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Solution of a non-homogeneous dynamic equation on a time scale

Ahmed Azaba[1], **Kamal R. Raslan**[2], **Khalid K. Ali**[2,3,*]
[1]Al-Ryada University for Science and Technology, Sadat City, Menoufia, Egypt.
[2]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt.
[3]Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman.

Abstract

We find the general solution to the non-homogeneous dynamic equation, which is a combination of discrete and continuous mathematics. The nature of this equation requires a careful approach, as it involves elements of both types of math, making it both versatile and challenging. To address this, we will derive the formula for the general solution of the non-homogeneous equation, incorporating the given initial conditions. During this process, we will define several critical points that may be either discrete, continuous, or a mix of both. By analyzing these points, we aim to capture the essence of the dynamic behavior of the system. Our approach involves finding an analytical solution to the equation and comparing it with a numerical approximation to evaluate their accuracy. We will graph both the analytical and numerical solutions to visualize their behavior and identify any discrepancies. Additionally, we will calculate the absolute error between the exact solution and the numerical solution to quantify the differences precisely. This comparison provides valuable insights into the accuracy and stability of numerical methods for solving such equations. Finally, we will demonstrate this approach by applying it to various examples, showcasing the methodology's effectiveness in solving a range of non-homogeneous dynamic equations with different initial conditions and parameters.

Keywords. Time scale, Non-Homogeneous dynamic equation, Numerical solution, Initial value problem, General solution.
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1. Introduction

Time scale (T.S) theory provides a powerful and versatile mathematical framework for studying dynamic systems that evolve over time, seamlessly integrating both continuous and discrete processes [14]. Introduced by Stefan Hilger in 1988, the theory unifies the analysis of continuous systems, typically described by differential equations, and discrete systems, modeled using difference equations [11]. This unification enables researchers to study processes that transition fluidly between continuous and discrete behaviors without switching mathematical frameworks, making it an indispensable tool for analyzing hybrid systems [6].

At its core, a T.S, denoted as $\mathbb{T}$, is a closed, non-empty subset of real numbers that can encompass isolated points (representing discrete events), continuous intervals (representing ongoing processes), or a mix of both [4]. For instance, when $\mathbb{T} = \mathbb{R}$, the framework aligns with differential calculus, while $\mathbb{T} = \mathbb{Z}$ corresponds to difference calculus [1, 9]. This versatility is achieved through the introduction of the delta derivative, a generalized form of differentiation that adapts to the structure of $\mathbb{T}$. This allows the framework to transition seamlessly between continuous and discrete dynamics, capturing the intricacies of hybrid systems [5].

Early extensions of time scale calculus by Aulbach and Bohner established rigorous frameworks for dynamic equations, bridging classical calculus and difference equations [6]. Over the decades, the field advanced through Gorenflo's work on mathematical analysis, highlighting the versatility of time scales [14]. Recent studies by Atici and Uysal explored advanced applications in dynamical systems [4]. Liu and Zhang provided new insights into modeling hybrid

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* Corresponding author. Email: khalidkaram2012@azhar.edu.eg.

phenomena using time scales [17]. Harris's 2018 work unified approaches to complex system modeling [21–23]. These developments emphasize time scale calculus's growing role in addressing hybrid dynamics across disciplines.

One of the most compelling applications of T.S theory lies in biology, where many natural systems exhibit hybrid behaviors. Consider population dynamics: bacterial growth, which is continuous, can be effectively modeled using differential equations such as:

$$\frac{d\Upsilon}{d\psi} = r\Upsilon$$

where $\Upsilon(\psi)$ represents the population size at time ψ, and r is the growth rate [2, 12]. On the other hand, species like insects, which reproduce in fixed intervals, require discrete-time models:

$$\Upsilon_{n+1} = \Upsilon_n + r\Upsilon_n.$$

Here, Υ_n denotes the population at the n-th generation. T.S theory enables researchers to model such systems holistically by defining hybrid T.S like $\mathbb{T} = \mathbb{R} \cup \mathbb{Z}$, where continuous growth and discrete reproduction coexist within a single framework [17].

In economics, financial systems often blend continuous and discrete behaviors. Stock prices, for instance, fluctuate continuously during trading hours, a phenomenon described by stochastic differential equations such as the Black-Scholes model [3]. However, discrete events like quarterly earnings releases, policy changes, or sudden market shocks cause abrupt changes in prices, which are better captured by difference equations [16]. By leveraging T.S theory, economists can develop models that incorporate both gradual fluctuations and discrete shifts, providing a more accurate representation of financial markets [19].

Engineering systems often involve hybrid dynamics, making T.S theory especially relevant. Continuous processes, such as fluid flow in pipelines or the gradual heating of a furnace, are governed by differential equations [10]. At the same time, discrete events, such as the opening of a valve or the activation of a thermostat, introduce sudden changes that are better described using difference equations [13]. Using T.S like $\mathbb{T} = [0, 1] \cup \{2, 3, 4\}$, engineers can represent smooth operations interrupted by discrete actions, improving the design and control of such systems [21].

In physics, the theory facilitates the study of systems that bridge quantum and classical behaviors. Classical mechanics, rooted in Newton's laws, involves continuous dynamics described by differential equations [18]. Quantum mechanics, in contrast, deals with discrete phenomena, such as quantized energy levels, requiring difference equations or operator-based methods. T.S theory unifies these approaches, allowing researchers to analyze semi-classical systems where continuous motion and discrete transitions coexist, such as in laser dynamics or nanoscale systems [15].

Beyond these domains, T.S theory has found applications in a range of fields. In control theory, it enables the modeling of hybrid systems that operate continuously under normal conditions but switch to discrete controls during disturbances [22]. In environmental science, processes like soil erosion exhibit continuous degradation interrupted by discrete rainfall events, which can be effectively studied using hybrid T.S. In epidemiology, disease spread may follow continuous growth patterns but also show discrete spikes due to seasonal factors or vaccination campaigns[18].

T.S theory provides a unified framework for studying hybrid systems that evolve across both continuous and discrete temporal structures. By bridging these domains, the theory overcomes the limitations of traditional approaches, offering a more comprehensive understanding of complex dynamics.[7, 8] Whether in biology, economics, engineering, or physics, T.S theory equips researchers with a versatile toolkit for analyzing systems that cannot be fully described by differential or difference equations alone. Its ability to integrate diverse temporal behaviors has unlocked new insights into the functioning of natural, technological, and social systems, underscoring its transformative potential across disciplines[11].

2. Preliminaries

T.S $\mathbb{T}$ is defined as an arbitrary, non-empty closed subset of the real numbers $\mathbb{R}$. Consider a specific element ψ that belongs to the set $\mathbb{T}$.

- The *forward jump operator*, $\Gamma : \mathbb{T} \to \mathbb{T}$ is defined as: [9, 10]

  $\Gamma(\psi) = \inf\{s \in \mathbb{T} : s > \psi\}.$

- The *backward jump operator* $\mathfrak{p} : \mathbb{T} \to \mathbb{T}$ is defined as: [9, 10]

  $\mathfrak{p}(\psi) = \sup\{s \in \mathbb{T} : s < \psi\}.$
- A point $\psi \in \mathbb{T}$ is classified as follows:[9, 10]
  - **Left-scattered**: if $\mathfrak{p}(\psi) < \psi$.
  - **Right-scattered**: if $\Gamma(\psi) > \psi$.
  - **Left-dense**: if $\mathfrak{p}(\psi) = \psi$.
  - **Right-dense**: if $\Gamma(\psi) = \psi$.
  - **Isolated**: if $\mathfrak{p}(\psi) < \psi < \Gamma(\psi)$.
- We define the following adjusted T.S: [9, 11]

  $$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus \{\sup \mathbb{T}\}, & \text{if } \sup \mathbb{T} \text{ is LEFT-scattered,} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

  Similarly,

  $$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus \{\inf \mathbb{T}\}, & \text{if } \inf \mathbb{T} \text{ is RIGHT-scattered,} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$
- The *forward graininess function*, $\gamma : \mathbb{T} \to \mathbb{R}$ is defined as: [9, 11]

  $\gamma(\psi) = \Gamma(\psi) - \psi.$
- The *backward graininess function*, $\nu : \mathbb{T} \to \mathbb{R}$ is defined as: [9, 11]

  $\nu(\psi) = \psi - \mathfrak{p}(\psi).$

Definition 2.1. *Let $\mathbb{T}$ represent a time scale, and let $\Xi : \mathbb{T} \to \mathbb{R}$. The delta derivative of Ξ at a point $\psi \in \mathbb{T}$, denoted as $\Xi^{\Delta}(\psi)$, is defined as the value that fulfills the following criterion:[8–10]*

For any $\varepsilon > 0$, there exists a corresponding $\delta_\varepsilon > 0$ such that for every $s \in \mathbb{T}$, if $|s - \psi| < \delta_\varepsilon$, then

$$|\Xi(\Gamma(\psi)) - \Xi(s) - \Xi^{\Delta}(\psi)(\Gamma(\psi) - s)| \leq \varepsilon|\Gamma(\psi) - s|.$$

If such a value cannot be found, we say that Ξ is not delta differentiable at the point ψ.

