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New numerical method via RBF approach to the price of fixed-rate mortgages

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Abstract

In this paper, we introduce a novel fixed-rate mortgage (FRM) pricing model that overcomes the limitations of existing approaches. Unlike traditional models, which rely on deterministic interest rate volatility and thus produce inaccurate valuations in volatile markets, our model incorporates stochastic volatility to more accurately reflect the dynamic nature of interest rate risk. This yields a pricing formula derived from a stochastic volatility framework, providing a strong theoretical basis for understanding volatility's impact on FRM prices. We use the efficient and accurate Radial Basis Function (RBF) method to solve the resulting partial differential equation (PDE), effectively handling complex boundary conditions. Our numerical experiments demonstrate the model's practical application and illustrate how FRM prices react to varying volatility across different market conditions. Our findings underscore the critical need for stochastic volatility in FRM valuation and offer valuable insights for improved hedging strategies, ultimately contributing to more realistic and accurate mortgage pricing and enhanced risk management for financial institutions and investors.

Keywords. Fixed-rate mortgages, Stochastic volatility, Stochastic interest rate, Heston model, Radial Basis Function.
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1. Introduction

This paper addresses the pricing of fixed-rate mortgages (FRMs). A mortgage is a financial contract in which a borrower receives funds, typically from a bank or financial institution, using a risky asset, such as a house, as collateral. FRMs are characterized by fixed interest rates and predetermined payment schedules throughout the loan's term. This paper develops a pricing model for FRMs that considers the dependence of interest rates and house prices on the contract amount. Specifically, we incorporate stochastic volatility to model the fluctuations of house prices and interest rates, which are crucial underlying factors for accurate FRM valuation.

The Black-Scholes model, while widely used for option pricing due to its simplicity, assumes constant volatility [2, 6]. This assumption, however, often fails to accurately reflect real-world market conditions, where volatility exhibits significant fluctuations. Consequently, stochastic volatility models, which treat volatility as a stochastic process, have emerged as valuable alternatives. Pioneering work in this area includes the models developed by Hull and White [10] and Heston [8]. These models offer a more realistic representation of market dynamics and improved accuracy in option pricing. Among stochastic volatility models, the Heston model [8] is widely favored due to its closed-form solution. Several models have also been proposed in the literature for stochastic interest rates, including those by Cox, Ingersoll, Ross, and Vasicek [19]. This paper focuses on the well-established Heston model for stochastic volatility and the Vasicek model for stochastic interest rates [8, 19].

The numerical solution of partial differential equations (PDEs) is crucial due to its relevance in various fields of applied analysis [4]. For instance, Azevedo-Pereira et al. [1] utilized an explicit finite-difference scheme to address this

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challenge. Additionally, Safaei et al. [16] introduced a new splitting scheme for solving three-dimensional PDEs. In this context, we employ Radial Basis Functions (RBFs), a well-known meshless technique, to compute the necessary contract values related to the PDE.

In 1990, Euan Kansa introduced a novel method for solving partial differential equations (PDEs) using radial basis functions (RBFs) [13]. This innovative approach represented a significant advancement in numerical methods, leveraging the flexibility and effectiveness of RBFs to tackle complex PDEs that arise in various scientific and engineering disciplines. Kansa's method has since been applied to a wide range of engineering problems, demonstrating its versatility and practical utility in real-world applications.

Given the numerous advantages of RBFs—such as their ability to handle irregularly spaced data and their spectral convergence properties—the Kansa collocation method has gained popularity, particularly in the financial sector. For instance, it has been employed for solving various option pricing problems, as discussed in the prestigious Wilmott magazine [15]. This highlights the growing interest in RBFs as a tool for addressing financial modeling challenges.

Building on this foundation, Hon and Mao extended the application of RBFs to option pricing in 1999 [9]. Their research provided concrete examples that showcased the effectiveness of RBFs in this domain, further validating Kansa's initial findings. Additionally, Franke's review paper [5] offers a critical examination of Hardy's multi-quadratic (MQ) interpolation method, evaluating it against established criteria such as accuracy, computational speed, and ease of implementation. This review is particularly relevant as it establishes a benchmark for the performance of RBF methods.

In this study, we aim to utilize the RBF form based on the MQ method in our numerical approach, which allows for improved accuracy and efficiency when solving PDEs. Specifically, our focus will be on demonstrating the application of RBFs for pricing financial contracts. We emphasize the capacity of this method to solve the PDE without reliance on the locations of the approximation nodes, thereby enhancing its applicability in scenarios where node placement may be challenging.

