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Enhancing the Legendre-Gauss-Radau pseudospectral method with sigmoid-based control parameterization for solving bang-bang optimal control problems

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Abstract

In bang-bang optimal control problems, the control function is inherently piecewise constant. This feature creates substantial difficulties for the standard Legendre-Gauss-Radau pseudospectral method, which relies on polynomial approximation for the control function. This study introduces a simplified approach that seamlessly integrates sigmoid-based control parameterization with the traditional Legendre-Gauss-Radau pseudospectral method. This integration enables precise approximation of discontinuous control profiles while maintaining the polynomial approximation for state variables. The proposed method significantly minimizes the number of decision variables in the optimization problem while precisely determining both the number and locations of switching points. This leads to notable enhancements in computational efficiency and solution accuracy. Numerical experiments conducted on two benchmark problems, a bridge crane system and a robotic arm control problem, demonstrate the exceptional precision and efficiency of the proposed method. Despite its simplicity, the method delivers results that are on par with those produced by more advanced and intricate techniques.

Keywords. Bang-bang optimal control problems, Legendre-Gauss-Radau pseudospectral methods, Sigmoid function.
2010 Mathematics Subject Classification. 49M25, 49M37, 65M70.

1. Introduction

Bang-bang optimal control problems, where the control switches between two extreme values, are prevalent in many engineering and scientific applications [4, 11, 36, 49]. These problems are especially important in fields such as aerospace [9, 12, 13, 26], where they are used to optimize fuel-efficient trajectories for spacecraft or aircraft, and in robotics [2, 39, 40], where they model minimum-time maneuvers for robotic arms and autonomous vehicles. Similarly, in economics and resource management, bang-bang controls are applied to determine optimal switching between different operational modes or investment strategies [31, 35].

The ability to accurately solve bang-bang optimal control problems is crucial for reducing operational costs and achieving accurate solutions. However, the discontinuous nature of the control variables poses significant challenges for numerical methods, making the development of robust and efficient solution techniques essential for advancing these applications. Various numerical methods have been developed to address these challenges, broadly categorized into indirect methods, direct methods, and hybrid approaches.

Indirect methods involve deriving the necessary conditions for optimality using the Pontryagin maximum principle [49], resulting in a boundary-value problem that is then solved numerically by collocation or (multiple) shooting methods [5, 32, 33, 41, 42, 48]. Direct methods, including direct collocation and pseudospectral methods [7, 9, 15, 23, 34, 50], direct shooting [3, 10, 16, 17, 27], and direct control parameterization [37], transform the optimal control problem into a nonlinear programming problem.

Direct and indirect methods for solving bang-bang optimal control problems often face significant challenges, including sensitivity to initial guesses, insufficient accuracy, and slow convergence. These difficulties are more prominent in indirect methods, primarily due to the discontinuous nature of the optimal control function. To overcome these

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challenges, various strategies have been developed. Some approaches focus on smoothing the control function to enhance numerical stability [8, 44, 52, 54], while others treat the switch points as decision variables to improve solution accuracy [45, 46, 51]. Additionally, certain techniques aim to mitigate the sensitivity to initial guesses [22, 30, 43]. Techniques involving mesh refinements have also been employed to tackle these problems effectively [1, 29, 47].

Direct methods are the most commonly used due to their straightforwardness, effectiveness, and efficiency. Within the family of direct methods, pseudospectral methods are recognized as some of the most accurate and frequently applied approaches. Several types of pseudospectral methods are employed in solving optimal control problems, including Legendre-Gauss-Lobatto, Gauss, and Legendre-Gauss-Radau (LGR) methods. Of these, the LGR pseudospectral method is considered one of the most advanced and widely utilized due to its exceptional accuracy and computational efficiency [19, 24, 53]. Moreover, recent advancements in LGR pseudospectral methods, including hp-LGR and hybrid techniques, enhance their accuracy and efficiency for solving bang-bang optimal control problems by effectively identifying switching points [14, 20, 21, 38, 47].

