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Symmetry properties, exact solution and conservation laws of time-fractional Zeldovich-Frank-Kamenetskii equation

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Abstract

The article focuses on investigating Lie symmetry analysis of the time-fractional Zeldovich-Frank-Kamenetskii equation with Riemann-Liouville derivative. The fractional reaction-diffusion equation describes how planar laminar premixed flames spread in combustion theory. The use of the Lie method is also illustrated to obtain Lie symmetry generators, symmetry reduction solutions, invariant properties, and conservation laws. Furthermore, we convert the time-fractional Zeldovich-Frank-Kamenetskii equation to a nonlinear fractional ordinary differential equation (ODE) with Erdélyi-Kober derivative using its Lie point symmetries. This decreased fractional ODE is investigated by explicit power series. In addition, some figures for the obtained explicit solution are presented.

Keywords. Fractional differential equation, Zeldovich-Frank-Kamenetskii equation, Lie symmetry, Explicit solutions, Ibragimov method, Conservation laws.

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1. Introduction

In recent years, fractional differential equations (FDEs), as a generalization of partial differential equations (PDEs), have been applied in physics, engineering, chemistry, biology, blood flow phenomena, quantum mechanics, etc. In fact, many physical and natural phenomena may depend on their current and historical situations that can be formulated by the theory of fractional derivatives and integrals. Fractional calculus is the most suitable tool for describing long-memory processes. The most popular models of this calculus are differential equations with fractional-order derivatives. Also, there are numerous methods available in the literature for solving FDEs. Both mathematicians and physicists have made many efforts to discover effective methods to find exact solutions to FDEs. For example, see analytical methods like unified model, extended unified model, and variational model [1, 20], or numerical methods like Chebyshev series method, Tikhonov regularization method, and ABC-fractional technique [3, 7, 21, 23]. However, Lie symmetry analysis, as an analytic method introduced by Lie, provides an effective way to derive exact solutions for PDEs and FDEs. Finding explicit solutions is a great and interesting problem while dealing with a problematic system of PDEs or FDEs.

The Lie symmetry group of a system of differential equations has numerous practical applications. One of the most important efforts is its ability to generate new solutions from known ones. This is achieved by applying the defined properties of the symmetry group. Additionally, a Lie symmetry group provides a method for categorizing different classes of symmetrical solutions. Two solutions are considered equivalent if one can be converted to the other with a group element. Moreover, these symmetry groups can be used to classify a set of differential equations based on given parameters or functions. Another valuable application of Lie group theory is to identify the conservation laws of equations, playing a pivotal role in examining solution properties, including their existence, uniqueness, and stability [4].

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In the present article, we show that Lie symmetry techniques can also be applied to discover solutions of a reaction-diffusion equation, named Zeldovich-Frank-Kamenetskii (ZFK). Further, we concentrate on one of the easiest variants of the ZFK equation [5]. However, we envisage applying our method to more complex patterns for planar flame propagation away from flammability limits, explained by a system of reaction-diffusion equations for which an Arrhenius law with high activation energy determines reaction rate. There are very few papers that have dealt with ZFK equation in fractional form, and among them, almost all of them have solved this equation with numerical methods and have provided approximate solutions for it ([17], [27]). In this paper, however, an attempt has been made to solve this equation with the Lie method and obtain exact solutions. Also, for the first time, we have obtained the conservation laws of this fractional equation.

The time-fractional differential equation is defined by

$$\partial_t^\eta \theta = \theta_{xx} + \frac{\beta^2}{2}\theta(1-\theta)e^{-\beta(1-\theta)}, \tag{1.1}$$

in which $t \geq 0$, $x \in R$, β is the Zeldovich number, and θ is a non-dimensional variable that quantifies the ratio of burnt to unburnt gas in an easy reaction including two modes and is called reduced temperature. The term $\omega(\theta,\beta) = \frac{\beta^2}{2}\theta(1-\theta)e^{-\beta(1-\theta)}$ is called reaction term which depends on Zeldovich number $\beta \gg 1$ [5]. The focus of the article is on investigating Lie point symmetries, similarity reduction, formal Lagrangian, and conservation laws for the time-fractional ZFK equation.

The continuation of this paper is 7 sections, including a conclusion. Section 2 recalls some primary definitions and features of fractional calculus. Section 3 explains how to determine Lie point symmetries of FDEs. In section 4, Lie point symmetries of the time-fractional ZFK equation are found. The next section is devoted to applying Lie symmetries to construct novel invariant solutions for this equation. In Section 6, we obtain the power series solutions of (1.1). Section 7 explains how, by using Lie symmetry generators, conservation laws can be constructed for the fractional ZFK equation. Some conclusions are also presented in the last section.

2. Preliminaries

First, some preliminary notations and definitions of fractional calculus are presented. We should mention that there are various definitions for fractional derivatives like Caputo, Riemann-Liouville, Riesz, Grünwald-Letnikov, Miller-Ross, Hadamard, and Erdélyi-Kober fractional derivatives. Here, we employ Riemann-Liouville and Erdélyi-Kober fractional derivative to obtain symmetries and exact solutions of system (1.1) (see [2, 8, 26]).

Definition 2.1. [6, 22] Presume $f(x)$ is an integrable function and $-\infty < a < b < \infty$. Then left-sided and right-sided fractional Riemann-Liouville integrals of order $\eta > 0$ will be

$$\begin{aligned} {}_aI_t^\eta f(t) &= \frac{1}{\Gamma(\eta)}\int_a^t \frac{f(s)}{(t-s)^{1-\eta}}ds, \quad x > a, \\ {}_tI_b^\eta f(t) &= \frac{1}{\Gamma(\eta)}\int_t^b \frac{f(s)}{(s-t)^{1-\eta}}ds, \quad x < b, \end{aligned} \tag{2.1}$$

respectively, in which $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}$ is the Gamma function [6].

