.3 and 5.6, we establish the following relationship between the Caputo and Riemann-Liouville (ψ,ϕ)-fractional derivatives.

Proposition 5.7. *Let* $\varkappa \in]0,1]$, $\vartheta \in \mathbb{C}$ *where* $\mathrm{Re}(\vartheta) > 0$ *and* $n_\vartheta = [\mathrm{Re}(\vartheta)] + 1$, *then*

$$
\left({}_a^CD^{\vartheta,\varkappa}x\right)(\Theta) = \left({}_aD^{\vartheta,\varkappa}x\right)(\Theta) - \sum_{k=0}^{n_\vartheta-1}\frac{\psi(\varkappa)^{\vartheta-k}(\Theta-a)^{k-\vartheta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}}{\Gamma(k+1-\vartheta)}\left(D^{k,\varkappa}x\right)(a), \tag{5.5}
$$

and

$$
\left({}^CD_b^{\vartheta,\varkappa}x\right)(\Theta) = \left(D_b^{\vartheta,\varkappa}x\right)(\Theta) - \sum_{k=0}^{n_\vartheta-1}\frac{\psi(\varkappa)^{\vartheta-k}(b-\Theta)^{k-\vartheta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(b-\Theta)}}{\Gamma(k+1-\vartheta)}\left(D^{k,\varkappa}x\right)(b). \tag{5.6}
$$

Remark 5.8. We have

- $\left({}_a^CD^{\vartheta,\varkappa}c\right)(\Theta) \neq 0, \quad \forall\varkappa \in]0,1[$.
- Let $n_\vartheta = [\mathrm{Re}(\vartheta)] + 1$ then $\mathcal{L}_a\left\{\left({}_a^CD^{\vartheta,\varkappa}1\right)(\Theta)\right\}(p) = \frac{\psi(\varkappa)^{n_\vartheta-\vartheta}(\phi(\varkappa))^{n_\vartheta}}{(\phi(\varkappa)+\psi(\varkappa)p)^{n_\vartheta}}\frac{1}{p}, \quad \forall\varkappa \in]0,1[$.
- $D^\varkappa e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta} = 0$ implies that ${}_a^CD^{\vartheta,\varkappa}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} = 0$.

Theorem 5.9. *Consider the linear Caputo (ψ, ϕ)-fractional differential equation:*

$$\begin{cases} {}_a^C D^{\vartheta,\varkappa} x(\Theta) = \lambda x(\Theta) + y(\Theta), \quad 0 < \vartheta, \varkappa \leq 1, \\ x(a^+) = x_a. \end{cases} \tag{5.7}$$

Then the solution of (5.7) is:

$$\begin{aligned} x(\Theta) &= e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right) x_a \\ &\quad + \psi(\varkappa)^{-\vartheta} \int_a^{\Theta} e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)} (\Theta-s)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-s)^{\vartheta}\right) y(s)ds. \end{aligned} \tag{5.8}$$

Proof. Applying the operator $\mathcal{L}_a$ to equation (5.7) and employing Theorem 5.6 with $n_\vartheta = 1$, we obtain:

$$(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} \mathcal{L}_a\{x(\Theta)\}(p) - \psi(\varkappa)(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta-1} x_a = \lambda \mathcal{L}_a\{x(\Theta)\}(p) + \mathcal{L}_a\{y(\Theta)\}(p).$$

Which is equivalent to

$$\left[(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} - \lambda\right] \mathcal{L}_a\{x(\Theta)\}(p) = \psi(\varkappa)(\psi(\varkappa)p + \phi(\varkappa))^{\vartheta-1} x_a + \mathcal{L}_a\{y(\Theta)\}(p).$$

Thus,

$$\mathcal{L}_a\{x(\Theta)\}(p) = \left[(\phi(\varkappa) + \psi(\varkappa)p)^{\vartheta} - \lambda\right]^{-1} \psi(\varkappa)(\psi(\varkappa)p + \phi(\varkappa))^{\vartheta-1} x_a + \left[(\phi(\varkappa) - \psi(\varkappa)p)^{\vartheta}\lambda\right]^{-1} \mathcal{L}_a\{y(\Theta)\}(p). \tag{5.9}$$