To address these PDEs, the solution is initiated from an established initial condition, ensuring a robust framework for the numerical method. By applying the RBF approximation, we derive a system of linear constant-coefficient ordinary differential equations (ODEs), which can be efficiently solved using established numerical techniques. In this context, we will employ the fourth-order Runge-Kutta Method (RK4) as a preferred numerical technique for option pricing, owing to its high order of accuracy and stability.

The organization of this paper is as follows: In section 2, we will describe the stochastic variables involved in option pricing and derive the pricing model for estimating the financial risk management (FRM) associated with various contracts. Section 3 will delve deeper into the RBF method, elaborating on its theoretical foundation and explaining how it can be effectively utilized to tackle the underlying PDEs governing option pricing. Following that, we will concentrate on the solution of the associated ordinary differential equations in section 4. To validate our proposed approach, we will present a numerical example demonstrating the applicability and effectiveness of the method. Finally, we will conclude with a summary of our findings and remarks on potential future work.

2. Fixed-Rate mortgage pricing

To estimate the pricing of the FRM model, we derive the pricing framework in which the underlying stochastic factors are the house price H_t and the interest rate r_t. We now consider the following stochastic differential equation (SDE), which describes the evolution of the house price H_t,

$$dH_t = (\mu - \delta)H_t dt + \sigma_H H_t dZ_t^H.$$

The instantaneous rate and the dividend yield are denoted as μ and δ, respectively. Moreover, Z_t^H represents a standardized Wiener process for the house price, and σ_H, the volatility of the underlying asset, is assumed to satisfy $v(t) = \sigma_H^2$. This indicates that the volatility σ_H is the square root of the variance v. The focus of this section is on continuous-time stochastic volatility models, particularly the derivation of a partial differential equation (PDE) from the Heston stochastic volatility model [8]. In what follows, we will present the equation that denotes the stochastic process of the variance as follows:

$$dv_t = \alpha(H_t, v_t, t)dt + \xi\beta(H_t, v_t, t)v^\gamma dZ_t^v, \tag{2.1}$$

where ξ is defined as the volatility and α and β will be specified next. Our focus herein is on incorporating another source of uncertainty into stochastic volatility models. To this end, it is assumed that the interest rate r_t at time t follows the Vasicek process [19]. So, we have

$$dr_t = a(b - r_t)dt + \sigma_r dZ_t^r,$$

that the parameters a, b, σ_r, and Z_t^r are the speed of reversion, long-term means level, the volatility of interest rate, and a standardized Wiener process, respectively. The correlation between three Wiener processes Z_t^H, Z_t^r, and Z_t^v is indicated by $dZ_t^H dZ_t^r = dZ_t^H$, $dZ_t^v = dZ_t^v$, $dZ_t^r = \rho dt$ where ρ is the correlation coefficient to incorporate possible correlation between interest rate and house price and volatility, respectively. The valuation of any asset $F = F(t, H_t, v_t, r_t)$ whose value is a function only of house price H_t, volatility v_t, interest rate r_t, and time t, is usually identified by a partial differential equation and is a stochastic process. Henceforth, the dependence on t is suppressed to simplify notation. The PDE is derived by applying standard techniques based on Ito formulas [11] for our stochastic volatility model. For this case, let F is a twice differentiable function H, v, r and differentiable in t. Therefore, we get

$$\begin{aligned} dF = {} & \partial_t F dt + \partial_H F dH + \partial_v F dv + \partial_r F dr + \frac{1}{2}\partial_{HH} F (dH)^2 + \frac{1}{2}\partial_{vv} F (dv)^2 \\ & + \frac{1}{2}\partial_{rr} F (dr)^2 + \partial_{Hv} F (dH dv) + \partial_{Hr} F (dH dr) + \partial_{vr} F (dv dr), \end{aligned} \tag{2.2}$$

where partial derivatives are specified by sub-indices in the symbol ∂. In Eq. (2.2), the values of $\partial_{Hv} F(dHdv)$, $\partial_{Hr} F(dHdr)$, and $\partial_{vr} F(dvdr)$ are obtained by computing the covariance of two stochastic processes. By substituting the obtained values, Eq. (2.2) simplifies to

$$\begin{aligned} dF = {} & (F_t + \frac{1}{2} v H^2 F_{HH} + \frac{1}{2}\xi^2 \beta^2 v^{2\gamma} F_{vv} + \frac{1}{2} v F_{rr} + v^{\gamma + \frac{1}{2}} H \xi \beta \rho_{Hv} F_{Hv} \\ & + v H \rho_{Hr} F_{Hr} + v^{\gamma + \frac{1}{2}} \xi \beta \rho_{vr} F_{vr}) dt + F_H dH + F_v dv + F_r dr. \end{aligned}$$