Definition 2.2. [6, 22] On $[a,b]$,

$$\begin{aligned} {}_aD_t^\eta f(t) &= D_t^n({}_aI_t^{n-\eta} f(t)) = \frac{1}{\Gamma(n-\eta)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(r)}{(t-r)^{\eta-n+1}}dr, \\ {}_tD_b^\eta f(t) &= (-1)^n D_t^n({}_tI_b^{n-\eta} f(t)) = \frac{(-1)^n}{\Gamma(n-\eta)}\left(\frac{d}{d}\right)^n \int_t^b \frac{f(r)}{(r-t)^{\eta-n+1}}dr, \end{aligned} \tag{2.2}$$

are left and right Riemann-Liouville fractional partial derivatives of order $\eta > 0$ for $f(x)$, where $n = [\eta] + 1$.

Some applicable features of the Riemann-Liouville partial derivative are as follows [22]:

- $D_t^\eta C = \dfrac{Ct^{-\eta}}{\Gamma(1-\eta)}$, C is a constant;

- $D_t^{\eta} t^{\beta} = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\eta)} t^{\beta-\eta}$;
- $D_t^{\eta}[h(t)g(t)] = \sum\limits_{m=0}^{\infty} \binom{\eta}{m} D_t^{\eta-m} h(t) D_t^m g(t)$, $\eta > 0$.

The third feature refers to the generalized Leibnitz rule in which

$$\binom{\eta}{m} = \frac{(-1)^{m-1}\eta\Gamma(m-\eta)}{\Gamma(1-\eta)\Gamma(m+1)}.$$

Further, we need the Erdélyi-Kober fractional integral operator

$$(\mathcal{K}_{\delta}^{\tau,\eta} h)(z) = \begin{cases} \dfrac{1}{\Gamma(\eta)} \displaystyle\int_1^{\infty} (s-1)^{\eta-1} s^{-(\tau+\eta)} h(zs^{\frac{1}{\delta}}) ds, & \eta > 0, \\ h(z), & \eta = 0. \end{cases} \tag{2.3}$$

Using this notion, we can define the extended left-hand side of this operator with

$$\left(\mathcal{P}_{\delta}^{\tau,\eta} h\right)(z) = \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\delta} z \frac{\partial}{\partial z}\right) (\mathcal{K}_{\delta}^{\tau+\eta, n-\eta} h)(z), \tag{2.4}$$

where $z > 0$, $\delta > 0$, $\eta > 0$ and

$$n = \begin{cases} [\eta] + 1, & \eta \notin N, \\ \eta, & \eta \in N. \end{cases}$$

3. Finding Lie symmetries of time-fractional PDEs regarding Riemann-Liouville derivative

Lie symmetries for FDEs were first proposed by Gazizov et al. [11], and then many authors considered Lie group theory for analyzing FDEs in [10, 24] and references therein. This section presents several short points of Lie symmetry analysis for time-fractional PDEs with over independent variables. This procedure can be extended to other time-fractional PDEs or systems with more independent variables. Consider a scaler time-fractional PDE involving independent variables t and x and a dependent variable θ as follows:

$$\partial_t^{\eta} \theta(t,x) = F(t, x, \theta, \theta_x, \theta_{xx}, \cdots), \tag{3.1}$$

for the order $0 < \eta < 1$, where ∂_t^{η} indicates Riemann-Liouville fractional derivative and indexes are partial derivatives. According to Lie group theory, if (3.1) is invariant under a one-parameter Lie group of point transformations, it remains unchanged under invertible transformations of variables t, x, θ, and the derivatives of θ for independent variables are

$$\begin{aligned} &\bar{t} = t + \epsilon\tau(t,x,\theta) + \mathcal{O}(\epsilon^2), && \bar{x} = x + \epsilon\xi(t,x,\theta) + \mathcal{O}(\epsilon^2), \\ &\bar{\theta} = \theta + \epsilon\Phi(t,x,\theta) + \mathcal{O}(\epsilon^2), && \frac{\partial\bar{\theta}}{\partial\bar{x}} = \frac{\partial\theta}{\partial x} + \epsilon\Phi^{x}(t,x,\theta) + \mathcal{O}(\epsilon^2), \\ &\frac{\partial^2\bar{\theta}}{\partial\bar{x}^2} = \frac{\partial^2\theta}{\partial x^2} + \epsilon\Phi^{xx}(t,x,\theta) + \mathcal{O}(\epsilon^2), && \frac{\partial^{\eta}\bar{\theta}}{\partial\bar{t}^{\eta}} = \frac{\partial^{\eta}\theta}{\partial t^{\eta}} + \epsilon\Phi^{(\eta,t)}(t,x,\theta) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3.2}$$

in which τ, ξ and Φ are called infinitesimal and Φ^x, Φ^{xx}, and $\Phi^{(\eta,t)}$ are called generalized infinitesimals of order 1, 2, and η, respectively. The set of all these transformations for a continuous parameter ϵ is called a Lie group transformation G. Like properties of groups, G has an identity transformation. This group is known as the admitted symmetry group by (2.3), too. The associated infinitesimal generator for these group transformations is defined as:

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \Phi\frac{\partial}{\partial\theta}, \tag{3.3}$$

where

$$\left.\frac{d\bar{t}}{d\epsilon}\right|_{\epsilon=0} = \tau, \quad \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0} = \xi, \quad \left.\frac{d\bar{\theta}}{d\epsilon}\right|_{\epsilon=0} = \Phi.$$

Based the invariance criterion of Lie theory, (3.1) accepts Lie group transformation (3.2) and Lie symmetry vector field (3.3) as generator if prolonged vector field $Pr^{(\eta,t)}X$ annihilates (3.1) on its solution; that is,

$$Pr^{(\eta,t)}X(\Delta)\Big|_{\delta=0}=0, \qquad \Delta=\partial_t^\eta\theta(t,x)-F(t,x,\theta,\theta_x,\theta_{xx},\cdots).$$