In this study, we utilize the LGR pseudospectral method as a foundational framework and introduce an innovative enhancement through sigmoid-based control parameterization. The sigmoid function, widely recognized in machine learning [28], maps input values onto a smooth curve between 0 and 1, facilitating gradual transitions and improved numerical stability. These attributes make it particularly suitable for approximating bang-bang control functions. In the proposed approach, state functions are approximated using the traditional LGR pseudospectral method. However, the control function is modeled as a combination of sigmoid functions, offering a smooth and adaptable representation. This enhancement allows the method to be classified as a smoothing technique. The key features of the proposed method are as follows:

- (i) Simplicity: The method does not rely on intricate hp scenarios or mesh refinement techniques. Instead, it extends the traditional LGR pseudospectral framework with a novel control approximation strategy.
- (ii) Reduced decision variables: The proposed method requires fewer decision variables compared to traditional and hp LGR pseudospectral methods, enhancing computational efficiency.
- (ii) Automatic detection: The method automatically identifies the number of switching times and the structure of the optimal control function without requiring prior knowledge.
- (iv) Accurate switching time estimation: While maintaining simplicity, the switching times are determined with an accuracy that is comparable to, but not necessarily superior to, more complex methods.

The limitation of the proposed method is that, due to its simplicity, it cannot detect the switching points with the same accuracy as state-of-the-art methods. However, it provides reasonable accuracy that is sufficient for practical applications.

The structure of the paper is as follows: In section 2, the general formulation of bang-bang optimal control problems is stated. Section 3 provides the necessary preliminaries for developing the proposed method. In section 4, the proposed method is introduced. Finally, the proposed method is demonstrated on two benchmark bang-bang problems in section 5, and the results are reported and compared with other methods.

2. Problem Formulation

The general form of a bang-bang optimal control problem is expressed as finding the control function $\mathbf{u}(t) = [u_1(t), \dots, u_q(t)]$ and the state function $\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]$ that minimize the following functional:

$$J = \theta(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \left[L_1(\mathbf{x}(t), t) + L_2(\mathbf{x}(t), t)\mathbf{u}(t)\right] dt, \tag{2.1a}$$

subject to the dynamics

$$\dot{\mathbf{x}}(t) = f_1(\mathbf{x}(t), t) + f_2(\mathbf{x}(t), t)\mathbf{u}(t), \tag{2.1b}$$

with the initial and terminal conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f, \ \texttt{or free}, \tag{2.1c}$$

and control box constraints

$$u_i^{\min} \le u_i(t) \le u_i^{\max}, \quad i = 1, \dots, q. \tag{2.1d}$$

The final time t_f may be fixed or free. It is important to note that the control function appears linearly in both the Lagrangian and the system dynamics. Consequently, due to the presence of box constraints, the optimal control is either bang-bang or singular [36, 49]. In this article, we consider cases where the problem is bang-bang and not singular.

3. Preliminaries

An introductory discussion is presented, laying the groundwork for the development of the proposed method. These preliminaries establish the essential concepts and tools required for constructing the approach.

3.1. **Legendre-Gauss-Radau points and quadrature.** Legendre-Gauss-Radau (LGR) points are a specialized set of collocation points derived from the orthogonal polynomial theory [18, 25]. They are widely utilized in numerical methods for solving optimal control problems. Mathematically, LGR points of order n are defined as the roots of the Radau polynomial $R(x) = P_{n-1}(x) + P_n(x)$, where P_n and P_{n-1} are the Legendre polynomials of orders n and $n-1$, respectively [18].

The LGR points are distributed in the interval $[-1, 1)$. To extend the definition of LGR points to any arbitrary interval $[a, b)$, a linear transformation is applied, mapping the LGR points from $[-1, 1)$ to $[a, b)$:

$$\tilde{x}_i = \frac{b-a}{2} z_i + \frac{b+a}{2},$$

where z_i are the original LGR points in the interval $[-1, 1)$.

LGR points also play a crucial role in numerical integration, particularly through LGR quadrature. This quadrature method provides an efficient approximation of integrals of the form

$$\int_{-1}^{1} f(t)\, dt \approx \sum_{j=0}^{n} w_j f(z_j),$$

where $w_j{}_{j=0}^{n}$ are the associated quadrature weights [18]. These weights are defined as

$$w_0 = \frac{2}{(n+1)^2}, \quad w_j = \frac{1 - z_j}{(n+1)^2 \left[P_n(z_j)\right]^2}, \ j = 1, \dots, n. \tag{3.1}$$

These weights ensure exactness for polynomials of degree $2n-2$ or less, making the LGR quadrature particularly accurate for smooth integrands. This feature underscores its utility in solving computational problems requiring precise numerical integration.