At this point, suppose Π be the portfolio containing any asset's valuation $F = F(t, H, v, r)$ and the quantity $-\Delta$ of the underlying asset house and $-\Delta^1$ and $-\Delta^2$ of the asset $F^1 = F^1(t, H, v, r)$ and $F^2 = F^2(t, H, v, r)$, respectively. Afterward, the following equality indicates the value of such a portfolio

$$\Pi = F - \Delta H - \Delta^1 F^1 - \Delta^2 F^2. \tag{2.3}$$

Note that the proper variation in the value of this portfolio yields

$$\begin{aligned} d\Pi = {} & (F_t + \frac{1}{2} v H^2 F_{HH} + \frac{1}{2}\xi^2 \beta^2 v^{2\gamma} F_{vv} + \frac{1}{2} v F_{rr} + v^{\frac{1}{2} + \gamma} H \xi \beta \rho_{Hv} F_{Hv} + v H \rho_{Hr} F_{Hr} \\ & + v^{\frac{1}{2}+\gamma} \xi \beta \rho_{vr} F_{vr}) dt - \Delta^1 [(F_t^1 + \frac{1}{2} v H^2 F_{HH}^1 + \frac{1}{2}\xi^2 \beta^2 v^{2\gamma} F_{vv}^1 + \frac{1}{2} v F_{rr}^1 + v^{\frac{1}{2}+\gamma} H \xi \beta \rho_{Hv} F_{Hv}^1 \\ & + v H \rho_{Hr} F_{Hr}^1 + v^{\frac{1}{2}+\gamma} \xi \beta \rho_{vr} F_{vr}^1) dt] - \Delta^2 [((F_t^2 + \frac{1}{2} v H^2 F_{HH}^2 + \frac{1}{2}\xi^2 \beta^2 v^{2\gamma} F_{vv}^2 + \frac{1}{2} v F_{rr}^2 \\ & + v^{\frac{1}{2}+\gamma} H \xi \beta \rho_{Hv} F_{Hv}^2 + v H \rho_{Hr} F_{Hr}^2 + v^{\frac{1}{2}+\gamma} \xi \beta \rho_{vr} F_{vr}^2) dt] + (F_H - \Delta - \Delta^1 F_H^1 - \Delta^2 F_H^2) dH \\ & + (F_v - \Delta^1 F_v^1 - \Delta^2 F_v^2) dv + (F_r - \Delta^1 F_r^1 - \Delta^2 F_r^2) dr. \end{aligned} \tag{2.4}$$

Since the portfolio is risk-free, we assign a coefficient of zero to each risky term. Consequently, the PDE in Eq. (2.4) becomes risk-free by selecting $\Delta = F_H - (\frac{F_v}{F_v^1}) F_H^1$, $\Delta^1 = \frac{F_v}{F_v^1}$, and $\Delta^2 = \frac{F_r}{F_r^1}$. To ensure that the portfolio in Equation (2.3) is risk-free, its return must match the return from the risk-free rate r. Therefore, the change in the risk-free portfolio is given by

$$d\Pi = r \Pi dt = r(F - \Delta H - \Delta^1 F^1 - \Delta^2 F^2) dt.$$

Now we substitute our appropriate choices of Δ and Δ^1. The achieved quantity can only be valid if both sides are equal to a function of the form $-(\alpha - \Phi\beta)$ for some function Φ, where $\beta \neq 0$ [18].