This equation is named the determining equation. By keeping essential terms, the prolongation of operator $Pr^{(\eta,t)}X$ is

$$Pr^{(\eta,t)}X=X+\Phi^x\partial_{\theta_x}+\Phi^{xx}\partial_{\theta_{xx}}+\Phi^{(\eta,t)}\partial_{\partial_t^\eta\theta}. \tag{3.4}$$

The explicit expression for Φ^x and Φ^{xx}, coming from the classical prolongation formula for PDEs, is expressed by [19]:

$$\begin{aligned}\Phi^x&=D_x\Phi-\theta_tD_x\tau-\theta_xD_x\xi=\Phi_x+\Phi_\theta\theta_x-\tau_x\theta_t-\xi_x\theta_x-\tau_\theta\theta_t\theta_x-\xi_\theta\theta_x^2,\\ \Phi^{xx}&=D_x\Phi^x-\theta_{tx}D_x\tau-\theta_{xx}D_x\xi=\Phi_{xx}+(2\Phi_{x\theta}-\xi_{xx}-2\tau_{x\theta}\theta_t-3\xi_\theta\theta_{xx})\theta_x\\ &\quad-\tau_{xx}\theta_t+(\Phi_{\theta\theta}-\tau_{\theta\theta}\theta_t-2\xi_{x\theta})\theta_x^2-2(\tau_x+\tau_\theta\theta_x)\theta_{xt}-(\tau_\theta\theta_t-\Phi_\theta+2\xi_x)\theta_{xx}-\xi_{\theta\theta}\theta_x^3,\end{aligned} \tag{3.5}$$

in which D_i stands for total differentiation with respect to the independent variable i. Also, extended infinitesimal $\Phi^{(\eta,t)}$ corresponding to time-fractional Riemann-Liouville derivative is [12]:

$$\Phi^{(\eta,t)}=D_t^\eta\Phi+\xi D_t^\eta\theta_x-D_t^\eta(\xi\theta_x)+\tau D_t^\eta\theta_t+D_t^\eta(\theta D_t\tau)-D_t^{\eta+1}(\tau\theta),$$

where D_t^η is total fractional derivative operator of order η in related to t. Applying Leibnitz rule presented in the previous section, the generalized infinitesimal $\Phi^{(\eta,t)}$ is

$$\Phi^{(\eta,t)}=D_t^\eta\Phi-\eta.D_t\tau.\frac{\partial^\eta\theta}{\partial t^\eta}-\sum_{n=1}^{\infty}\binom{\eta}{n}.D_t^n\xi.D_t^{\eta-n}\theta_x-\sum_{n=1}^{\infty}\binom{\eta}{n+1}D_t^{n+1}\tau.D_t^{\eta-n}\theta.$$

Now, we state the generalized chain rule for the combination of two functions as below:

$$\frac{d^mg(h(t))}{dt^m}=\sum_{k=0}^{m}\sum_{l=0}^{k}\binom{k}{l}\frac{1}{k!}[-h(t)]^l\frac{d^m}{dt^m}[h(t)^{k-l}]\frac{d^kg(h)}{dh^k}.$$

Giving both generalized chain and Leibniz rules, we have $D_t^\eta\Phi$ in $\Phi^{(\eta,t)}$ as follows:

$$D_t^\eta(\Phi)=\frac{\partial^\eta\Phi}{\partial t^\eta}+\Phi_\theta\frac{\partial^\eta\theta}{\partial t^\eta}-\theta\frac{\partial^\eta\Phi_\theta}{\partial t^\eta}+\sum_{n=1}^{\infty}\binom{\eta}{n}\frac{\partial^n\Phi_\theta}{\partial t^n}D_t^{\eta-n}\theta+\mu,$$

in which μ is expressed as

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1}\binom{\eta}{n}\binom{n}{m}\binom{k}{l}\frac{t^{n-\eta}(-\theta)^l}{k!\Gamma(n+1-\eta)}\frac{\partial^m\theta^{k-l}}{\partial t^m}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial\theta^k}.$$

The explicit model of infinitesimal $\Phi^{(\eta,t)}$ will also be

$$\begin{aligned}\Phi^{(\eta,t)}&=\sum_{n=1}^{\infty}\left[\binom{\eta}{n}\partial_t^n\Phi_\theta-\binom{\eta}{n+1}D_t^{n+1}\tau\right]\partial_t^{\eta-n}\theta-\sum_{n=1}^{\infty}\binom{\eta}{n}D_t^n\xi.\partial_t^{\eta-n}\theta_x\\ &\quad+\partial_t^\eta\Phi+(\Phi_\theta-\eta D_t\tau)\partial_t^\eta\theta-\theta\partial_t^\eta\Phi_\theta+\mu.\end{aligned} \tag{3.6}$$

The lower limit of (2.2) is constant in $t=a$ and will also be invariant with regard to group transformations (3.2). Then the invariance condition yields as follows:

$$\tau(t,x,\theta)\Big|_{t=0}=0. \tag{3.7}$$

4. Application of Lie symmetry method to time-fractional ZFK equation

Here, the Lie symmetry theory is employed to get the symmetry group (1.1) with independent variables (t, x) being invariant under a one-parameter (ϵ) Lie group of (3.2) on an open set $M \subset X \times U \simeq R^{(2+1)}$. Using the second prolongation of generator $Pr^{(\eta,t)}X$ introduced in (3.4) for (1.1), we can obtain invariance criterion as follows:

$$\left[\Phi^{(\eta,t)} - \Phi_{xx} + \beta^2(\theta - \frac{1}{2})e^{-\beta(1-\theta)}\Phi + \frac{\beta^3}{2}\theta(\theta-1)e^{-\beta(1-\theta)}\Phi\right]_{(1.1)} = 0, \tag{4.1}$$