Equation (5.9) can be written as:

$$\begin{aligned} \mathcal{L}_a\{x(\Theta)\}(p) &= \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta} - \psi(\varkappa)^{-\vartheta}\lambda\right]^{-1} \left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta-1} x_a \\ &\quad + \psi(\varkappa)^{-\vartheta} \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta} - \psi(\varkappa)^{-\vartheta}\lambda\right]^{-1} \mathcal{L}_a\{y(\Theta)\}(p), \end{aligned} \tag{5.10}$$

using

$$\mathcal{L}_a\{e^{-c(\Theta-a)} x(\Theta)\}(p) = \mathcal{L}_a\{x(\Theta)\}(p+c),$$

as follows:

$$\begin{aligned} \mathcal{L}_a\left\{(\Theta-a)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right) &= \mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}\right. \\ &\quad \left.(\Theta-a)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p). \end{aligned} \tag{5.11}$$

And

$$\mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right) = \mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p). \tag{5.12}$$

Using Eqs. (4.10) and (4.10) as follows:

$$\mathcal{L}_a\{(\Theta-a)^{\vartheta-1} E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\}\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right) = \left[(p + \frac{\phi(\varkappa)}{\psi(\varkappa)})^{\vartheta} - \psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}. \tag{5.13}$$

And,

$$\mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right) = \left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta-1} \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta} - \psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}. \tag{5.14}$$

Hence, by using Equations (5.12) and (5.14), we have:

$$\begin{aligned} \left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta-1} \left[\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta} - \psi(\varkappa)^{-\vartheta}\right]^{-1} &= \mathcal{L}_a\left\{E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p + \frac{\phi(\varkappa)}{\psi(\varkappa)}\right) \\ &= \mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)} E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p). \end{aligned} \tag{5.15}$$

Further, by using Eqs. (5.11) and (5.13), we get

$$\begin{aligned}\left[\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)^{\vartheta}-\psi(\varkappa)^{-\vartheta}\lambda\right]^{-1}&=\mathcal{L}_a\left\{(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}\left(p+\frac{\phi(\varkappa)}{\psi(\varkappa)}\right)\\&=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p).\end{aligned}\tag{5.16}$$

We may get the following result by combining Eqs. (5.15) and (5.16) in Eq. (5.10), we get

$$\begin{aligned}\mathcal{L}_a[x(\Theta)](p)&=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)x_a\\&\quad+\psi(\varkappa)^{-\vartheta}\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)\mathcal{L}_a[y(\Theta)](p).\end{aligned}\tag{5.17}$$

From Eq.(5.17), we get

$$\begin{aligned}\mathcal{L}_a[x(\Theta)](p)&=\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)\right\}(p)x_a\\&\quad+\psi(\varkappa)^{-\vartheta}\mathcal{L}_a\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)*y(\Theta)\right\}(p).\end{aligned}\tag{5.18}$$

We use the Laplace inverse of Eq.(5.18), we get

$$\begin{aligned}x(\Theta)&=e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)x_a\\&\quad+\psi(\varkappa)^{-\vartheta}\left\{e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}(\Theta-a)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)*y(\Theta)\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}x(\Theta)&=e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-a)}E_{\vartheta,1}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-a)^{\vartheta}\right)x_a\\&\quad+\psi(\varkappa)^{-\vartheta}\int_a^{\Theta}e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}(\Theta-s)}(\Theta-s)^{\vartheta-1}E_{\vartheta,\vartheta}\left(\psi(\varkappa)^{-\vartheta}\lambda(\Theta-s)^{\vartheta}\right)y(s)ds.\end{aligned}\tag{5.19}$$

□

6. Conclusions

We've introduced (ψ,ϕ)-fractional derivatives of Riemann-Liouville and Caputo types that rely on parameters ϑ, $\varkappa$, and functions ψ, ϕ derived from proportional derivatives. When $\varkappa=1$, these yield Riemann-Liouville and Caputo fractional derivatives. We've explored the interplay between the integrals and derivatives proposed. The (ψ,ϕ)-integrals exhibit semigroup properties and, in conjunction with their respective derivatives, encompass exponential functions within their kernels. Notably, the function $x(\Theta)=e^{-\frac{\phi(\varkappa)}{\psi(\varkappa)}\Theta}$ stands as a non-constant function, having a proportional derivative that renders its left Caputo fractional derivative as zero. Utilizing the Laplace transform of (ψ,ϕ)-fractional derivatives and integrals, we've formulated linear (ψ,ϕ)-fractional differential equations. Moreover, these fractional operators are general to classical fractional operators, cotangent fractional operators, and generalized proportional operators.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics, *279* (2015), 5766.