3.2. **Function approximation and derivative approximation by interpolation.** Let $f : [0, 1] \to \mathbb{R}$. For approximating f, we can use the interpolation polynomial of f based on some nodes in $[0, 1]$. Let $t_0, \dots, t_n$ be distinct nodes in $[0, 1]$. Then, in the interval $[0, 1]$, we can approximate f as

$$f(t) \simeq \sum_{j=0}^{n} a_j \ell_j(t), \tag{3.2}$$

where $a_j = f(t_j)$ and $\ell_j(t), j = 0, \dots, n$, are the Lagrange polynomials corresponding to points $t_0, \dots, t_n$ defined as

$$\ell_j(t) = \prod_{k=0, k\neq j}^{n} \frac{t - t_k}{t_j - t_k}, \quad j = 0, \dots, n.$$

It is worthwhile to note that the distribution of nodes $t_0, \dots, t_n$ can affect the accuracy and stability of the above approximation [18, 25]. In this way, LGL points (together with point 1) are a good choice for interpolation nodes.

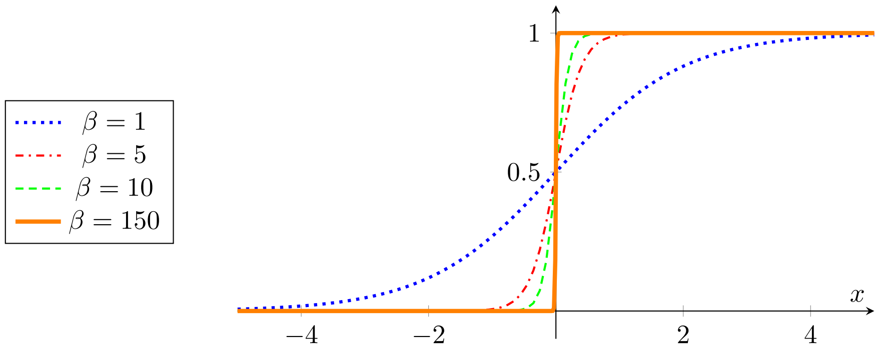

FIGURE 1. The sigmoid function with various values of β.

If f is differentiable and approximated as (3.2), then $\dot{f}$ could be approximated as

$$\dot{f}(t) \approx \sum_{j=0}^{n} b_j \ell_j(t), \quad b_j = \sum_{k=0}^{n} d_{jk} a_j, \tag{3.3}$$

where

$$d_{jk} = \begin{cases} \frac{\lambda_k}{\lambda_j (t_k - t_j)}, & j \neq k, \\ -\sum_{r=0, r\neq j}^{n} d_{jr}, & j = k, \end{cases} \qquad \lambda_j = \prod_{r=0, r\neq j}^{n} (t_j - t_r).$$

If the coefficients $a_0, \dots, a_n$ and $b_0, \dots, b_n$ are organized into vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, the relationship between $\mathbf{a}$ and $\mathbf{b}$ can be represented as

$$\mathbf{b} = \mathbf{D}\mathbf{a}, \tag{3.4}$$

where $\mathbf{D}$ is called the derivative matrix and defined as $[\mathbf{D}]_{jk} = d_{jk}$ [6, 55].

3.3. **The sigmoid function.** The sigmoid function is a smooth, differentiable mathematical function commonly used to approximate the step function [28]. Its general form is given by

$$\sigma(x) = \frac{1}{1 + e^{-\beta x}},$$

where $\beta > 0$ is a parameter that adjusts the transition between the two states of the step function. A larger value of β results in a sharper transition, effectively mimicking the discontinuities of a step function. Conversely, smaller values of β yield a smoother transition. To illustrate the sigmoid function, Figure 1 presents plots of the sigmoid function for $\beta = 1, 5$, and 10.

By shifting the argument of the sigmoid function, such as replacing x with $x - x_0$, it is possible to control the position of the step. This shift allows the step-like transition to occur at $x = x_0$ instead of at the origin. Adjusting x_0 provides additional flexibility in modeling scenarios where the step occurs at a specific location.