where Φ_{xx} is infinitesimal presented in (3.5) and $\Phi^{(\eta,t)}$ is generalized infinitesimal of order η expressed in (3.6). By inserting Φ_{xx} and $\Phi^{(\eta,t)}$ into the invariance criterion (4.1) and making equal the coefficients of partial derivatives of θ related to t and x, we have an over-determined system of linear PDEs and FDEs. By solving this system, infinitesimals can be derived as follows:

$$\tau = 4c_2t + c_4, \quad \xi = 2c_2\eta x + c_1, \quad \Phi = 3c_2\eta\theta + c_3\theta - 2c_2\theta + F(x,t),$$

where c_i for $i = 1, ..., 4$ and $F(x,t)$ are arbitrary constant and function, respectively. Regarding condition (3.7), the Lie symmetry group of the time-fractional ZFK equation is spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 4t\frac{\partial}{\partial t} + 2\eta x\frac{\partial}{\partial x} + \theta(3\eta-2)\frac{\partial}{\partial\theta}, \quad X_3 = \theta\frac{\partial}{\partial\theta}, \quad X_F = F(x,t)\frac{\partial}{\partial\theta}. \tag{4.2}$$

In order to find symmetries of FDEs and perform lengthy calculations, one can utilize the Maple symbolic computing platform. FracSym symmetry package [16], used together with MAPLE symmetry packages DESOLVII [15] and ASP [14], can assist in calculating infinitesimal generators and determining equations for the symmetries of FDEs.

5. Similarity reductions

In this section, we consider the similarity reduction method and describe how it can be used to find exact solutions for FDEs. This approach entails finding ways to simplify PDEs and FDEs by reducing the number of independent variables. A solution is considered invariant if it equates zero after applying infinitesimal generators of symmetries of equation [25]. By solving characteristic equations of the obtained vector fields, one can find the similarity reduction of the equation.

Case 1. For symmetry X_1, integrating the following characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{d\theta}{0},$$

yields invariant solution $\theta = \psi(t)$. Inserting this solution into (1.1) reduces ordinary FDE $\partial_t^\eta\psi(t) = 0$, connoting $\psi(t) = c_1t^{\eta-1}$ in which c_1 is arbitrary real constant.

Case 2. Similarity variables for infinitesimal generator X_2 can be concluded by integrating characteristic equation

$$\frac{dt}{4t} = \frac{dx}{2\eta x} = \frac{d\theta}{\theta(3\eta-2)},$$

which causes the following similarity variable:

$$\theta = t^{\frac{3\eta-2}{4}}\psi(\omega), \quad \omega = xt^{\frac{-\eta}{2}}. \tag{5.1}$$

Thus, by the above similarity transformation, (1.1) can be reduced into an ordinary FDE. This process is presented in the next theorem.

Theorem 5.1. *The similarity transformation* (5.1) *for* $\eta > 0$ *transforms time-fractional ZFK Equation* (1.1) *to the following nonlinear ordinary FDE:*

$$\left(\mathcal{P}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},\eta}\psi\right)(\omega) = \psi''(\omega) + \frac{\beta^2}{2}\psi(\omega)(1-\psi(\omega))e^{-\beta(1-\psi(\omega))}, \tag{5.2}$$

where $\psi''(\omega) = \frac{d\psi(\omega)}{d\omega}$ *and* $(\mathcal{P}_\delta^{\tau,\eta}\psi)$ *indicates extended left-hand sided Erdélyi-Kober fractional differential operator introduced by* (2.3) *and* (2.4).

Proof. Suppose that $n-1<\eta<n$ for $n=1,2,\cdots$. Applying Riemann-Liouville fractional derivative, we will obtain

$$\frac{\partial^\eta\theta}{\partial t^\eta}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\eta)}\int_0^t (t-s)^{n-\eta-1}s^{\frac{3\eta-2}{4}}\psi\left(xs^{-\frac{\eta}{2}}\right)ds\right]. \tag{5.3}$$

Assume that $\nu=\frac{t}{s}$. One can get $ds=\frac{-t}{\nu^2}d\nu$ and then (5.3) can be considered below:

$$\frac{\partial^\eta\theta}{\partial t^\eta}=\frac{\partial^n}{\partial t^n}\left[t^{n-\eta+\frac{3\eta-2}{4}}\frac{1}{\Gamma(n-\eta)}\int_1^\infty (\nu-1)^{n-\eta-1}\nu^{-(n-\eta+1+\frac{3\eta-2}{4})}\psi(\omega\nu^{\frac{\eta}{2}})d\nu\right].$$

Regarding (2.3), we obtain

$$\frac{\partial^\eta\theta}{\partial t^\eta}=\frac{\partial^n}{\partial t^n}\left[t^{n-\eta+\frac{3\eta-2}{4}}\left(\mathcal{K}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},n-\eta}\psi\right)(\omega)\right].$$

Taking into consideration $\omega=xt^{\frac{-\eta}{2}}$ and applying chain rule for differentiable functions, we get

$$t\frac{\partial}{\partial t}F(\omega)=tx(-\frac{\eta}{2})t^{-\frac{\eta}{2}-1}F'(\omega)=-\frac{\eta}{2}\omega\frac{\partial}{\partial\omega}F(\omega).$$

So,

$$\begin{aligned}\frac{\partial^\eta\theta}{\partial t^\eta}&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\eta+\frac{3\eta-2}{4}}\left(\mathcal{K}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},n-\eta}\psi\right)(\omega)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\eta+\frac{3\eta-2}{4}}\left(n-\eta+\frac{3\eta-2}{4}-\frac{\eta}{2}\omega\frac{\partial}{\partial\omega}\right)\left(\mathcal{K}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},n-\eta}\psi\right)(\omega)\right].\end{aligned}$$