Definition 2.2. *Let $\mathbb{T}$ represent a time scale and consider a function $\Xi : \mathbb{T} \to \mathbb{R}$. The nabla derivative of Ξ at a point $\psi \in \mathbb{T}$, which is denoted as $\Xi^{\nabla}(\psi)$, is defined as the value that meets the following condition: [9–11]*

For every $\varepsilon > 0$, there exists a corresponding $\delta_\varepsilon > 0$ such that for all values $s \in \mathbb{T}$, if $|s - \psi| < \delta_\varepsilon$, then it holds that

$$|\Xi(\mathfrak{p}(\psi)) - \Xi(s) - \Xi^{\nabla}(\psi)(\psi - s)| \leq \varepsilon|\psi - s|.$$

If such a value cannot be found, then Ξ is described as being not nabla differentiable at ψ.

Theorem 2.3. *[7, 20] Let $\Xi : \mathbb{T} \to \mathbb{R}$ be a continuous function, and suppose $\psi \in \mathbb{T}^\kappa$.*

If ψ is Right-dense, then

$$\Xi^{\Delta}(\psi) = \lim_{s \to \psi} \frac{\Xi(\psi) - \Xi(s)}{\psi - s} = \Xi'(\psi).$$

If ψ is Right-scattered, then

$$\Xi^{\Delta}(\psi) = \frac{\Xi(\Gamma(\psi)) - \Xi(\psi)}{\gamma(\psi)}.$$

Theorem 2.4. *[8, 10] Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function, and let $\psi \in \mathbb{T}_\kappa$.*

If ψ is Left-dense, then

$$g^{\nabla}(\psi) = \lim_{s \to \psi} \frac{g(\psi) - g(s)}{\psi - s} = g'(\psi).$$

If ψ is Left-scattered, then

$$g^{\nabla}(\psi) = \frac{g(\psi) - g(\mathfrak{p}(\psi))}{\nu(\psi)}.$$

We now move on to defining integration on T.S, starting with several foundational definitions. Let $\Xi : \mathbb{T} \to \mathbb{R}$ be a function that is [9–11]:

- **Regulated** if finite right-hand limits exist at right-dense points, and finite left-hand limits exist at left-dense points.
- **Right-dense continuous** if it is both regulated and continuous at right-dense points (denoted by $C_{\mathrm{rd}}(\mathbb{T})$).
- **Pre-differentiable** on the set D if $\mathbb{T}^{\kappa} \setminus D$ is countable and D has no right-scattered points.

Starting with the following theorem, we shall describe integration using a variant of the Fundamental Theorem of Calculus:

Theorem 2.5. *[9, 11] If Ξ is a regulated function, then there exists a function Ξ that is pre-differentiable on a set D such that*

$$\Xi^{\Delta}(\psi) = \Xi(\psi) \quad \forall \psi \in D.$$

The anti-derivative may now be used to define the integral.

Definition 2.6. *[1, 4] Let $\Xi : \mathbb{T} \to \mathbb{R}$ be a REGULATED function. The indefinite integral of Ξ is defined as*

$$\int \Xi(\psi)\Delta\psi = \Xi(\psi) + C,$$

where Ξ is the function specified in Theorem 2.3. The Cauchy integral of Ξ is then defined by

$$\int_r^s \Xi(\psi)\Delta\psi = \Xi(s) - \Xi(r), \quad \forall\ r, s \in \mathbb{T}.$$

We now provide the calculus fundamental theorem's analog.

Theorem 2.7. *[8, 18] Every rd-continuous function has an antiderivative. If $\psi_0 \in \mathbb{T}$, then*

$$\Xi(\psi) := \int_{\psi_0}^{\psi} \Xi(\psi)\Delta\psi,$$

is an antiderivative of Ξ for all $\psi \in \mathbb{T}$.

Theorem 2.8. *[1, 4] Every rd-continuous function has an antiderivative. If $\psi_0 \in \mathbb{T}$, then*

$$\Xi(\psi) := \int_{\psi_0}^{\psi} \Xi(\psi)\nabla\psi \quad \text{for } \psi \in \mathbb{T},$$

is an antiderivative of Ξ.

Theorem 2.9 (Variation of Constants)**.** *[9] Suppose $p \in \mathbb{R}$ and $\Xi \in C_{rd}$. Let $\psi_0 \in \mathbb{T}$ and $\mathcal{Y}_0 \in \mathbb{R}$. The unique solution of the IVP*

$$\mathcal{Y}^{\Delta} = p(\psi)\mathcal{Y} + \Xi(\psi), \quad \mathcal{Y}(\psi_0) = \mathcal{Y}_0$$

is given by

$$\mathcal{Y}(\psi) = e_p(\psi, \psi_0)\mathcal{Y}_0 + \int_{\psi_0}^{\psi} e_p(\psi, \Gamma(\psi))\Xi(\psi)\Delta\psi.$$

- **The Euler Method**

The Euler method is a basic and straightforward numerical method for solving initial value problems for first-order differential equations. It can also be generalized for dynamic equations on T.S. [15]

Given the differential equation

$\Upsilon^{\Delta}(\psi) = \Xi(\psi, \Upsilon(\psi)), \quad \Upsilon(\psi_0) = x_0.$

with known $\Xi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, the Euler method updates the solution by the rule:

For right-scattered times:

$\Upsilon_{i+1} = \Upsilon_i + (\Gamma(\psi_{i+1}) - \psi_i)\Xi(\psi_i, \Upsilon_i).$

For right-dense times:

$\Upsilon_{i+1} = \Upsilon_i + \Delta\psi_i \Xi(\psi_i, \Upsilon_i).$

3. Main result

Theorem 3.1. *Suppose $\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi)$ is a regressive equation, and let $\psi_0 \in \mathbb{T}$ with the initial condition $\mathcal{Y}(\psi_0) = \mathcal{Y}_0$ on $\mathbb{T}$, where*

$\mathbb{T} = [\psi_0, \psi_1] \cup [\psi_2, \psi_3].$

ψ_0———ψ_1 ψ_2———ψ_3

The unique solution to the initial value problem $\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi)$, with $\mathcal{Y}(\psi_0) = \mathcal{Y}_0$ on $\mathbb{T}$, is represented by the following piecewise function:

$$\mathcal{Y}(\psi) = \begin{cases} e_p(\psi, \psi_0)\left[\mathcal{Y}_0 + \int_{\psi_0}^{\psi} e_p(\psi_0, \Gamma(\psi))\Xi(\psi)\, d\psi\right], & \psi_0 \le \psi \le \psi_1, \\ e_p(\psi, \psi_2)\Big[\mathcal{Y}_0 e_p(\psi_1, \psi_0) + (1 + \gamma(\psi_1)p(\psi_1)) \int_{\psi_0}^{\psi_1} e_p(\psi_0, \Gamma(\psi))\Xi(\psi)\, d\psi & \\ +\gamma(\psi_1)\Xi(\psi_1) + \int_{\psi_2}^{\psi} e_p(\psi_2, \Gamma(\psi))\Xi(\psi)\, d\psi\Big], & \psi_2 \le \psi \le \psi_3. \end{cases}$$

Proof. Consider $\psi \in [\psi_0, \psi_1]$.
We rewrite the differential equation $\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi)$. The solution for this equation is given by the theorem.