4. The Present Method

To simplify the presentation of the solution method for the optimal control problem (OCP) (2.1), we first transform the time domain from $[t_0, t_f]$ to $[0, 1]$. Additionally, the box bounds for u_i, where $i = 1, \dots, q$, are changed to -1 and $+1$. This transformation converts the OCP (2.1) into the following problem:

$$\min \quad J[\mathbf{v}] = \theta(\mathbf{y}(1), t_f) + (t_f - t_0) \int_0^1 \left[\hat{L}_1(\mathbf{y}(t), t) + \hat{L}_2(\mathbf{y}(t), t)\mathbf{v}(t) \right] dt, \tag{4.1a}$$

$$\text{subject to:} \;\; \dot{\mathbf{y}}(t) = (t_f - t_0)\hat{f}_1(\mathbf{y}(t), t) + (t_f - t_0)\hat{f}_2(\mathbf{y}(t), t)\mathbf{v}(t), \quad t \in [0, 1], \tag{4.1b}$$

$$\mathbf{y}(0) = \mathbf{x}_0, \quad \mathbf{y}(1) = \mathbf{x}_f, \;\; \texttt{or free}, \tag{4.1c}$$

$$-1 \leq v_i(t) \leq +1, \quad i = 1, \ldots, q, \quad t \in [0,1], \tag{4.1d}$$

where:

$$\begin{aligned}
&\mathbf{y}(t) = \mathbf{x}(t_0 + t(t_f - t_0)), \quad t \in [0,1],\\
&v_i(t) = \frac{1}{u_i^{\max} - u_i^{\min}} \left[u_i(t_0 + t(t_f - t_0)) - (u_i^{\max} + u_i^{\min}) \right], \quad t \in [0,1],\\
&\hat{L}_k(\mathbf{a}, t) = L_k(\mathbf{a}, t_0 + t(t_f - t_0)), \quad k \in \{1,2\}, \quad t \in [0,1],\\
&\hat{f}_k(\mathbf{a}, t) = f_k(\mathbf{a}, t_0 + t(t_f - t_0)), \quad k \in \{1,2\}, \quad t \in [0,1].
\end{aligned}$$

Note that in the OCP (4.1), the domain of the problem is independent of t_f, which simplifies the discretization, especially when t_f is free.

To discretize the OCP (4.1) using the pseudospectral method, we first need to approximate the state and control functions. The state functions are approximated using the traditional LGR method. However, in this paper, we introduce a new adaptive approximation for the control functions.

4.1. **Approximation of control functions.** We propose a sigmoid-based parameterization for $v_i(t)$ as

$$v_i(t) = \pm \prod_{j=1}^{m_i} \hat{\sigma}(\beta(t - s_{ij})), \quad i = 1, \ldots, q, \tag{4.2}$$

where β is a known parameter that adjusts the steepness of the transition and $s_{ij}, j = 1, \ldots, m_i$, are decision variables that are considered for the unknown switching times of v_i. We note that the value of $\prod_{j=1}^{m_i} \hat{\sigma}(\beta(t - s_{ij}))$ at $t = t_0$ is $(-1)^{m_i}$. However, to ensure that the value of v_i at t_0 is equal to -1 or $+1$, we use the $\pm$ symbol to achieve this.

By selecting a large enought β, this formulation provides smooth transitions that effectively approximate the ideal bang-bang profile with switching points $s_{ij}, j = 1, \ldots, m_i$. It is worthwhile to note that in the above approximation, the number of switching points, say m_i, is considered known. Moreover, the value of control at t_0 is not known. However, as we know, the number of switching points and the value of control at the initial time are not known *a priori*. This issue will be addressed in the proposed method. Accordingly, for now, in the approximation (4.2), we treat m_i as a fixed parameter.

The following points are some key considerations regarding the approximation (4.2), which are crucial for understanding the proposed method:

- We emphasize that approximation (4.2) serves a dual purpose: on one hand, it provides a smooth approximation, which is a valuable property for ensuring the stability and robustness of direct methods. On the other hand, it effectively captures the behavior of bang-bang functions.
- We note that $s_{ij}, j = 1, \ldots, m_i$, is considered as the switching points. These switching points may occur at any position. But we should force these switching points to lie in $[0,1]$. Accordingly, we consider the following conditions on s_{ij}:

$$0 \leq s_{ij} \leq 1, \quad i = 1, \ldots, q, \quad j = 1, \ldots, m_i. \tag{4.3}$$

- In the approximation (4.2), under the conditions (4.3), the number of switching points is not exactly equal to m_i. Instead, m_i represents the maximum possible number of switching points. For instance, if $s_{i1} = s_{i2}$, then s_{i1} and s_{i2} are not truly switching points. Additionally, if the conditions (4.3) are not imposed, some s_{ij} may lie outside the interval $[0,1]$. In this case, the number of distinct switching points within $[0,1]$ can be fewer than m_i. This characteristic of the approximation (4.2) is not a drawback; rather, it is a beneficial feature that can help identify the correct number of switching points.