The function Φ , which depends on the variables H, v, r, t, represents the expected return of F explained by the asset pricing risk. Reordering the terms in Eq. (2.4) gives the following equation:

$$\begin{aligned} F_t &+ \frac{1}{2}vH^2F_{HH} + \frac{1}{2}\xi^2\beta^2v^{2\gamma}F_{vv} + \frac{1}{2}vF_{rr} + v^{\frac{1}{2}+\gamma}H\xi\beta\rho_{Hv}F_{Hv} + vH\rho_{Hr}F_{Hr} + v^{\frac{1}{2}}\xi\beta\rho_{vr}F_{vr} \\ &- rF - rHF_H = -(\alpha - \Phi\beta)F_v. \end{aligned} \tag{2.5}$$

Now we consider the Heston model [8], replacing the values of $\alpha = -\lambda(v - \overline{v})$, $\beta = 1$, and $\gamma = \frac{1}{2}$ in Eq. (2.1). Thus, we have

$$dv = -\lambda(v - \overline{v})dt + \xi\sqrt{v}dZ_t^v,$$

where $\overline{v}$ is the long-term price variance, λ is the rate at which v reverts to $\overline{v}$, and ξ is defined as the volatility of volatility.

Practically, we will now use the Heston model with the new values of α and β and reorder the terms in Eq. (2.5). The following transformation yields

$$\begin{aligned} F_t &+ \frac{1}{2}vH^2F_{HH} + \frac{1}{2}\xi^2\beta^2vF_{vv} + \frac{1}{2}vF_{rr} + vH\xi\beta\rho_{Hv}F_{Hv} + vH\rho_{Hr}F_{Hr} + v\xi\beta\rho_{vr}F_{vr} \\ &- rF + rHF_H - \lambda(v - \overline{v})F_v = 0. \end{aligned} \tag{2.6}$$

Eq. (2.6) represents the valuation of any asset, where the value is a function solely of the house price H, volatility v, interest rate r, and time t, particularly for FRM with appropriate initial and boundary conditions.

Moreover, let T_m denote the total time in one month and t_m the time that has elapsed in the month; thus, $\tau_m = T_m - t_m$ defines the time remaining until the month's payment date m. This time variable definition transforms Eq. (2.6) into an initial value problem, which is explained starting from the initial condition.

Therefore, the valuation PDE for $F(\tau_m, H, v, r)$ must be solved for $0 \leq \tau_m \leq T_m$, $0 \leq H < \infty$, $0 \leq v < \infty$ and $0 \leq r < \infty$. Similarly, the functions representing the mortgage values $V(\tau_m, H, v, r)$ for the lender during the month m also satisfy this PDE.

Details of the remaining mortgage contract, including formulas for the monthly payment (MP), the outstanding balance following each payment $P(m-1)$, the usual monthly payment-date conditions, and other relevant information, can be found in the works of Azevedo-Pereira et al. [1] and Sharp et al. [17].

The mortgage pricing model begins with the amount of the mortgage at maturity $t = T_M$, where the last payment is the minimum of MP and the value of the house, expressed as $V(\tau_M = 0, H, v, r) = \min(MP, H)$. At other payment dates, the value is given by

$$V(\tau_m = 0, H, v, r) = \min(v(\tau_{m+1} = T_{m+1}, H, v, r) + MP, H),\ 1 \leq m \leq M - 1.$$

The boundary conditions are as follows:

$$\begin{aligned} \lim_{H\to\infty} \frac{\partial F(\tau, H, v, r)}{\partial H} &= 0, \qquad (\tau, v, r) \in [0, T] \times [0, +\infty) \times [0, +\infty), \\ \lim_{H\to-\infty} F(\tau, H, v, r) &= 1, \qquad (\tau, v, r) \in [0, T] \times [0, +\infty) \times [0, +\infty), \\ F(\tau, H, 0, r) &= 0, \qquad (\tau, H, r) \in [0, T] \times [0, +\infty) \times [0, +\infty), \\ \lim_{v\to\infty} \frac{\partial F(\tau, H, v, r)}{\partial v} &= 0, \qquad (\tau, H, r) \in [0, T] \times [0, +\infty) \times [0, +\infty), \\ F(\tau, H, v, 0) &= \min(MP, H), \qquad (\tau, H, v) \in [0, T] \times [0, +\infty) \times [0, +\infty), \\ \lim_{r\to\pm\infty} \frac{\partial F(\tau, H, v, r)}{\partial r} &= 0, \qquad (\tau, H, v) \in [0, T] \times [0, +\infty) \times [0, +\infty). \end{aligned}$$