By continuing $n-1$ times same process, we get

$$\begin{aligned}\frac{\partial^\eta\theta}{\partial t^\eta}&=t^{n-\eta+\frac{3\eta-2}{4}}\prod_{j=0}^{n-1}\left(1-\eta+\frac{3\eta-2}{4}+j-\frac{\eta}{2}\omega\frac{\partial}{\partial\omega}\right)\left(\mathcal{K}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},n-\eta}\psi\right)(\omega)\\&=t^{-\eta+\frac{3\eta-2}{4}}\left(\mathcal{P}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},\eta}\psi\right)(\omega).\end{aligned} \tag{5.4}$$

Using (5.4) and substituting the phrase of partial derivative of θ_{xx}, it can be derived that ZFK Equation (1.1) reduces to nonlinear ordinary FDE (5.2), and the proof ends. □

6. Exact power series solutions

Here, we address the exact solutions of (5.2) with power series. This method is used to find a power series solution to ODEs. In fact, such solutions are considered as power series with unknown coefficients, substituting these solutions into the differential equation to find a recursive expression for the coefficients. Let

$$\psi(\omega)=\sum_{n=0}^\infty a_n\omega^n. \tag{6.1}$$

Thus, we have

$$\psi'(\omega)=\sum_{n=0}^\infty na_n\omega^{n-1},\quad \psi''(\omega)=\sum_{n=0}^\infty n(n-1)a_n\omega^{n-2}. \tag{6.2}$$

Putting (6.1) and (6.2) in (5.2), we get

$$\begin{aligned}\sum_{n=0}^\infty\frac{\Gamma(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}{\Gamma(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}a_n\omega^n&=\sum_{n=0}^\infty(n+2)(n+1)a_{n+2}\omega^n+\frac{\beta^2}{2}\sum_{n=0}^\infty a_n\omega^n e^{-\beta(1-\sum_{n=0}^\infty a_n\omega^n)}\\&\quad-\frac{\beta^2}{2}\sum_{n=0}^\infty a_n\omega^n\sum_{n=0}^\infty a_n\omega^n e^{-\beta(1-\sum_{n=0}^\infty a_n\omega^n)}.\end{aligned} \tag{6.3}$$

Setting n = 0 in (6.3) and comparing coefficients, we conclude

$$a_2 = \frac{1}{2}\left(\frac{\Gamma\left(2+(\frac{3\eta-2}{4})\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})\right)}a_0 - \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0 + \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0^2\right), \tag{6.4}$$

For $n \geq 1$, we have

$$a_{n+2} = \frac{1}{(n+2)(n+1)}\left\{\frac{\Gamma\left(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}a_n - \frac{\beta^2}{2}e^{-\beta(1-a_n)}\left(a_n - \sum_{j=0}^{n}a_j a_{n-j}\right)\right\}. \tag{6.5}$$

Hence, each coefficient a_n for $n \geq 1$ in (6.1) is found by given constants a_i for $i = 0, 1, 2$, i.e. exact power series solution for ODE (5.2) exists by coefficients depending on (6.4) and (6.5). Hence, the exact power series solution for (5.2) is

$$\begin{aligned}\psi(\omega) &= a_0 + a_1\omega + a_2\omega^2 + \sum_{n=1}^{\infty}a_{n+2}\omega^{n+2}\\ &= a_0 + a_1\omega + \frac{1}{2}\left(\frac{\Gamma\left(2+(\frac{3\eta-2}{4})\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})\right)}a_0 - \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0 + \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0^2\right)\omega^2\\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}\left\{\frac{\Gamma\left(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}a_n - \frac{\beta^2}{2}e^{-\beta(1-a_n)}\left(a_n - \sum_{j=0}^{n}a_j a_{n-j}\right)\right\}\omega^{n+2}.\end{aligned}$$

As a result, we can obtain exact power series solution for (1.1) as

$$\begin{aligned}\theta(t,x) &= a_0 t^{\frac{3\eta-2}{4}} + a_1 x t^{\frac{\eta-2}{4}} + a_2 x^2 t^{\frac{-\eta-2}{4}} + \sum_{n=1}^{\infty}a_{n+2}x^{n+2}t^{-\frac{2\eta n+\eta+2}{4}}\\ &= a_0 t^{\frac{3\eta-2}{4}} + a_1 x t^{\frac{\eta-2}{4}} + \frac{1}{2}\left(\frac{\Gamma\left(2+(\frac{3\eta-2}{4})\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})\right)}a_0 - \frac{\beta^2}{2}e^{-\beta(1-a_0)}(a_0 - a_0^2)\right)x^2 t^{\frac{-\eta-2}{4}}\\ &+ \sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}\left\{\frac{\Gamma\Big(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\Big)}{\Gamma\Big(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\Big)}a_n - \frac{\beta^2}{2}e^{-\beta(1-a_n)}\Big(a_n - \sum_{j=0}^{n}a_j a_{n-j}\Big)\right\}.x^{n+2}t^{-\frac{2\eta n+\eta+2}{4}}.\end{aligned} \tag{6.6}$$

6.1. **Physical explanation of power series for solution (6.6).** A graphical display of (6.6) for various values of η is presented in Figure 1((a) and (b)). Additionally, Figure 2 shows connection between θ and a_0 and a_1 for $t \in [-10, 10]$ in (6.6).

7. Conservation laws for time-fractional ZFK equation

One of the significant roles of Lie symmetries in the analysis of PDEs and FDEs is building conservation laws. From a physical viewpoint, they state that the total amount of a specific physical quantity doesn't change during the evolution of an isolated system and stays constant. In mathematics, conservation laws provide conserved quantities for each solution, can perform integrability, describe linearization, and demonstrate the existence and uniqueness. In FDEs, conservation laws display a powerful concept of their integrability.