For $\psi \in [\psi_2, \psi_3]$, we have the same differential equation:

$$\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi), \quad \mathcal{Y}(\psi_0) = \mathcal{Y}_0 \text{ on } [\psi_2, \psi_3]$$

The solution is given by:

$$\mathcal{Y}(\psi) = \mathcal{Y}(\psi_2)e_p(\psi, \psi_2) + \int_{\psi_2}^{\psi} e_p(\psi_2, \Gamma(\psi))\Xi(\psi)\, d\psi.$$

Now, consider $\mathcal{Y}(\psi_2)$:

$$\mathcal{Y}(\psi_2) = \mathcal{Y}(\Gamma(\psi_1)) = \mathcal{Y}(\psi_1) + \gamma(\psi_1)\mathcal{Y}^{\Delta}(\psi_1).$$

Then,

$$\mathcal{Y}(\psi_2) = \mathcal{Y}(\psi_1)[1 + \gamma(\psi_1)p(\psi_1)] + \gamma(\psi_1)\Xi(\psi_1).$$

Now, substitute the value of $\mathcal{Y}(\psi_1)$ from the earlier interval $[\psi_0, \psi_1]$:

$$\mathcal{Y}(\psi_2) = \left(\mathcal{Y}_0 e_p(\psi_1, \psi_0) + \int_{\psi_0}^{\psi_1} e_p(\psi_0, \Gamma(\psi))\Xi(\psi)\, d\psi\right)[1 + \gamma(\psi_1)p(\psi_1)] + \gamma(\psi_1)\Xi(\psi_1)$$

Finally, we substitute $\mathcal{Y}(\psi_2)$ into the solution for $\psi \in [\psi_2, \psi_3]$:

$$\mathcal{Y}(\psi) = e_p(\psi, \psi_2)\left[\left(\mathcal{Y}_0 e_p(\psi_1, \psi_0) + \int_{\psi_0}^{\psi_1} e_p(\psi_0, \Gamma(\psi))\Xi(\psi)\, d\psi\right)(1 + \gamma(\psi_1)p(\psi_1)) + \gamma(\psi_1)\Xi(\psi_1)\right]$$

$$+\int_{\psi_2}^{\psi} e_p(\psi_2, \Gamma(\psi))\Xi(\psi)\, d\psi.$$

This completes the proof. □

Corollary 3.2. *Consider a regressive equation $\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi)$ with an initial value condition $\mathcal{Y}(\psi_0) = \mathcal{Y}_0$ defined on the time scale $\mathbb{T} = [\psi_0, \psi_1] \cup [\psi_2, \psi_3] \cup [\psi_4, \psi_5]$. Here, $\psi_0 \in \mathbb{T}$ serves as the starting point.*

$$\psi_0 \text{———} \psi_1 \quad \psi_2 \text{———} \psi_3 \quad \psi_4 \text{———} \psi_5$$

We define the step sizes by $\gamma(\psi_1) = \psi_2 - \psi_1$ and $\gamma(\psi_3) = \psi_4 - \psi_3$. The solution to this initial value problem $\mathcal{Y}^{\Delta}(\psi) = p(\psi)\mathcal{Y} + \Xi(\psi)$, with the condition $\mathcal{Y}(\psi_0) = \mathcal{Y}_0$ on $\mathbb{T}$, is presented as a piecewise function:

$$\mathcal{Y}(\psi) = \begin{cases} e_p(\psi,\psi_0)\left[\mathcal{Y}_0 + \int_{\psi_0}^{\psi} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi\right], & \psi_0 \le \psi \le \psi_1, \\ e_p(\psi,\psi_2)\Big\{ \left(\mathcal{Y}_0 e_p(\psi_1,\psi_0) + \int_{\psi_0}^{\psi_1} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi\right)(1+\gamma(\psi_1)p(\psi_1)) & \\ +\gamma(\psi_1)\Xi(\psi_1) + \int_{\psi_2}^{\psi} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\Big\}, & \psi_2 \le \psi \le \psi_3, \\ e_p(\psi,\psi_4)\Big\{ \mathcal{Y}_0 e_p(\psi_1,\psi_0)e_p(\psi_3,\psi_2)(1+\gamma(\psi_1)p(\psi_1))(1+\gamma(\psi_3)p(\psi_3)) & \\ \quad +e_p(\psi_3,\psi_2)(1+\gamma(\psi_1)p(\psi_1))(1+\gamma(\psi_3)p(\psi_3))\int_{\psi_0}^{\psi_1} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi & \\ \quad +e_p(\psi_3,\psi_2)(1+\gamma(\psi_3)p(\psi_3))\int_{\psi_2}^{\psi_3} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi & \\ \quad +e_p(\psi_3,\psi_2)(1+\gamma(\psi_3)p(\psi_3))\gamma(\psi_1)\Xi(\psi_1) + \gamma(\psi_3)\Xi(\psi_3) & \\ \quad + \int_{\psi_4}^{\psi} e_p(\psi_4,\Gamma(\psi))\Xi(\psi)\,d\psi\Big\}, & \psi_4 \le \psi \le \psi_5. \end{cases}$$

Proof. For $\psi_0 \le \psi \le \psi_1$ and $\psi_2 \le \psi \le \psi_3$, as established in Theorem2.10, a similar proof can be constructed.

For $\psi \in [\psi_4, \psi_5]$, the same differential equation applies:

The solution is given by:

$$\mathcal{Y}(\psi) = \mathcal{Y}(\psi_4)e_p(\psi,\psi_4) + \int_{\psi_4}^{\psi} e_p(\psi_4,\Gamma(\psi))\Xi(\psi)\,d\psi.$$

Now, consider $\mathcal{Y}(\psi_4)$:

$$\mathcal{Y}(\psi_4) = \mathcal{Y}(\Gamma(\psi_3)) = \mathcal{Y}(\psi_3) + \gamma(\psi_3)\mathcal{Y}^{\Delta}(\psi_3).$$

Then,

$$\mathcal{Y}(\psi_4) = \mathcal{Y}(\psi_3)[1 + \gamma(\psi_3)p(\psi_3)] + \gamma(\psi_3)\Xi(\psi_3).$$

Now, substitute the value of $\mathcal{Y}(\psi_3)$ from the earlier interval $[\psi_2, \psi_3]$:

$$\mathcal{Y}(\psi_4) = \Bigg(e_p(\psi,\psi_2)\Big\{ \left(\mathcal{Y}_0 e_p(\psi_1,\psi_0) + \int_{\psi_0}^{\psi_1} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi\right)(1+\gamma(\psi_1)p(\psi_1))$$
$$+\gamma(\psi_1)\Xi(\psi_1) + \int_{\psi_2}^{\psi} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\Big\}\Bigg)[1+\gamma(\psi_3)p(\psi_3)] + \gamma(\psi_3)\Xi(\psi_3).$$

Now, substitute the value of $\mathcal{Y}(\psi_4)$

$$\mathcal{Y}(\psi) = \Bigg(e_p(\psi,\psi_2)\Bigg\{\left(\mathcal{Y}_0 e_p(\psi_1,\psi_0) + \int_{\psi_0}^{\psi_1} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi\right)(1+\gamma(\psi_1)p(\psi_1)) + \gamma(\psi_1)\Xi(\psi_1)$$

$$+\int_{\psi_2}^{\psi} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\Bigg\}\left[1+\gamma(\psi_3)p(\psi_3)\right]+\gamma(\psi_3)\Xi(\psi_3)\Bigg)\, e_p(\psi,\psi_4)$$
$$+\int_{\psi_4}^{\psi} e_p(\psi_4,\Gamma(\psi))\Xi(\psi)\,d\psi.$$

After simplifying:

$$\begin{aligned}
e_p(\psi,\psi_4)\Bigg\{&\mathcal{Y}_0 e_p(\psi_1,\psi_0)e_p(\psi_3,\psi_2)(1+\gamma(\psi_1)p(\psi_1))(1+\gamma(\psi_3)p(\psi_3))\\
&+e_p(\psi_3,\psi_2)(1+\gamma(\psi_1)p(\psi_1))(1+\gamma(\psi_3)p(\psi_3))\int_{\psi_0}^{\psi_1} e_p(\psi_0,\Gamma(\psi))\Xi(\psi)\,d\psi\\
&+e_p(\psi_3,\psi_2)(1+\gamma(\psi_3)p(\psi_3))\int_{\psi_2}^{\psi_3} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\\
&+e_p(\psi_3,\psi_2)(1+\gamma(\psi_3)p(\psi_3))\gamma(\psi_1)\Xi(\psi_1)+\gamma(\psi_3)\Xi(\psi_3)\\
&+\int_{\psi_4}^{\psi} e_p(\psi_4,\Gamma(\psi))\Xi(\psi)\,d\psi\Bigg\}
\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Assume that the differential equation $\mathcal{Y}^{\Delta}(\psi)=p(\psi)\mathcal{Y}+\Xi(\psi)$ is regressive, with an initial condition $\mathcal{Y}(\psi_0)=\mathcal{Y}_0$ at some point $\psi_0\in\mathbb{T}$. Here, the time scale $\mathbb{T}$ is defined as $[\psi_0,\psi_1]\cup[\psi_2,\psi_3]\cup[\psi_4,\psi_5]\cup\cdots\cup[\psi_{2n},\psi_{2n+1}]$, where $\psi_{2n+1}=\eta$, representing the maximum value of ψ in $\mathbb{T}$ (with n jumps).*

$$\psi_0 \text{———} \psi_1 \quad \psi_2 \text{———} \psi_3 \quad \psi_4 \text{———} \psi_5 \quad \dots \quad \psi_{2n} \text{———} \psi_{2n+1}$$