By converting the coefficients to constants and applying a variable transformation, the PDE can be solved. We employ a logarithmic transformation: $x = Ln(H)$. The transformed house price variable, x, is bounded by $x \in [0, Ln(\overline{H})]$, where $(\overline{H})$ represents the upper bound of the house price [3]. For volatility (v) and the interest rate (r), the natural boundaries are 0 and ∞, respectively. The transformation of the original function is given by: $F(t, H, v, r) = u(\tau, x, v, r)$. For simplicity, the details of the variable transformation and coefficient conversion are omitted here; we simply note that ξ

is equivalent to σ_v. Consequently, the valuation PDE must now be solved for the transformed variables: $\tau_m \in [0, T_m]$ and $x, v, r \in \Upsilon$.

$$\begin{aligned} &-\frac{\partial u}{\partial \tau_m} + \frac{1}{2}v\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 u}{\partial v^2} + \frac{1}{2}v\frac{\partial^2 u}{\partial r^2} + v\sigma_v\rho_{Hv}\frac{\partial^2 u}{\partial x\partial v} + v\rho_{Hr}\frac{\partial^2 u}{\partial x\partial r} \\ &+ v\sigma_v\rho_{vr}\frac{\partial^2 u}{\partial v\partial r} + (r - \frac{1}{2}v)\frac{\partial u}{\partial x} + \lambda(\overline{v} - v)\frac{\partial u}{\partial v} - ru = 0, \end{aligned} \tag{2.7}$$

where Υ is $[0, Ln(\overline{H})] \times [0, \infty) \times [0, \infty)$. Similarly, the other boundary conditions are also transformed. This section provides a basic overview of FRM pricing models. In the next section, we will highlight a popular numerical method and explain how we plan to use it to solve pricing problems.

3. Applying RBF

We will now demonstrate how Radial Basis Functions (RBFs), a well-established meshless technique, can be used both for reconstructing an unknown function from scattered data and for pricing financial contracts by solving the governing PDE. Utilizing an RBF interpolant, we approximate the unknown function $u = u(\tau, x, v, r)$ to determine suitable interpolation points for the initial value. Given the wide variety of available RBF types, readers are encouraged to consult reference [14] for a comprehensive overview. Having selected the interpolation points, we then interpolate the unknown function u using a radial basis function $\phi(\big|\big|x - x_k, v - v_l, r - r_j\big|\big|)$, where φ represents the basis function.

$$u(\tau, x, v, r) \cong \sum_{k=1}^{M}\sum_{l=1}^{L}\sum_{j=1}^{N}\psi_{klj}(\tau)\phi(\big|\big|x - x_k, v - v_l, r - r_j\big|\big|), \tag{3.1}$$

where $\psi_{klj} \in \mathbb{R}$ is estimated by the interpolation condition $u(x_k, v_l, r_j) = \tilde{u}(x_k, v_l, r_j)$, and $L_{k'l'j'klj} = \phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)$,$k, k^{'} = 1, ..., M$, $l, l^{'} = 1, ..., L$ and $j, j^{'} = 1, ..., N$. We use the Euclidean norm, with $MLN = \Lambda$. Substituting the right-hand side of Eq. (3.1) into Eq. (2.7) yields a system of linear equations. This system can be expressed in matrix form as $L\psi = \tilde{u}$, where $\psi = [\psi_1, ..., \psi_\Lambda]^T$ is the vector of unknown coefficients. This paper utilizes the multiquadric (MQ) interpolation method, as described by Hardy:

$$\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|, \varsigma) = \sqrt{\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|^2 + \varsigma^2}.$$

It is used in Eq. (3.1) as a spatial approximation for the unknown function $\tilde{u}$. Here, ς is a positive parameter, and in the case of MQ, it is referred to as the shape parameter. While theoretical research on selecting the optimal shape parameter is still ongoing, we follow Hardy's suggestion [7] by choosing $\varsigma = 4d_{min}$, where d_{min} represents the minimum distance.

Since the RBF does not depend on time, the time derivative of $\tilde{u}$ is simply the time derivative of the coefficients.

$$\frac{\partial \tilde{u}(\tau, x_{k'}, v_{l'}, r_{j'})}{\partial \tau} \approx \sum_{k=1}^{M}\sum_{l=1}^{L}\sum_{j=1}^{N}\frac{d\psi_{klj}(\tau)}{d\tau}.\phi(\|x - x_k, v - v_l, r - r_j\|).$$

Therefore, we need to find the partial derivatives of $\tilde{u}$ with respect to the underlying assets. Additionally, we must compute the first and second partial derivatives of φ for the MQ radial basis function concerning the underlying assets x, v, and r, respectively. By substituting the expansions for the partial derivatives of $\tilde{u}$ into the FRM pricing model and applying the collocation method, we obtain the following relation.