The famous Notether's theorem connects conservation laws and symmetries of Euler-Lagrange equations. A lot of PDEs do not have fractional Lagrangians. There exist some approaches for acquiring conservation laws of PDEs, which do not have a Lagrangian, and one of the best theorems presented by Ibragimov [13], considering a formal Lagrangian for these equations. For equations with fractional derivatives, some literature computes conservation laws by the Noether theorem. In [18], based on Ibragimov's method, Lukashchuk has made a good investigation to discover conservation laws for FDEs which do not have a fractional Lagrangian. Here, we apply the Ibragimov method

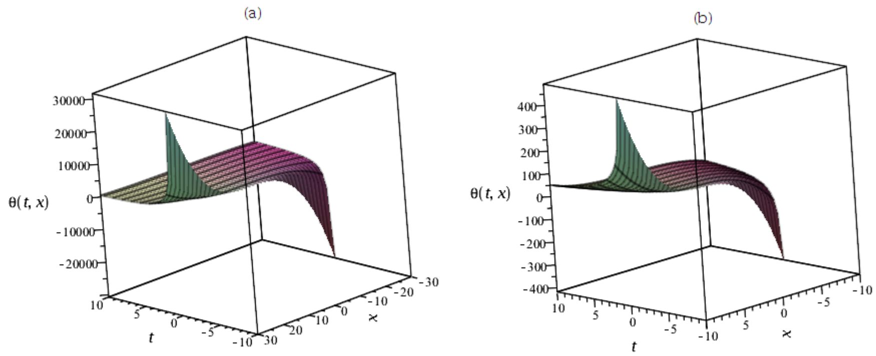

FIGURE 1. (a) 3-D plot of effect of η on $\theta(t,x)$ in (6.6) with $a_0 = 0.5$, $a_1 = 1$, $\eta = 0.75$, $\beta = 2$. (b) 3-D plot of effect of η on $\theta(t,x)$ in (6.6) with $a_0 = 0.5$, $a_1 = 1$, $\eta = 0.125$, $\beta = 2$.

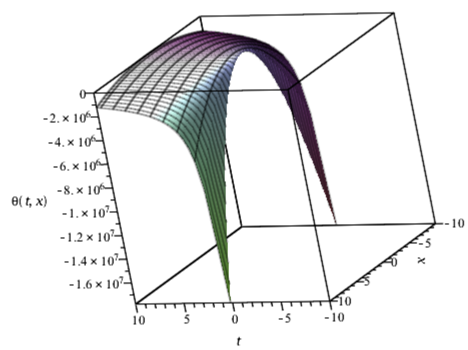

FIGURE 2. 3-D plot of effect of a_0 and a_1 on $\theta(t,x)$ in (6.6) with $a_0 = 5$, $a_1 = 1$, $\eta = 0.75$, $\beta = 2$.

to construct conservation laws for (1.1). A conservation law for fractional PDE with variables x, t is a continuous equation

$$D_t(C^t) + D_x(C^x) \equiv 0\Big|_{Eq.(1.1)=0}. \tag{7.1}$$

So, formal Lagrangian for (1.1) is

$$\mathcal{L} = \nu(t,x,\theta)\left[\partial_t^\eta\theta - \theta_{xx} - \frac{\beta^2}{2}\theta(1-\theta)e^{-\beta(1-\theta)}\right], \tag{7.2}$$

in which $\nu(t,x,\theta)$ is new variable. Thus, the adjoint form of (1.1) will be

$$\mathcal{F}^* = \frac{\delta\mathcal{L}}{\delta\theta} = 0, \tag{7.3}$$

in which $\delta/\delta\theta$, Euler-Lagrange operator for θ, is

$$\frac{\delta}{\delta\theta} = \frac{\partial}{\partial\theta} + (D_t^\eta)^*\frac{\partial}{\partial(D_t^\eta\theta)} - D_x\frac{\partial}{\partial\theta_x} + D_{xx}\frac{\partial}{\partial\theta_{xx}} - \cdots, \tag{7.4}$$

where $(D_t^\eta)^*$ is the adjoint operator of D_t^η. For the Riemann-Liouville derivative, the adjoint operator is

$$({}_0D_t^\eta)^* = (-1)^n {}_tI_T^{n-\eta}(D_t^n) \equiv {}_t^C D_T^\eta,$$

where ${}_tI^{n-\eta}$ is right-sided Riemann-Liouville fractional integral operator defined by (2.1) and ${}_t^CD_\theta^\eta$ is right-sided Caputo time-fractional derivative by the following definition:

$${}_t^CD_b^\eta f(t) = (-1)^n {}_tI_b^{n-\eta}(D_t^n f(t)) = \frac{(-1)^n}{\Gamma(n-\eta)}\int_t^b \frac{f^{(n)}(s)}{(s-t)^{\eta-n+1}}ds.$$

(1.1) is referred to a non-linearly self-adjoint equation if (7.3) is held for all solutions of (1.1) after substitution $\nu(t,x,\theta)$ provided that $\nu(t,x,\theta)\neq 0$. Substituting (7.2) into (7.4), we have the following adjoint fractional ZFK equation [9]:

$$\begin{aligned}\mathcal{F}^* &= {}_t^CD_T^\eta(\nu) + (\beta^2 - \frac{\beta^3}{2})\nu\theta e^{-\beta(1-\theta)} - \frac{\beta^2}{2}\nu e^{-\beta(1-\theta)} + \frac{\beta^3}{2}\nu\theta^2 e^{-\beta(1-\theta)} \\ &- \nu_{xx} - 2\nu_{x\theta}\theta_x - \nu_{\theta\theta}\theta_x^2 - \nu_\theta\theta_{xx} = 0. \end{aligned} \tag{7.5}$$

We solve (7.5) to obtain $\nu(t,x)$ to investigate the self-adjointness of the time-fractional ZFK equation and then $\nu(t,x,\theta) = \chi(t)\Upsilon(x,\theta)$, in which function $\chi(t)$ is a consequence from

$$({}_0D_t^\eta)^*\chi(t) = ({}_t^CD_T^\eta\chi(t)) = 0 \quad \Rightarrow \chi(t) = c,$$

with given constant c. The substitution $\Upsilon(x,\theta)$ resembles the substitution for the partial differential ZFK equation [13].