[2] T. Abdeljawad and D. Baleanu, *Monotonicity results for fractional difference operators with discrete exponential kernels*, Advances in Difference Equations, *2017*(1) (2017), 19.

[3] T. Abdeljawad and D. Baleanu, *On fractional derivatives with exponential kernel and their discrete versions*, Reports on Mathematical Physics, *80*(1) (2017), 1127.

[4] Y. Adjabi, F. Jarad, D. Baleanu and T. Abdeljawad, *On Cauchy problems with Caputo Hadamard fractional derivatives*, Journal of Computational Analysis and Applications, *21*(1) (2016), 661681.

[5] D. R. Anderson and D. J. Ulness, *Newly defined conformable derivatives*, Advances in Dynamical Systems and Applications, *10*(2) (2015), 109137.

[6] D. R. Anderson, *Second-order self-adjoint differential equations using a proportional-derivative controller*, Communications in Applied Nonlinear Analysis, *24* (2017), 1748.

[7] A. Atangana and D. Baleanu, *New fractional derivative with non-local and non-singular kernel*, Thermal Science, *20* (2016), 757.

[8] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation & Applications, *1*(2) (2015), 7385.

[9] L. Debnath, *Recent applications of fractional calculus to science and engineering*, International Journal of Mathematics and Mathematical Sciences, *2003*(54) (2003), 34133442.

[10] Y. Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, *On Caputo modification of the Hadamard fractional derivatives*, Advances in Difference Equations, *2014*(1) (2014), 112.

[11] R. Hilfer (Ed.), *Applications of fractional calculus in physics*, World Scientific (2000).

[12] F. Jarad, T. Abdeljawad and D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Advances in Difference Equations, *2012* (2012), 18.

[13] F. Jarad, T. Abdeljawad and D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, Journal of Nonlinear Sciences and Applications, *10*(5) (2017), 26072619.

[14] F. Jarad, T. Abdeljawad and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives*, The European Physical Journal Special Topics, *226* (2017), 34573471.

[15] F. Jarad, E. Ugurlu, T. Abdeljawad and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, *2017*(1) (2017), 116.

[16] F. Jarad and T. Abdeljawad, *Generalized fractional derivatives and Laplace transform*, Discrete and Continuous Dynamical Systems - Series S, *13*(3) (2020), 709722.

[17] F. Jarad, M. A. Alqudah and T. Abdeljawad, *On more general forms of proportional fractional operators*, Open Mathematics, *18*(1) (2020), 167176.

[18] U. N. Katugampola, *New approach to a generalized fractional integral*, Applied Mathematics and Computation, *218*(3) (2011), 860865.

[19] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bulletin of Mathematical Analysis and Applications, *6* (2014), 115.

[20] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, *264* (2014), 6570.

[21] A. A. Kilbas, M. Saigo and R. K. Saxena, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transforms and Special Functions, *15*(1) (2004), 3149.

[22] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, *204* (2006).

[23] A. N. Kochubei, *General fractional calculus, evolution equations, and renewal processes*, Integral Equations and Operator Theory, *71*(4) (2011), 583600.

[24] J. Losada and J. J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progress in Fractional Differentiation & Applications, *1*(2) (2015), 8792.

[25] R. L. Magin, *Fractional calculus in bioengineering*, Begell House Publishers (2006).

[26] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, California, USA (1999).
[27] L. Sadek, *A Cotangent Fractional Derivative with the Application*, Fractal and Fractional, *7*(6) (2023), 444.
[28] L. Sadek and T. A. Lazar, *On Hilfer cotangent fractional derivative and a particular class of fractional problems*, AIMS Mathematics, *8*(12) (2023), 2833428352.
[29] L. Sadek, *Controllability, observability, and stability of ϕconformable fractional linear dynamical systems*, Asian Journal of Control, *26*(5) (2024), 2476-2494.
[30] L. Sadek, D. Baleanu, M. S. Abdo and W. Shatanawi, *Introducing novel θ-fractional operators: Advances in fractional calculus*, Journal of King Saud UniversityScience, *36*(9) (2024), 103352.
[31] L. Sadek and F. Jarad, *The general CaputoKatugampola fractional derivative and numerical approach for solving the fractional differential equations*, Alexandria Engineering Journal, *121* (2025), 539-557.
[32] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon, (1993).