4.2. **Approximation of state functions.** For a given integer $n > 1$, let $\tau_1, \dots, \tau_n$ represent the LGR nodes associated with the interval $[0, 1]$. Recall that $\tau_n < 1$. We extend the LGR nodes by adding $\tau_{n+1} = +1$ to them and define ℓ_j, $j = 1, \dots, n$, as the Legendre polynomials based on $\tau_1, \dots, \tau_{n+1}$. The state function $\mathbf{y}(t)$ is then approximated using the traditional LGR pseudospectral method as

$$\mathbf{y}(t) \simeq \sum_{j=1}^{n+1} \mathbf{a}_j \ell_j(t), \tag{4.4}$$

where $\mathbf{a}_j$, $j = 1, \dots, n$, are unknown p-vectors that represent the approximate values of $\mathbf{y}(t)$ at the LGR points $\tau_1, \dots, \tau_{n+1}$, i.e.

$$\mathbf{y}(\tau_j) \simeq \mathbf{a}_j, \quad j = 1, \dots, n. \tag{4.5}$$

Using the approximation (4.4) and derivative matrix, we can approximate $\dot{\mathbf{y}}(t)$ as:

$$\dot{\mathbf{y}}(t) \simeq \sum_{j=1}^{n+1} \mathbf{D}\mathbf{a}_j \ell_j(t), \tag{4.6}$$

and consequently

$$\dot{\mathbf{y}}(\tau_j) = \mathbf{D}\mathbf{a}_j, \quad j = 1, \dots, n. \tag{4.7}$$

4.3. **Problem discretization.** We are now at a position that can discretize the OCP (4.1) to a Non-Linear Programming (NLP). To this end, we begin with discretization of the performance index. By using LGR quadrature, the performance index (4.1a) is approximated as

$$J \simeq \theta(\mathbf{y}(1), t_f) + (t_f - t_0) \sum_{j=1}^{n} \left[\hat{L}_1(\mathbf{y}(\tau_j), \tau_j) + \hat{L}_2(\mathbf{y}(\tau_j), \tau_j)\mathbf{v}(\tau_j) \right].$$

Using (4.2) and (4.5), finally the performance index is converted to

$$J \simeq \theta(\mathbf{a}_{n+1}, t_f) + (t_f - t_0) \sum_{j=1}^{n} \left[\hat{L}_1(\mathbf{a}_j, \tau_j) + \hat{L}_2(\mathbf{a}_j), \tau_j)\mathbf{w}_j \right], \tag{4.8}$$

where

$$\mathbf{w}_j = \mathbf{v}(\tau_j) = \begin{bmatrix} \pm \prod_{k=1}^{m_1} \hat{\sigma}(\beta(\tau_j - s_{1k})) \\ \vdots \\ \pm \prod_{k=1}^{m_q} \hat{\sigma}(\beta(\tau_j - s_{qk})) \end{bmatrix} \in \mathbb{R}^q. \tag{4.9}$$

Note that the values of the state and control functions at the point $\tau_{n+1} = 1$ are not used in approximating the integral part of the performance index.

To discretize the dynamic equations, by evaluating (4.1b) at $t = \tau_j$ for $j = 1, 2, \dots, n$ and using (4.5), (4.7), and (4.9), we finally obtain

$$\mathbf{D}\mathbf{a}_j = (t_f - t_0)\hat{f}_1(\mathbf{a}_j, \tau_j) + (t_f - t_0)\hat{f}_2(\mathbf{a}_j, \tau_j)\mathbf{w}_j, \quad j = 1, \dots, n. \tag{4.10}$$

It is important to note that the dynamic constraints (4.1b) are not collocated at the endpoint $\tau_{n+1} = 1$.

The initial and possibly terminal conditions (4.1c) are discretized as

$$\mathbf{a}_1 = \mathbf{x}_0, \quad \mathbf{a}_{n+1} = \mathbf{x}_f \texttt{ or free}. \tag{4.11}$$

It should be noted that there is no need to explicitly discretize the box constraints (4.1d), as the control function approximation (4.2) inherently satisfies constraint (4.1d). However, we should consider the constraints (4.3).