Based on the fundamental fractional Noether theorem [18], the elements of conserved vector are acquired by utilizing the Noether operators in the Lagrangian. Considering independent variables t and x and dependent variable $\theta(t,x)$, this important identity can be expressed by

$$\tilde{X} + D_t(\tau)\mathcal{I} + D_x(\xi)\mathcal{I} = W\frac{\delta}{\delta\theta} + D_t\mathcal{N}^t + D_x\mathcal{N}^x. \tag{7.6}$$

In the above identity, we have $\tilde{X}$ as a proper prolongation of Lie point group generators (4.2). Here, $\mathcal{I}$, $\mathcal{N}^t$ and $\mathcal{N}^x$ represent identity and Noether operators, respectively. Furthermore, $\frac{\delta}{\delta u}$ denotes Euler-Lagrange operator and W will be Lie characteristic for X introduced by $W = \Phi - \tau\theta_t - \xi\theta_x$. Also, fractional Noether operator $\mathcal{N}^t$ is

$$\mathcal{N}^t = \tau\mathcal{I} + D_t^{\eta-1}(W).\frac{\partial}{\partial({}_0D_t^\eta\theta)} + J\left(W, D_t\frac{\partial}{\partial(D_t^\eta\theta)}\right),$$

where integral J defined by

$$J(h,\ell) = \frac{1}{\Gamma(n-\eta)}\int_0^t\int_t^T \frac{h(\tau,x)\ell(\mu,x)}{(\mu-\tau)^{\eta+1-n}}d\mu d\tau,$$

has the following feature:

$$D_tJ(h,\ell) = h_tI_T^{n-\eta}\ell - \ell\ {}_0I_t^{n-\eta}h.$$

Noether operator $\mathcal{N}^x$ is also presented as follows:

$$\begin{aligned}\mathcal{N}^x &= \xi\mathcal{I} + W.\left(\frac{\partial}{\partial\theta_x} - D_x\frac{\partial}{\partial\theta_{xx}} + D_x^2\frac{\partial}{\partial\theta_{xxx}} - \cdots\right) \\ &+ D_x(W)\left(\frac{\partial}{\partial\theta_{xx}} - D_x\frac{\partial}{\partial\theta_{xxx}} + \cdots\right) + D_x^2(W)\left(\frac{\partial}{\partial\theta_{xxx}} - D_x\frac{\partial}{\partial\theta_{xxxx}} + \cdots\right) + \cdots.\end{aligned}$$

By applying both sides of (7.6) on (7.2), for each symmetry generator X of (1.1) and any solution of the attended equation, the left-hand side of the Noether identity equates to zero, and the other side is given by

$$D_t(\mathcal{N}^t\mathcal{L}) + D_x(\mathcal{N}^x\mathcal{L})\Big|_{(1.1)=0} = 0. \tag{7.7}$$

It follows by comparing (7.1) and (7.7) that there is a conserved vector for any Lie symmetry generator of (1.1) with components

$$C^t = \mathcal{N}^t\mathcal{L}, \qquad C^x = \mathcal{N}^x\mathcal{L}. \tag{7.8}$$

Thus, (7.8) causes conservation laws below:

Conservation laws for symmetry X_1

For symmetry X_1 in (4.2), the characteristic is $W^1 = -\theta_x$. Thus, using (7.8), symmetry X_1 gives the following conserved vectors:

$$C^t = \nu D_t^{\eta-1}(-\theta_x) + J(-\theta_x, D_t(\nu)),$$
$$C^x = (-\theta_x)(-D_x(\nu)) + D_x(-\theta_x)(\nu) = \theta_x(\nu_x + \nu_\theta\theta_x) - \theta_{xx}\nu.$$

Conservation laws for symmetry X_2

For generator X_2 in (4.2), the characteristic is $W = \theta(3\eta - 2) - 4t\theta_t - 2\eta x\theta_x$. Thus, using (7.8), conserved vectors are given by

$$C^t = \nu D_t^{\eta-1}\left(\theta(3\eta - 2) - 4t\theta_t - 2\eta x\theta_x\right) + J\big(\theta(3\eta - 2) - 4t\theta_t - 2\eta x\theta_x, D_t(\nu)\big),$$
$$C^x = (\theta(3\eta - 2) - 4t\theta_t - 2\eta x\theta_x)(-D_x(\nu)) + (\theta_x(3\eta - 2) - 4t\theta_{tx} - 2\eta\theta_x - 2\eta x\theta_{xx})(\nu).$$

Conservation laws for symmetry X_3

For Lie symmetry generator X_3 in (4.2), the characteristic is $W = \theta$. Therefore, conserved vectors are

$$C^t = \nu D_t^{\eta-1}(\theta) + J(\theta, D_t(\nu)),$$
$$C^x = (\theta)(-D_x(\nu)) + D_x(\theta)(\nu) = \theta(\nu_x + \nu_\theta\theta_x) - \theta_x\nu.$$

8. Conclusion

This paper has presented a method to determine Lie point symmetries of time-fractional PDEs with Riemann-Liouville fractional derivative. Furthermore, we showed the efficiency of the Lie symmetry method for solving this type of equation. The efficacy of this process was demonstrated via the time-fractional Zeldovich-Frank-Kamenetskii equation. Applying the obtained Lie point symmetries showed that the stated time-fractional PDE can be converted to a fractional ODE. Exact solutions of the Zeldovich-Frank-Kamenetskii equation were deduced wherever possible. In addition, power series solutions of the resulting fractional ODE (5.2) have also been established by using this method. Ultimately, we determined how conservation laws are derived for the proposed PDE by way of the Ibragimov conservation theorem. This method helps us establish conservation laws for FDEs with Riemann-Liouville derivatives of order $\eta \in (0, 1)$, which don't have Lagrangian in classical form. To summarize, the emergence of nonlinear FDEs as models in fields like mathematical medicine and biology necessitates an investigation into the methods of solving such equations. Our research was a step in this direction, and we hope it contributes to the development of solutions for such equations. Using Lie group analysis method can be favourably generalized to other FDEs and effectively employed to construct exact solutions for them.