The step sizes are defined by $\gamma(\psi_1)=\psi_2-\psi_1$, $\gamma(\psi_3)=\psi_4-\psi_3$, …, $\gamma(\psi_{2n-1})=\psi_{2n}-\psi_{2n-1}$. The solution to the initial value problem $\mathcal{Y}^{\Delta}(\psi)=p(\psi)\mathcal{Y}+\Xi(\psi)$, with $\mathcal{Y}(\psi_0)=\mathcal{Y}_0$ on $\mathbb{T}$, is as follows:

$$\mathcal{Y}(\psi)=\begin{cases}
e_p(\psi,\psi_o)\left[\mathcal{Y}_o+\int_{\psi_o}^{\psi} e_p(\psi_o,\Gamma(\psi))\Xi(\psi)\,d\psi\right], & \psi_o\le\psi\le\psi_1,\\
e_p(\psi,\psi_2)\Big[\mathcal{Y}_o e_p(\psi_1,\psi_o)(1+\gamma(\psi_1)p(\psi_1))+(1+\gamma(\psi_1)p(\psi_1))\int_{\psi_o}^{\psi_1} e_p(\psi_o,\Gamma(\psi))\Xi(\psi)\,d\psi & \\
+\gamma(\psi_1)\Xi(\psi_1)+\int_{\psi_2}^{\psi} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\Big], & \psi_2\le\psi\le\psi_3,\\
\dots\dots\dots & \\
e_p(\psi,\psi_{2n})\Big[\mathcal{Y}_o\prod_{i=1}^{n} e_p(\psi_{2i-1},\psi_{2i-2})(1+\gamma(\psi_{2i-1})p(\psi_{2i-1})) & \\
+\sum_{k=1}^{n}\left(\prod_{m=k,m\neq 1}^{n} e_p(\psi_{2m-1},\psi_{2m-2})\prod_{j=k}^{n}(1+\gamma(\psi_{2j-1})p(\psi_{2j-1}))\right) & \\
\times\int_{\psi_{2k-2}}^{\psi_{2k-1}} e_p(\psi_{2k-2},\Gamma(\psi))\Xi(\psi)\,d\psi & \\
+\sum_{k=1}^{n-1}\left(\prod_{i=k+1}^{n} e_p(\psi_{2i-1},\psi_{2i-2})\prod_{j=k}^{n-1}(1+\gamma(\psi_{2j+1})p(\psi_{2j+1}))\right)\gamma(\psi_{2k-1})\Xi(\psi_{2k-1}) & \\
+\Xi(\psi_{2n-1})\gamma(\psi_{2n-1})+\int_{\psi_{2n}}^{\psi} e_p(\psi_2,\Gamma(\psi))\Xi(\psi)\,d\psi\Big], & \psi_{2n}\le\psi\le\psi_{2n+1}.
\end{cases}$$

Proof. We will prove by mathematical induction. The case at $n=1$ has been verified in Theorem 2.9.

Now, let us suppose the result is true for k jumps. Then for $\psi_0\le\psi\le\psi_1$, we have:

$$\mathcal{Y}(\psi) = e_p(\psi, \psi_0)\left[\mathcal{Y}_0 + \int_{\psi_0}^{\psi} e_p(t_0, \Gamma(\psi))\Xi(\psi)d\psi\right]. \tag{3.1}$$

For $\psi_2 \le \psi \le \psi_3$, the solution becomes:

$$\begin{aligned}\mathcal{Y}(\psi) = e_p(\psi, \psi_2)\Bigg[&\mathcal{Y}_0 e_p(\psi_1, \psi_0)\left(1+\gamma(\psi_1)p(\psi_1)\right) + \left(1+\gamma(\psi_1)p(\psi_1)\right)\int_{\psi_0}^{\psi_1} e_p(\psi_0, \Gamma(\psi))\Xi(\psi)d\psi \\ &+ \gamma(\psi_1)\Xi(\psi_1) + \int_{\psi_2}^{\psi} e_p(\psi_2, \Gamma(\psi))\Xi(\psi)d\psi\Bigg].\end{aligned} \tag{3.2}$$

Continuing this pattern for z jumps, we get:

$$\begin{aligned}\mathcal{Y}(\psi) = e_p(\psi, \psi_{2z})\Bigg[&\mathcal{Y}_0\prod_{i=1}^{z} e_p(\psi_{2i-1}, \psi_{2i-2})\left(1+\gamma(\psi_{2i-1})p(\psi_{2i-1})\right) \\ &+ \sum_{k=1}^{z}\left(\prod_{m=k, m\neq 1}^{z} e_p(\psi_{2m-1}, \psi_{2m-2})\prod_{j=k}^{z}\left(1+\gamma(\psi_{2j-1})p(\psi_{2j-1})\right)\right)\int_{\psi_{2k-2}}^{\psi_{2k-1}} e_p(\psi_{2k-2}, \Gamma(\psi))\Xi(\psi)d\psi \\ &+ \sum_{k=1}^{z-1}\left(\prod_{i=k+1}^{z} e_p(\psi_{2i-1}, \psi_{2i-2})\prod_{j=k}^{z-1}\left(1+\gamma(\psi_{2j+1})p(\psi_{2j+1})\right)\right) \\ &\gamma(\psi_{2k-1})\Xi(\psi_{2k-1}) + \Xi(\psi_{2z-1})\gamma(\psi_{2z-1}) + \int_{\psi_{2z}}^{\psi} e_p(\psi_{2z}, \Gamma(\psi))\Xi(\psi)d\psi\Bigg],\end{aligned} \tag{3.3}$$

where $\psi_{2z} \le \psi \le \psi_{2z+1}$.

Now, we will show for $z+1$ jumps. Let ψ_{2z+1} be the endpoint of our sequence, denoted as η. Without loss of generality, subdivide the last interval $[\psi_{2z}, \psi_{2z+1}]$ and define a new sequence:

$$t_i = \tilde{\psi}_i \quad \text{for } i = 0, 1, \dots, \psi_{2z}.$$

Let $\tilde{\psi}_{2z+1}, \tilde{\psi}_{2(z+1)} \in (\psi_{2z}, \psi_{2z+1})$ and $\eta = \tilde{\psi}_{2(z+1)+1} = \psi_{2z+1}$.

Now, for $\psi \in [\tilde{\psi}_{2z}, \tilde{\psi}_{2z+1}]$, the solution of the first dynamic equation is:

$$\begin{aligned}\mathcal{Y}(\psi) = e_p(\psi, \tilde{\psi}_{2z})\Bigg[&\mathcal{Y}_0\prod_{i=1}^{z} e_p(\psi_{2i-1}, \psi_{2i-2})\left(1+\gamma(\psi_{2i-1})p(\psi_{2i-1})\right) \\ &+ \sum_{k=1}^{z}\left(\prod_{m=k, m\neq 1}^{z} e_p(\psi_{2m-1}, \psi_{2m-2})\prod_{j=k}^{z}\left(1+\gamma(\psi_{2j-1})p(\psi_{2j-1})\right)\right)\int_{\psi_{2k-2}}^{\psi_{2k-1}} e_p(\psi_{2k-2}, \Gamma(\psi))\Xi(\psi)dt \\ &+ \sum_{k=1}^{z-1}\left(\prod_{i=k+1}^{z} e_p(\psi_{2i-1}, \psi_{2i-2})\prod_{j=k}^{z-1}\left(1+\gamma(\psi_{2j+1})p(\psi_{2j+1})\right)\right) \\ &\gamma(\psi_{2k-1})\Xi(\psi_{2k-1}) + \Xi(\psi_{2w-1})\gamma(\psi_{2z-1}) + \int_{\psi_{2z}}^{\psi} e_p(\psi_{2z}, \Gamma(\psi))\Xi(\psi)d\psi\Bigg].\end{aligned} \tag{3.4}$$