Finally, the OCP (4.1) is discretized into the following NLP:

$$\begin{cases} \text{Find: } \mathbf{a}_j, \ j=1,\dots,n, \ \text{and } s_{ik}, \ i=1,\dots,q, \ k=1,\dots,m_i, \\ \text{to minimize the objective function(4.8)}, \\ \text{subject to the constraints (4.10), (4.11), and (4.3).} \end{cases} \tag{4.12}$$

4.4. **Selecting the number of switching times.** In the proposed method, the number of switching times in each control v_i, denoted by m_i, is assumed to be known. However, in practice, the number of switching times in each control is not known a priori. Determining the optimal number of switching times is, in fact, a key challenge in bang-bang optimal control problems.

To address this, it is necessary to devise an effective strategy for selecting the optimal number of switching times. The following two observations about the proposed method play a crucial role in developing such a strategy:

- If the number of switches in the presented method, say m_i, is set to be less than the optimal number, the corresponding NLP (4.12) may either become infeasible or yield a suboptimal solution.
- On the other hand, if the number of switches is assumed to be greater than the optimal number, some switching points may overlap in the solution, or they may appear with negligible differences due to rounding errors or solver precision.

Based on these observations, to determine the optimal number of switches, one can start with a large assumed number of switching points, solve the problem, and then analyze the solution to identify the correct number of switching points. This approach is further illustrated in Example 5.1.

5. Numerical Experiments

The proposed method in this paper has been implemented in Python, utilizing the Pyomo optimization library for formulating and for solving the optimization problem. The IPOPT solver was employed, which is well-suited for large-scale nonlinear optimization tasks. The experiments were conducted in a Google Colab environment with a standard Python configuration.

To evaluate the method and demonstrate its capabilities, two case studies are presented. These examples are used to assess the performance of the proposed approach and provide insights into its applicability and effectiveness in different settings.

Example 5.1 (The optimal control of a bridge crane system)**.** As the first example, the optimal control of a bridge crane system is considered. This problem models the motion of a crane carrying a load, where the goal is to transfer the load from an initial position to a specified final position in minimal time. The formulation of this problem is as follows

$$\min J[u] = t_f,$$

subject to:

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t), && x_1(0)=0, && x_1(t_f)=15, \\
&\dot{x}_2(t) = u(t), && x_2(0)=0, && x_2(t_f)=0, \\
&\dot{x}_3(t) = x_4(t), && x_3(0)=0, && x_3(t_f)=0, \\
&\dot{x}_4(t) = -0.98x_3(t)+0.1u(t), && x_4(0)=0, && x_4(t_f)=0, && -1 \le u(t) \le +1.
\end{aligned}$$

This problem is particularly challenging because of its nonlinear nature and the presence of three switching points. This problem is chosen because it was thoroughly analyzed in [51], where the switching points were precisely identified. Consequently, it serves as a suitable benchmark for assessing the accuracy of our proposed method in determining the switching points.

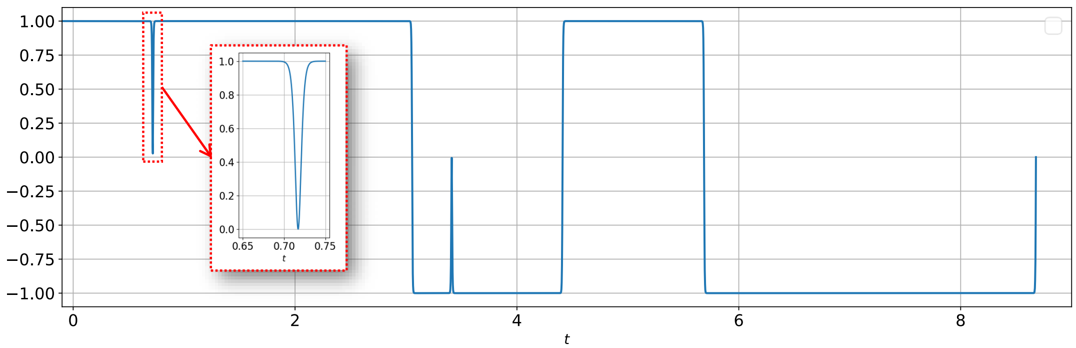

FIGURE 2. The control function obtained for the bridge crane problem using the proposed method with $m = 8$, $\beta = 400$, and $n = 150$.