References

[1] H. I. Abdel-Gawad and M. S. Osman, *On the variational approach for analyzing the stability of solutions of evolution equations*, Kyungpook Math. J., *53*(4) (2013), 661–680.

[2] A. D. Rawya, O. A. Arqub, M. Al-Smadi, and S. Momani. *Lie symmetry analysis, explicit solutions, and conservation laws of the time-fractional Fisher equation in two-dimensional space*, Journal of Ocean Engineering and Science, *7*(4) (2022), 345–352.

[3] K. K. Ali, M. A. Abd El Salam, E. M. H. Mohamed, B. Samet, S. Kumar, and M. S. Osman, *Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series*, Adv. Diff. Equ. *494* (2020).

[4] G. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, New York, 2002.

[5] P. Clavin and G. Searby, *Combustion Waves and Fronts in Flows: Flames, Shocks, Detonations, Ablation Fronts and Explosion of Stars*, Cambridge University Press, London, 2016.

[6] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition using Differential Operators of Caputo Type*, Springer Science & Business Media, 2010.

[7] S. Djennadi, N. Shawagfeh, M. Inc, M. S. Osman, J. F. Gömez-Aguilar, and O. Abu Arqub, *The Tikhonov regularization method for the inverse source problem of time fractional heat equation in the view of ABC-fractional technique*, Phys. Scr., *96* (2021), 9:094006.

[8] E. Bahi, M. Ihsane, and K. Hilal, *Lie Symmetry Analysis, Exact Solutions, and Conservation Laws for the Generalized Time-Fractional KdV-Like Equation*, Journal of Function Spaces 2021, *1* (2021), 6628130.

[9] R. K. Gazizov, N. H. Ibragimov, and S. Y. Lukashchuk, *Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations*, Commun. Nonlinear. Sci. Numer. Simulat., *23* (2015), 153–163.

[10] R. K. Gazizov, A. A. Kasatkin, and S. Y. Lukashchuk, *Continuous transformation groups of fractional differential equations*, Vestn. USATU., *9* (2007), 125–135.

[11] R. K. Gazizov, A. A. Kasatkin, and S. Y. Lukashchuk, *Symmetry properties of fractional diffusion equations*, Phys. Scr., *T136* (2009), 014016.

[12] Q. Huanga and R. Zhdanov, *Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann-Liouville derivative*, Physica A. *409* (2014), 110–118.

[13] N. H. Ibragimov, *A new conservation theorem*, J. Math. Anal. Appl., *333* (2007), 311–328.

[14] G. F. Jefferson and J. Carminati, *ASP: automated symbolic computation of approximate symmetries of differential equations*, Computer. Physics. Commun., *184* (2013), 1045–1063.

[15] G. F. Jefferson and J. Carminati, *Finding generalised symmetries of differential equations using the MAPLE package DESOLVII*, Computer. Physics. Commun., *183* (2012), 1044–1054.

[16] G. F. Jefferson and J. Carminati, *FracSym: Automated symbolic computation of Lie symmetries of fractional differential equations*, Computer. Physics. Commun., *185* (2014), 430–441.

[17] J. Samuel, K. U. Kristiansen, and P. Szmolyan. *Travelling Waves and Exponential Nonlinearities in the Zeldovich-Frank-Kamenetskii Equation*, arXiv preprint arXiv, *2405* (2024).

[18] S. Yu. Lukashchuk, *Conservation laws for time-fractional subdiffusion and diffusion-wave equations*, Nonlinear Dym., *80* (2015), 791–802.

[19] P. J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1986.

[20] M. S. Osman, *Multiwave solutions of time-fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equations*, Pramana J. Phys., *88*(4) (2017), 67–75.

[21] C. Park, R. I. Nurudden, K. K. Ali, L. Muhammad, M. S. Osman, and D. Baleanu, *Novel hyperbolic and exponential ansatz methods to the fractional fifth-order Korteweg-de Vries equations*, Adv. Diff. Equ. *627* (2020).

[22] I. Podlubny, *Fractional Differential Equations, An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*, Academic Press, San Diego, 1999.

[23] N. Raza, M. S. Osman, A. H. Abdel-Aty, S. Abdel-Khalek, and H. R. Besbes, *Optical solitons of space-time fractional Fokas-Lenells equation with two versatile integration architectures*, Adv. Diff. Equ. *517* (2020).

[24] R. Sahadevan and T. Bakkyaraj, *Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations*, J. Math. Anal. Appl., *393* (2012), 341–347.

[25] R. Sahadevan and P. Prakash, *Exact solution of certain time fractional nonlinear partial differential equations*, Nonlinear Dyn., *85* (2016), 659–673.

[26] S. Sait and E. Yaşar, *On the Lie symmetry analysis, analytic series solutions, and conservation laws of the time fractional Belousov–Zhabotinskii system*, Nonlinear Dynamics, *109*(4) (2022), 2997-3008.

[27] Z. Zhen, Z. Zou, E. Kuhl, and G. E. Karniadakis, *Discovering a reaction–diffusion model for Alzheimer's disease by combining PINNs with symbolic regression*, Computer Methods in Applied Mechanics and Engineering, *419* (2024), 116647.