Now, for $[\tilde{\psi}_{2(z+1)}, \tilde{\psi}_{2(z+1)+1}]$, the solution is given by:

$$\mathcal{Y}(\psi) = \mathcal{Y}(\tilde{\psi}_{2(k+1)})e_p(\psi, \tilde{\psi}_{2(k+1)}) + \int_{\tilde{\psi}_{2(k+1)}}^{\psi} e_p(\psi, \Gamma(\psi))\Xi(\psi)d\psi, \tag{3.5}$$

where:

$$\begin{aligned}
\mathcal{Y}(\tilde{\psi}_{2(k+1)}) &= \mathcal{Y}(\Gamma(\tilde{\psi}_{2(k+1)})) \\
&= \mathcal{Y}(\tilde{\psi}_{2k+1}) + \gamma(\tilde{\psi}_{2k+1})\mathcal{Y}^{\Delta}(\tilde{\psi}_{2k+1}) \\
&= \mathcal{Y}(\tilde{\psi}_{2k+1}) + \gamma(\tilde{\psi}_{2k+1}) \left[p(\tilde{\psi}_{2k+1})\mathcal{Y}(\tilde{\psi}_{2k+1}) + \Xi(\tilde{\psi}_{2k+1})\right] \\
&= \mathcal{Y}(\tilde{\psi}_{2k+1}) + \gamma(\tilde{\psi}_{2k+1})p(\tilde{\psi}_{2k+1})\mathcal{Y}(\tilde{\psi}_{2k+1}) + \gamma(\tilde{\psi}_{2k+1})\Xi(\tilde{\psi}_{2k+1}) \\
&= \mathcal{Y}(\tilde{\psi}_{2k+1}) \left[1 + \gamma(\tilde{\psi}_{2k+1})p(\tilde{\psi}_{2k+1})\right] + \gamma(\tilde{\psi}_{2k+1})\Xi(\tilde{\psi}_{2k+1})
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}(\tilde{\psi}_{2(k+1)}) = \ & \mathcal{Y}_o \prod_{n+1}^{i=1} \left[e_p(\tilde{\psi}_{2i-1}, \tilde{\psi}_{2i-2}) \left(1 + \gamma(\tilde{\psi}_{2i-1})p(\tilde{\psi}_{2i-1})\right)\right] \\
& + \sum_{k}^{n+1} \left(\prod_{m=k, m\neq 1}^{m=k} e_p(\tilde{\psi}_{2m-1}, \tilde{\psi}_{2m-2}) \prod_{j=k}^{n+1} \left(1 + \gamma(\tilde{\psi}_{2j-1})p(\tilde{\psi}_{2j-1})\right) \int_{\tilde{\psi}_{2k-2}}^{\tilde{\psi}_{2k-1}} e_p(\tilde{\psi}_{2k-2}, \Gamma(\psi))d\psi \right) \\
& + \sum_{k=1}^{n} \left[\prod_{L=k+1}^{n+1} e_p(\tilde{\psi}_{2L-1}, \tilde{\psi}_{2L-2}) \prod_{j=k}^{n} \left(1 + \gamma(\tilde{\psi}_{2j-1})p(\tilde{\psi}_{2j-1})\right) \gamma(\tilde{\psi}_{2k-1})\Xi(\tilde{\psi}_{2k-1}) \right] \\
& + \gamma(\tilde{\psi}_{2k-1})\Xi(\tilde{\psi}_{2k-1}).
\end{aligned}$$

Finally, the solution becomes:

$$\begin{aligned}
\mathcal{Y}(\tilde{\psi}_{2(k+1)}) = \ & e_p(\psi, \tilde{\psi}_{2(k+1)}) \Bigg[\mathcal{Y}_o \prod_{n+1}^{i=1} \left[e_p(\tilde{\psi}_{2i-1}, \tilde{\psi}_{2i-2}) \left(1 + \gamma(\tilde{\psi}_{2i-1})p(\tilde{\psi}_{2i-1})\right)\right] \\
& + \sum_{k}^{n+1} \left(\prod_{m=k, m\neq 1}^{m=k} e_p(\tilde{\psi}_{2m-1}, \tilde{\psi}_{2m-2}) \prod_{j=k}^{n+1} \left(1 + \gamma(\tilde{\psi}_{2j-1})p(\tilde{\psi}_{2j-1})\right) \int_{\tilde{\psi}_{2k-2}}^{\tilde{\psi}_{2k-1}} e_p(\tilde{\psi}_{2k-2}, \Gamma(\psi))d\psi \right) \\
& + \sum_{k=1}^{n} \left[\prod_{L=k+1}^{n+1} e_p(\tilde{\psi}_{2L-1}, \tilde{\psi}_{2L-2}) \prod_{j=k}^{n} \left(1 + \gamma(\tilde{\psi}_{2j-1})p(\tilde{\psi}_{2j-1})\right) \gamma(\tilde{\psi}_{2k-1})\Xi(\tilde{\psi}_{2k-1}) \right] \\
& + \gamma(\tilde{\psi}_{2n-1})\Xi(\tilde{\psi}_{2n-1}) \Bigg] + \int_{\tilde{\psi}_{2(k+1)}}^{\psi} e_p(\tilde{\psi}_{2(k+1)}, \Gamma(\psi))\Xi(\psi)d\psi.
\end{aligned}$$

□

4. Examples

Example 4.1. *Let the initial value problem specified for the given time scale:*

$$\mathcal{Y}^{\Delta} = 2\mathcal{Y} + \psi^2, \quad \mathcal{Y}(0) = 1, \quad \mathbb{T} = [0, 1] \cup [3, 4].$$

4.1. **Solution.** For $0 \leq \psi \leq 1$:

$$\mathcal{Y}(\psi) = e_2(\psi, 0)\left[\mathcal{Y}(0) + \int_0^{\psi} \psi^2 \cdot e_2(0, \Gamma(\psi))\, d\psi\right].$$

Now, substituting the given initial condition $\mathcal{Y}(0) = 1$ and solving step by step:

$$\mathcal{Y}(\psi) = e^{2\psi}\left[1 + \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}\psi^2 e^{-2\psi} - \frac{1}{2}\psi e^{-2\psi}\right].$$

Simplifying the expression:

$$\mathcal{Y}(\psi) = \frac{5}{4}e^{2\psi} - \frac{1}{4} - \frac{1}{2}\psi - \frac{1}{2}\psi^2.$$

Thus, the solution to the initial value problem on the given time scale for $0 \leq \psi \leq 1$ is:

$$\mathcal{Y}(\psi) = \frac{5}{4}e^{2\psi} - \frac{1}{4} - \frac{1}{2}\psi - \frac{1}{2}\psi^2.$$

For $3 \leq \psi \leq 4$:
Given:

$$p(\psi) = 2, \quad \Xi(\psi) = \psi^2, \quad \gamma(\psi_1) = 2, \quad \psi_0 = 1, \quad \psi_1 = 1, \quad \psi_2 = 3.$$

The solution for $3 \leq \psi \leq 4$ is:

$$\mathcal{Y}(\psi) = e_2(\psi, 3)\left[\left(e_2(1, 0) + \int_0^1 \psi^2 \cdot e_2(0, \Gamma(\psi))\, d\psi\right) \cdot (1 + 2 \cdot 2) + 2 \cdot 1 + \int_3^{\psi} \psi^2 \cdot e_2(3, \Gamma(\psi))\, d\psi\right].$$

Substituting and simplifying:

$$\mathcal{Y}(\psi) = e^{\psi - 3}\left[\left(5e^2 - \frac{5}{4}e^{-2} + \frac{1}{4}\right) + 2 + \frac{25}{4} - \frac{1}{4}e^{-2t+6} - \frac{1}{2}\psi^2 e^{-2t+6} - \frac{1}{2}\psi e^{-2t+6}\right].$$

$$\mathcal{Y}(\psi) = 5e^{2\psi-4} - \frac{25}{4}e^{2\psi-8} + \frac{5}{4}e^{2\psi-6} + \frac{33}{4}e^{2t-6} - \frac{1}{4} - \frac{1}{2}\psi^2 - \frac{1}{2}\psi.$$

Thus, the solution for $3 \leq \psi \leq 4$ is:

$$\mathcal{Y}(\psi) = 5e^{2\psi-4} - \frac{25}{4}e^{2\psi-8} + \frac{5}{4}e^{2\psi-6} + \frac{33}{4}e^{2\psi-6} - \frac{1}{4} - \frac{1}{2}\psi^2 - \frac{1}{2}\psi.$$

Table 1 presents a comparison between the exact solution ($\mathcal{Y}_e$) and the numerical solution ($\mathcal{Y}_1$) for a range of time values (ψ). The absolute error, representing the difference between the exact and numerical values, is also provided to quantify the accuracy of the numerical. The data highlights how the numerical solution closely follows the exact solution, with the absolute error remaining minimal across the listed time points. This comparison underscores the reliability of the numerical method, even as the values increase significantly at higher time intervals.