$$\begin{aligned} \sum_{k=1}^{M}\sum_{l=1}^{L}\sum_{j=1}^{N}\Big(&-\frac{d\psi_{klj}(\tau)}{d(\tau)}\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|) + \frac{1}{2}v\psi_{klj}(\tau)\frac{\partial^2\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)}{\partial x^2} \\ &+ \frac{\sigma_v^2 v}{2}\psi_{klj}(\tau)\frac{\partial^2\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)}{\partial v^2} + \frac{v}{2}\psi_{klj}(\tau)\frac{\partial^2\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)}{\partial r^2} \\ &+ v\sigma_v\rho_{Hv}\psi_{klj}(\tau)\frac{\partial^2\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)}{\partial x\partial v} + v\rho_{Hv}\psi_{klj}(\tau)\frac{\partial^2\phi(\|x_{k'} - x_k, v_{l'} - v_l, r_{j'} - r_j\|)}{\partial x\partial r} \end{aligned}$$

$$+ v\sigma_v\rho_{vr}\psi_{klj}(\tau).\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial v\partial r}+(r-\frac{1}{2}v)\psi_{klj}(\tau)\frac{\partial\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial x}$$
$$+\lambda(\overline{v}-v)\psi_{klj}(\tau)\frac{\partial\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial v}-r\psi_{klj}(\tau)\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)\Big)=0. \tag{3.2}$$

Eq. (3.2) can be expressed as following matrix form

$$L\dot{\psi}=\frac{1}{2}vL_{xx}\psi+\frac{1}{2}\sigma_v^2vL_{vv}\psi+\frac{1}{2}vL_{rr}\psi+v\sigma_v\rho_{Hv}L_{xv}\psi+v\rho_{Hr}L_{xr}\psi+v\sigma_v\rho_{vr}L_{vr}\psi+(r-\frac{1}{2}v)L_x\psi$$
$$+\lambda(\overline{v}-v)L_v\psi-rL\psi, \tag{3.3}$$

where $\dot{\psi}$ denotes $\frac{d\psi_{klj}}{d\tau}$. For notational simplicity, we use the following symbolizations.

$$L_x=\frac{\partial\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial x},\ L_{xx}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial x^2},$$
$$L_v=\frac{\partial\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial v},\ L_{vv}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial v^2},$$
$$L_r=\frac{\partial\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial r},\ L_{rr}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial r^2},$$
$$L_{xv}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial x\partial v},\ L_{xr}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial x\partial r},$$
$$L_{vr}=\frac{\partial^2\phi(\|x_{k'}-x_k,v_{l'}-v_l,r_{j'}-r_j\|)}{\partial v\partial r}.$$

These expressions represent the first and second partial derivatives of φ for the MQ radial basis function with respect to the underlying assets x, v, and r, respectively. In paper [9], Hon and Mao noted that the matrix L is invertible in the case of equally spaced RBFs. Consequently, Eq. (3.3) can be rewritten as follows:

$$\dot{\psi}=(\frac{1}{2}vL_{xx}+\frac{1}{2}\sigma_v^2vL_{vv}+\frac{1}{2}vL_{rr}+v\sigma_v\rho_{Hv}L_{xv}+v\rho_{Hr}L_{xr}+v\sigma_v\rho_{vr}L_{vr}+(r-\frac{1}{2}v)L_x+\lambda(\overline{v}-v)L_v-rL)L^{-1}\psi.$$

In which

$$\dot{\psi}=\Omega\psi, \tag{3.4}$$

where Ω can also be expressed as

$$\Omega=(\frac{1}{2}vL_{xx}+\frac{1}{2}\sigma_v^2vL_{vv}+\frac{1}{2}vL_{rr}+v\sigma_v\rho_{Hv}L_{xv}+v\rho_{Hr}L_{xr}+v\sigma_v\rho_{vr}L_{vr}+(r-\frac{1}{2}v)L_x+\lambda(\overline{v}-v)L_v-rL)L^{-1}.$$