The top plot illustrates the comparison between the exact solution ($\mathcal{Y}_e$, dashed red line) and the numerical solution ($\mathcal{Y}_1$, solid blue line with markers) over time (ψ). The close overlap between the two curves demonstrates the accuracy of the numerical. The bottom plot displays the absolute error between $\mathcal{Y}_e$ and $\mathcal{Y}_1$, showing that the error remains minimal across the time range, with a slight increase at higher values of ψ.

Example 4.2.

$$\mathcal{Y}^{\Delta} = 2\mathcal{Y} + \sin_2(2\psi, \psi), \quad \mathcal{Y}(0) = 1, \quad \mathbb{T} = [0, 0.5] \cup [2, 3] \cup [4, 4.5].$$

TABLE 1. Comparison of values for $\mathcal{Y}_e$ and $\mathcal{Y}_1$ over time ψ.

ψ	$\mathcal{Y}_e$	$\mathcal{Y}_1$	**Absolute Error**
0.19507	1.47992332963609	1.47991731882619	6.01081E-06
0.25397	1.66809490214266	1.66808600535816	8.89678E-06
0.55698	3.12439448593903	3.12435718307892	3.73029E-05
3.09372	48.4177059973905	48.4176034220305	1.05475E-04
3.26469	70.2108912024092	70.2104830952691	4.08107E-04
3.62945	151.92931349112	151.927298286734	2.01520E-03
3.81158	221.72747498596	221.723733347833	3.74164E-03
3.82673	228.772990946834	228.769062088803	3.92886E-03
3.91494	274.356752363013	274.351564538498	5.18782E-03
3.92969	282.801137105241	282.795707680298	5.42942E-03

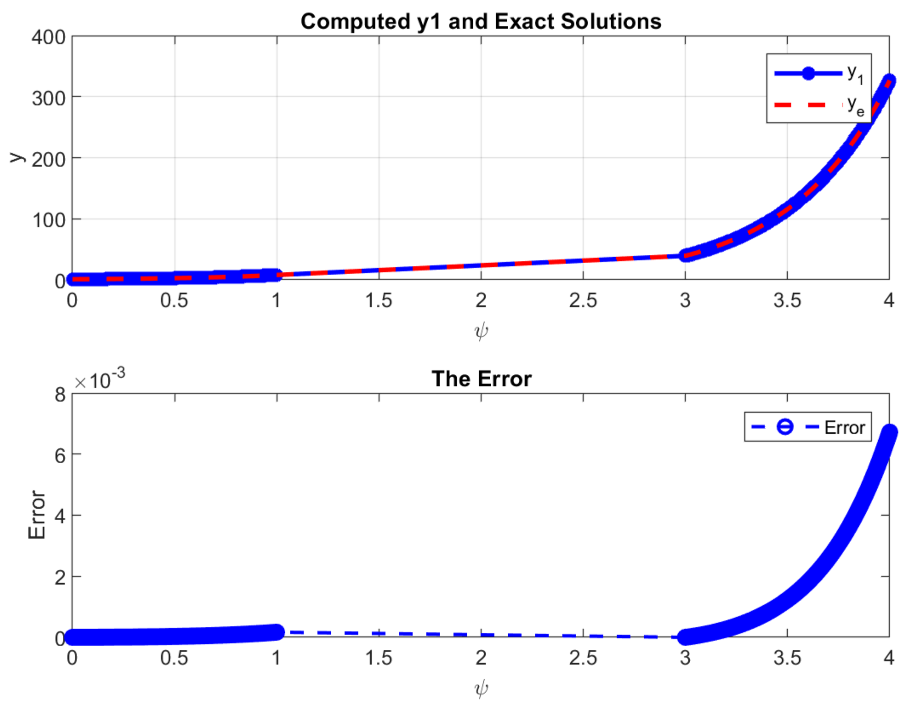

FIGURE 1. The first image displays both the exact and numerical solutions, while the second image illustrates the error by subtracting the exact form the numerical solution.

4.2. **Solution.** For $0 \leq \psi \leq 0.5$:

$$
\begin{aligned}
\mathcal{Y}(\psi) &= e_2(\psi, t_0)\left[1 + \int_0^{\psi} \sin_2(2\psi, \psi) \cdot e_2(0, \Gamma(\psi))\, d\psi\right] \\
&= e^{2\psi}\left[1 + \int_0^{\psi} \sin_2(2\psi, \psi) \cdot e^{-2\psi}\, d\psi\right]
\end{aligned}
$$

$$= e^{2\psi}\left[1+\int_0^{\psi}\sin(2\psi)\cdot e^{-2\psi}\,d\psi\right] = e^{2\psi}\left[\frac{5}{4}-\frac{1}{4}e^{-2\psi}\sin(2\psi)-\frac{1}{4}e^{-2\psi}\cos(2\psi)\right].$$

For $2 \le \psi \le 3$:

$$\begin{aligned}
\mathcal{Y}(\psi) &= e_2(\psi,2)\left[\left(e_2(0.5,0)+\int_0^{0.5} e_2(0,\Gamma(\psi))\sin_2(2t,\psi)\,d\psi\right)\right.\\
&\quad \left.\times\left(1+\frac{3}{2}(2)\right)+\frac{3}{2}\sin_2(1,\frac{1}{2})+\int_2^{\psi}\sin_2(2\psi,\psi)\cdot e_2(2,\Gamma(\psi))\,d\psi\right]\\
&= e^{2(\psi-2)}\left[4\left(e^1+\frac{1}{4}-\frac{1}{4}e^{-1}\sin(1)-\frac{1}{4}e^{-1}\cos(1)\right)\right.\\
&\quad \left.+\frac{3}{2}\sin(1)-\frac{1}{4}e^{4-2t}\sin(2\psi)-\frac{1}{4}e^{4-2\psi}\cos(2\psi)+\frac{1}{4}\sin(4)+\frac{1}{4}\cos(4)\right].
\end{aligned}$$

For $4 \le \psi \le 4.5$:

$$\begin{aligned}
\mathcal{Y}(\psi) &= e_2(\psi,4)\left[e_2\left(\frac{1}{2},0\right)e_2(3,2)\left(1+(\frac{3}{2})(2)\right)(1+2)\right.\\
&\quad + e_2(3,2)\left(1+(\frac{3}{2})(2)\right)(1+2)\int_0^{\frac{1}{2}} e_2(0,\psi)\sin_2(2\psi,\psi)\,d\psi\\
&\quad + e_2(3,2)\left(\frac{3}{2}\sin_2\left(1,\frac{1}{2}\right)\right)(1+2)+\sin_2(6,3)\\
&\quad \left.+ e_2(3,2)\,(1+2)\int_2^{3} e_2(2,\psi)\sin_2(2\psi,\psi)\,d\psi+\int_4^{\psi} e_2(4,\psi)\sin_2(2\psi,\psi)\,d\psi\right]\\
&= e^{2(\psi-4)}\left[e^3\left(1+\frac{3}{2}(2)\right)(1+2)+e^2\left(1+\frac{3}{2}(2)\right)(1+2)\left(\frac{1}{4}-\frac{1}{4}e^{-1}\sin(1)-\frac{1}{4}e^{-1}\cos(1)\right)\right.\\
&\quad + e^2\left(\frac{3}{2}\sin(1)\right)(1+2)+\sin(6)+e^2\,(1+2)\left(\frac{1}{4}\sin(4)+\frac{1}{4}\cos(4)-\frac{1}{4}e^{-2}\sin(6)\right.\\
&\quad \left.\left.-\frac{1}{4}e^{-2}\cos(6)\right)\quad -\frac{1}{4}e^{8-2\psi}\sin(2\psi)-\frac{1}{4}e^{8-2\psi}\cos(2\psi)+\frac{1}{4}\sin(8)+\frac{1}{4}\cos(8)\right].
\end{aligned}$$

Table 2 presents a comparison between the exact solution ($\mathcal{Y}_e$) and the numerical solution ($\mathcal{Y}_1$) for a range of time values (ψ). The absolute error, representing the difference between the exact and numerical values, is also provided to quantify the accuracy of the numerical. The data highlights how the numerical solution closely follows the exact solution, with the absolute error remaining minimal across the listed time points. This comparison underscores the reliability of the numerical method, even as the values increase significantly at higher time intervals.

The top plot illustrates the comparison between the exact solution ($\mathcal{Y}_e$, dashed red line) and the numerical solution ($\mathcal{Y}_1$, solid blue line with markers) over time (ψ). The close overlap between the two curves demonstrates the accuracy of the numerical. The bottom plot displays the absolute error between $\mathcal{Y}_e$ and $\mathcal{Y}_1$, showing that the error remains minimal across the time range, with a slight increase at higher values of ψ.

Example 4.3.

$$\mathcal{Y}^{\Delta} = 3\mathcal{Y} + e_3(2\psi,\psi), \quad \mathcal{Y}(0)=1, \quad \mathbb{T}=[0,1]\cup[3,3.75]\cup[4,4.25]$$

4.3. **Solution.** For $0 \le \psi \le 1$:

$$\mathcal{Y}(\psi) = e_3(\psi,\psi_0)\left[1+\int_0^{\psi} e_3(2\psi,\psi)\cdot e_3(0,\Gamma(\psi))\,d\psi\right]$$

TABLE 2. Comparison of values for $\mathcal{Y}_e$ and $\mathcal{Y}_1$ over time ψ.

ψ	$\mathcal{Y}_e$	$\mathcal{Y}_1$	**Absolute Error**
0.23799	1.67522507045728	1.67521369210418	1.13784E-05
0.32522	2.0454122996647	2.04512291739079	2.89482E-04
0.44761	2.70841937028611	2.70838742220548	3.19481E-05
2.01015	12.8781281813851	12.8781257108709	2.47051E-06
2.05204	13.9687893534093	13.9687755617609	1.37916E-05
2.18074	17.9400135644073	17.9399513613781	6.22030E-05
2.4967	33.3166447789803	33.3163206207931	3.24158E-04
2.51185	34.3268766264641	34.3265321951899	3.44431E-04
2.67049	46.977627602105	46.9770061112502	6.21491E-04
2.81019	62.0039717889716	62.0029766409122	9.95148E-04
2.8594	68.3870034094058	68.3858378940835	1.16552E-03
2.89807	73.8657698282552	73.8645331353551	1.23669E-03
4.00934	276.423428795103	276.423377082211	5.17129E-05
4.28177	476.983216821561	476.980527733125	2.68909E-03
4.3584	556.043563763531	556.039577152425	3.98661E-03
4.41939	628.219073249459	628.213803311938	5.26994E-03

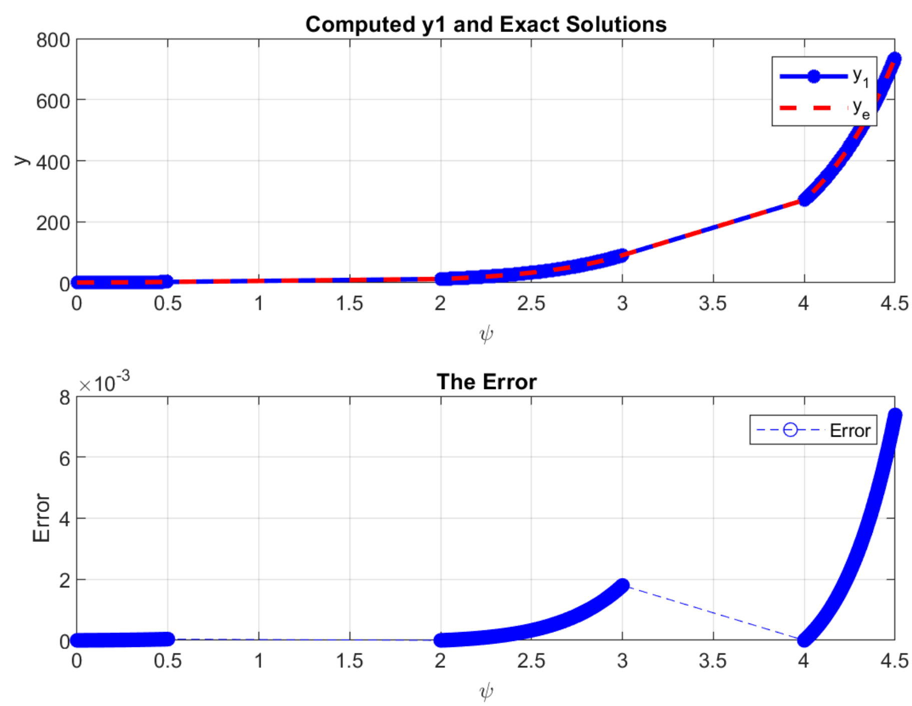

FIGURE 2. The first image displays both the exact and numerical solutions, while the second image illustrates the error by subtracting the exact form the numerical solution.

$$= e^{3\psi}\left[1 + \psi\right]$$

For $3 \leq \psi \leq 3.75$:

$$\begin{aligned}
\mathcal{Y}(\psi) &= e_3(\psi, 3)\left[\left((1)e_3(1,0) + \int_0^1 e_3(0, \Gamma(\psi))e_3(2\psi, \psi)\, d\psi\right)\right. \\
&= e^{3(\psi-3)}\left[7\left(e^3 + \frac{1}{6}(e^6 - 1)\right) + 2e^3 + te^9 - 3e^9\right] \\
&\left.\times\ (1 + (2)(3)) + 2e^3 + \int_3^{\psi} e_3(2\psi, \psi) \cdot e_3(3, \Gamma(\psi))\, d\psi\right] \\
&= e^{3(\psi-3)}\left[9e^3 - 3e^9 + te^9 + 7\right]
\end{aligned}$$

For $4 \leq \psi \leq 4.25$:

$$\begin{aligned}
\mathcal{Y}(\psi) &= e_3(\psi, 4)\Bigg[e_3\,(1,0)\,e_3(3.75, 3)\,(1 + (2)(3))\left(1 + \frac{1}{4}(3)\right) \\
&+ e_3(3.75, 3)\,(1 + (2)(3))\left(1 + \frac{1}{4}(3)\right)\int_0^1 e_3(0, \Gamma(\psi))e_3(2t, \psi)\, d\psi \\
&+ e_3(3.75, 3)\left(1 + \frac{1}{4}(3)\right)\int_3^{3.75} e_3(3, \Gamma(\psi))e_3(2\psi, \psi)\, d\psi \\
&+ e_3(3.75, 3)\left(1 + \frac{1}{4}(3)\right)(2)e_3(2, 1) \\
&+ \frac{1}{4}e_3(7.5, 3.75) + \int_4^{\psi} e_3(4, \Gamma(\psi))e_3(2\psi, \psi)\, d\psi\Bigg] \\
&= e^{3(\psi-4)}\Bigg[e^3 e^{\frac{9}{4}}\,(1+6)\left(1 + \frac{3}{4}\right) \\
&+ 7\left(\frac{7}{4}\right)e^{\frac{9}{4}}\int_0^1 e^{-3\psi}e^{3\psi}\, d\psi + \frac{7}{4}e^{\frac{9}{4}}\int_3^{3.75} e^{3(3-\psi)}e^{3\psi}\, d\psi + 2\left(\frac{7}{4}\right)e^{\frac{9}{4}}e^3 + \frac{1}{4}e^{\frac{45}{4}} + \int_4^{\psi} e^{3(4-\psi)}e^{3\psi}\, d\psi\Bigg] \\
&= e^{3(\psi-4)}\left[\frac{63}{4}e^{\frac{21}{4}} + \frac{49}{4}e^{\frac{9}{4}} + \frac{25}{16}e^{\frac{45}{4}} + e^{12}(\psi - 4)\right]
\end{aligned}$$

Table 3 presents a comparison between the exact solution ($\mathcal{Y}_e$) and the numerical solution ($\mathcal{Y}_1$) for a range of time values (ψ). The absolute error, representing the difference between the exact and numerical values, is also provided to quantify the accuracy of the numerical. The data highlights how the numerical solution closely follows the exact solution, with the absolute error remaining minimal across the listed time points. This comparison underscores the reliability of the numerical method, even as the values increase significantly at higher time intervals.