In Eq. (3.4), we can employ any backward time integration scheme, such as the fourth-order Runge-Kutta (RK4) method, to calculate the unknown coefficients ψ at each time step $T-\Delta t$. Let $\tilde{u}^n=L\psi^n$ represent the approximation $\tilde{u}(T-n\Delta t,x_{k'},v_{l'},r_{j'})$ at each time step $T-n\Delta t$. The explicit fourth-order backward time integration scheme is then as follows:

$$\Im_1=-\Delta t\Omega\psi^{n-1},\ \Im_2=-\Delta t\Omega(\psi^{n-1}+\frac{1}{2}\Im_1),$$
$$\Im_3=-\Delta t\Omega(\psi^{n-1}+\frac{1}{2}\Im_2),\ \Im_4=-\Delta t\Omega(\psi^{n-1}+\Im_3), \tag{3.5}$$
$$\psi^n=\psi^{n-1}+\frac{1}{6}(\Im_1+2\Im_2+2\Im_3+\Im_4).$$

For both explicit and implicit iterative approximations, we set $\Delta t=\frac{T}{M}$. For $n=1,2,...,M$, the coefficients ψ^n can be calculated based on the previous results. At each time step n, we need to update ψ^n to ensure that the boundary condition is satisfied. The numerical computations for the FRM valuation problem are presented in the following example, and all of our numerical simulations are conducted using MATLAB.

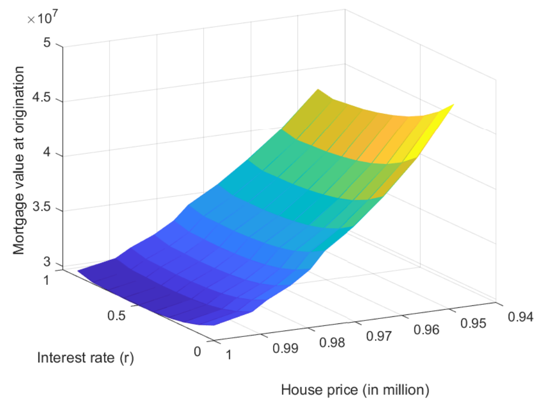

FIGURE 1. RBF approximation for the mortgage values at origination for the initial house price and the interest rate.

4. Implementation

This section presents a numerical experiment to evaluate the prices of the FRM model under the Heston framework. The experiment, which incorporates stochastic volatility and interest rate volatility, was conducted using the scheme described in section 4 and implemented in MATLAB. Additionally, a set of parameters must be specified to obtain the solution to the FRM valuation problem.

The following details present some of the contract parameters, which are fixed: $\sigma_v = 10\%$, $\sigma_r = 7\%$, $\sigma_H = 10\%$, $\theta = 10\%$, $k = 25\%$, $\delta = 7.5\%$, $\rho = 0$, $r_0 = 8\%$, $v_0 = 9\%$. The mortgage contract parameters include $H_{initial} = 100,000$ (the value of the house at origination), with a mortgage maturity of $T = 300$ months or 25 years, and $c_0 = 10\%$.

Accordingly, the RBF method yields results for pricing FRM models. A basic set of parameters has been chosen based on those reported in the literature (see Azevedo-Pereira et al. [1], Sharp et al. [17], and Kau et al. [12], for example). Specifying these parameters is essential for finding the FRM valuation problem and illustrating all necessary calculations.

Furthermore, some additional parameters and spatial discretization values are collected, which are necessary for the numerical methods employed to solve the problem: $H_\infty = 200,000$, $r_\infty = 40\%$, $\Delta v = 0.1$, $\Delta r = 0.1$. The number of time steps per month is 30. In terms of meshless interpolation methods, we use the MQ (Multiquadric) function as the basis.

In Figure 1, we display the contour of mortgage value at mortgage origination as a function of underlying house value and interest rates. Computations are based on the fixed parameters of the mentioned model. The FRM carries $100,000$, and the initial value of the house with a 25-year mortgage term is considered. Moreover, we utilized logarithmic scaling for the house value axis and labeled it with the actual house value.

Figure 2 displays the FRM price functions $\tilde{u}$ corresponding to a set of benchmark parameters for a loan term of 25 years, with various values of volatility $v = 5\%$ and interest rate $r = 3\%$. Figure 3 presents the FRM price functions $\tilde{u}$ associated with the same benchmark parameters in the domain (T, v) for a loan term of 25 years, for different values of interest rate $r = 3\%$.

The mortgage values $\tilde{u}$ under the Heston model at time $\tau = T, (t = 0)$ are compared with the mortgage values $\tilde{u}$ under the Heston model at the time T, as indicated in Figures 4 and 5, respectively.