The top plot illustrates the comparison between the exact solution ($\mathcal{Y}_e$, dashed red line) and the numerical solution ($\mathcal{Y}_1$, solid blue line with markers) over time (ψ). The close overlap between the two curves demonstrates the accuracy of the numerical. The bottom plot displays the absolute error between $\mathcal{Y}_e$ and $\mathcal{Y}_1$, showing that the error remains minimal across the time range, with a slight increase at higher values of ψ.

5. Discussion

Our investigation into non-homogeneous dynamic equations across various time scales yielded analytical solutions that aligned well with theoretical expectations. By integrating both continuous and discrete components, the derived general solutions effectively captured the hybrid nature of these problems. Numerical solutions confirmed our analytical findings, and error analysis revealed minimal discrepancies. Graphical comparisons between the exact and numerical

Table 3. Comparison of values for $\mathcal{Y}_e$ and $\mathcal{Y}_1$ over time ψ.

ψ	$\mathcal{Y}_e$	$\mathcal{Y}_1$	**Absolute Error**
0.0660666	1.2997547597689	1.29975475360797	6.16093E-09
0.26248973	2.77473414076989	2.77473409409384	4.66760E-08
0.31669535	3.40488486260814	3.40488479534993	6.72582E-08
0.72561995	15.2175022192617	15.21750163487	5.84492E-07
0.78025838	18.4956145211699	18.4956137708806	7.50289E-07
0.87748868	26.1121602383988	26.1121590821307	1.15627E-06
3.21263402	3616.12758894372	3616.12747211987	1.16824E-04
3.21312027	3628.8740605589	3628.87394326117	1.17298E-04
3.25565998	4865.06973290888	4865.06956877513	1.64134E-04
3.57088917	26686.0330080659	26686.031882556	1.12551E-03
3.64691563	37812.7285228827	37812.7268583064	1.66458E-03
3.72788745	54035.2953986739	54035.292915365	2.48331E-03
4.05063094	153050.812873901	153050.812248368	6.25533E-04
4.08312829	175510.787711257	175510.786566369	1.14489E-03
4.15044403	231992.459235629	231992.456641694	2.59394E-03

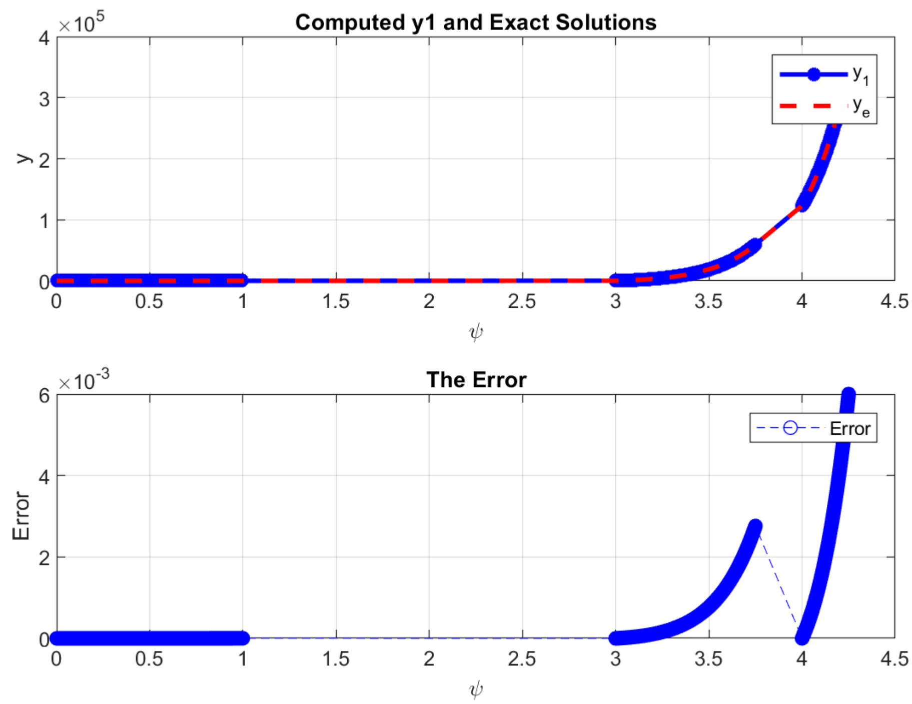

Figure 3. The first image displays both the exact and numerical solutions, while the second image illustrates the error by subtracting the exact form the numerical solution.

solutions further validated the accuracy of the numerical methods employed. The use of piecewise-defined time scales allowed us to address systems with abrupt transitions, demonstrating the flexibility of our proposed methods in dealing with complex dynamics. Our work builds on the foundational theories of time scale calculus introduced by Hilger in 1988, extending these concepts to more complex scenarios involving non-homogeneous equations with initial conditions.

Unlike previous studies that often concentrated on either continuous or discrete systems, this research illustrates the applicability of time scale theory to unified hybrid models. This represents a significant advancement in solving equations that bridge discrete and continuous domains. The methods developed in this research are particularly well-suited for modeling hybrid systems, such as those encountered in biology, economics, and engineering, where discrete events interact with continuous processes. The comparison of exact and numerical solutions provides valuable insights into the reliability of numerical methods when applied to dynamic equations on time scales. Error Reduction: The error analysis conducted in this study offers practical insights into the limitations of numerical approximations, contributing to the development of more precise computational tools. The piecewise-defined approach to time scales ensures that solutions remain applicable across diverse and complex temporal structures. The integration of analytical and numerical methods creates a comprehensive framework for addressing non-homogeneous equations, making the approach both versatile and robust. This study primarily focused on relatively simple time scale structures, while future investigations could explore more intricate and irregular time scales. Although the numerical methods employed were accurate, they may require optimization to effectively handle larger-scale systems or those with high-frequency transitions. This research underscores the potential of time scale calculus as a unifying framework for addressing hybrid dynamic systems, providing a solid foundation for future studies in this domain. Also we calculate the absolute error between exact $(\mathcal{Y}_e)$ and numerical $(\mathcal{Y}_1)$ solutions for non-homogeneous dynamic equations on time scales formula is $|\mathcal{Y}_e - \mathcal{Y}_1|$, as seen in Tables 1–3. In continuous intervals (e.g., $[0,1]$ in Example 4.1), errors are small due to smooth dynamics, ranging from 10^{-9} to 10^{-5}. At discrete jumps (e.g., $\gamma(t_1) = 2$ in Example 4.1), errors grow, reaching 10^{-4} to 10^{-3} in later intervals like $[3,4]$. Complex time scales with multiple jumps (Examples 4.2, 4.3) show larger errors in later segments due to error accumulation and exponential solution growth driven by $p(\psi)$. The forcing function's nature (e.g., polynomial in 4.1, trignometric in 4.2, exponential in 4.3) also affects errors, with rapid changes amplifying discrepancies.

6. Conclusion

In this paper, we explore the comprehensive solution to the non-homogeneous dynamic equation, a critical topic in the field of applied mathematics. Initially, we formulate and solve the non-homogeneous dynamic equation while incorporating specific initial conditions. Our study encompasses both discrete and continuous domains, allowing for a robust understanding of the equation's behavior in various contexts. After establishing the foundational theorem for our solution, we demonstrate its application through a series of illustrative examples. Each example is approached in two distinct ways: first, by directly applying our established theorem to derive the solution, and second, by utilizing numerical methods to obtain approximate solutions. To enhance our analysis, we graphically represent both the exact and approximate solutions derived from each approach. This visual comparison allows us to meticulously examine the discrepancies between the two methods, as we analyze the error associated with the numerical approximations. By doing so, we gain valuable insights into the effectiveness and accuracy of our theorem in solving non-homogeneous dynamic equations across different scenarios.

7. Limitations and Future Work

This study successfully derives solutions for non-homogeneous dynamic equations on time scales, but it is limited to relatively simple time scale structures, such as unions of disjoint intervals. The numerical Euler method, while effective, exhibits increasing errors in later intervals due to its first-order accuracy, particularly for complex time scales with large jumps. Additionally, the analysis assumes constant or simple $p(\psi)$ and $\Xi(\psi)$, which may not fully capture real-world hybrid systems. Future research could explore irregular time scales, such as fractal-like structures, to model more intricate dynamics. Developing higher-order numerical methods or adaptive step-size techniques could reduce errors, especially for rapidly growing solutions. Finally, deriving analytical error bounds for time scale methods would enhance their reliability across diverse applications.

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