5. Conclusion

Given the numerous advantages of RBF, this method has been implemented for the valuation of the FRM under the Heston model. We developed a pricing model for FRM, where the value of this contract depends on house price and interest rate as underlying factors, with volatility treated as a stochastic process. We demonstrated that the pricing model of FRM under the underlying assets is typically determined by PDEs.

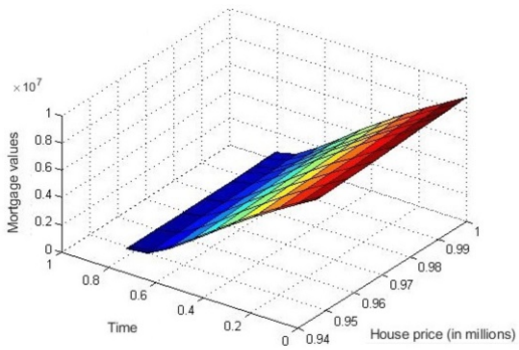

FIGURE 2. RBF approximation for the mortgage values under the Heston model for benchmark parameters and the various values of volatility $v = 5\%$ and interest rate $r = 3\%$.

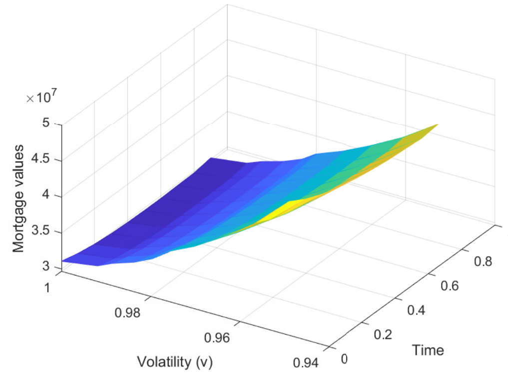

FIGURE 3. RBF approximation for the mortgage values under the Heston model for the parameters and the various values and interest rate $r = 3\%$.

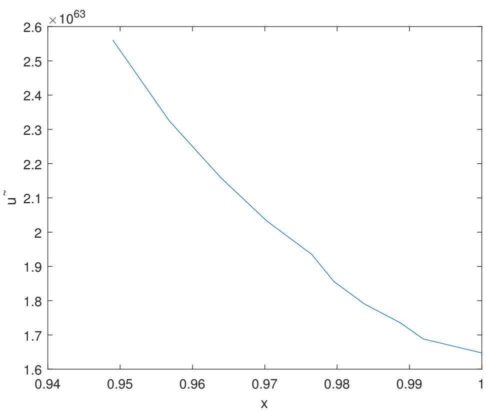

FIGURE 4. The mortgage values under the Heston model at time $\tau = T, (t = 0)$.

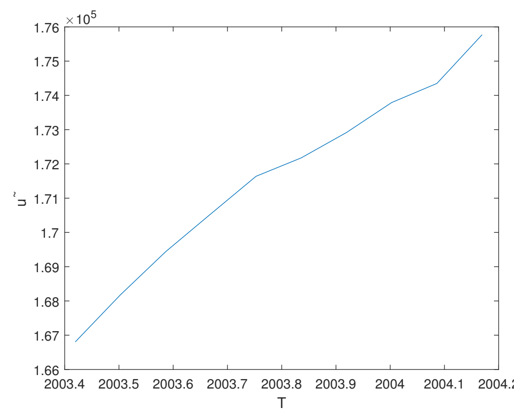

FIGURE 5. The mortgage values $\tilde{u}$ under the Heston model at time T.

The PDEs derived from the FRM pricing model are effectively addressed using RBF due to the lack of closed-form solutions. A linear combination of RBF has been defined as one potential method for solving problems related to scattered data functions. We adopted RBF in greater detail for the pricing of financial contracts by solving initial and boundary value problems.

Finally, based on the derived ODEs, we designed the RK4 method. A basic set of economic parameters, drawn from the literature, was specified for the mortgage as an example to demonstrate the potential application of the proposed approach. Furthermore, numerical results illustrate the expected behavior of mortgage values in relation to the stochastic processes of house prices, volatility, and interest rates. Ultimately, we assessed the value of the contract using the model and the proposed solution method.

Given that information on financial market data is often incomplete, future work could involve using artificial neural networks to estimate the parameters of the model. Since jumps can occur in the real world for unknown reasons, further research could extend the ideas presented in this paper to jump-diffusion models for house price evolution.

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