Initially, to identify the structure and the number of switches in the solution, the proposed method is applied to the problem with $m = 8$ and the following control characterization:

$$u(t) = +\prod_{j=1}^{8} \hat{\sigma}(400(t - s_j)).$$

The following switching times are determined using the proposed method:

$$s_1 = 0.71707968,\ s_2 = 0.71707972,\ s_3 = 3.05483114,\ s_4 = 3.4120997,$$
$$s_5 = 3.41210003,\ s_6 = 4.4114645,\ s_7 = 5.68716613,\ s_8 = 8.67867246,$$
$$t_f = 8.67867237.$$

As we see, the switching times s_1 and s_2 are almost the same. Also, s_4 and s_5 are almost the same. In addition, s_8 and t_f are also almost the same. This result shows that s_1, s_2, s_4, s_5, and s_8 are not really switching points and are extra. Indeed, the control does not switch at these points. For better vision, in Figure 2, the obtained control is plotted.

Based on these results, we detect that the true number of switching times is 3. Moreover, the value of the control at the initial point $t = 0$ is $+1$. Consequently, the true approximation for the control function is given by

$$u(t) = -\prod_{j=1}^{3} \hat{\sigma}\left(400(t - s_j)\right).$$

In the next step, we apply the presented method to the bridge crane problem, incorporating the above control parameterization. Additionally, to demonstrate the impact of β, we perform the method for values of $\beta = 20, 50$, and 400. After applying the proposed method with the different values of β to this problem, the obtained control function is depicted in Figure 3.

Also, in Table 1, the obtained values for the switches (with $\beta = 400$) and the results of [51] are reported. As we can see, the switch values have a good match with the values obtained in [51]. In Figure 4, the obtained state functions by the method with $n = 150$ and $\beta = 400$ are plotted.

Example 5.2 (Robot arm problem)**.** Robot arm problems are a fundamental topic in robotics, focusing on optimizing the motion of robotic arms to perform tasks where precise and efficient motion planning is essential. These problems are typically formulated as optimal control or motion planning problems. Here, we examine a specific optimal control problem that involves minimizing the time t_f required for a robotic arm to move between two points in three-dimensional space as follows

$$\min J[u_1, u_2, u_3] = t_f,$$

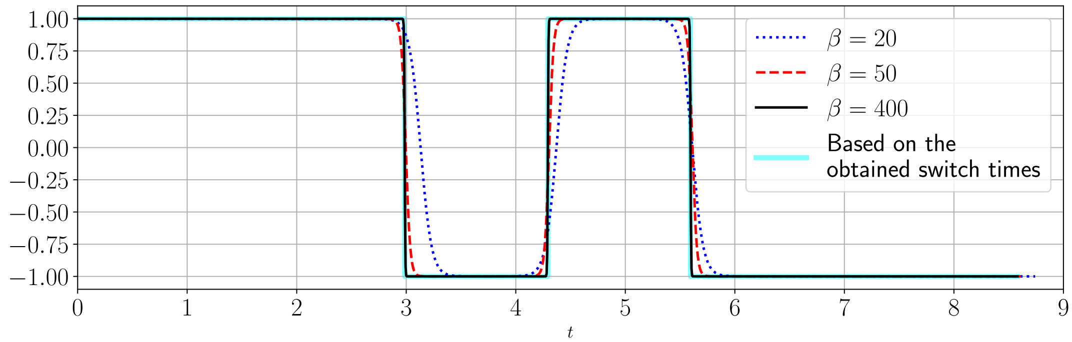

FIGURE 3. The optimal control functions obtained using the proposed method with $m = 3$, $n = 150$, and various values of $\beta = 20, 50$, and 400.

TABLE 1. Switching times and final time obtained using the proposed method with $\beta = 400$ for various values of n, compared with the results from [51] for the bridge crane problem.

n	s_1	s_2	s_3	t_f	CPU Time
50	3.0144397	4.3362332	5.6480158	8.5821765	0.086
100	2.9896016	4.3021411	5.6037928	8.5806433	0.198
150	2.9856529	4.2930080	5.5953170	8.5805476	0.431
200	2.9853438	4.2908345	5.5949048	8.5805439	2.314
250	2.9854319	4.2903836	5.5949980	8.5805436	3.982
300	2.9854751	4.2902938	5.5950415	8.5805436	6.228
350	2.9854862	4.2902761	5.5950526	8.5805436	7.867
400	2.9854883	4.2902727	5.5950547	8.5805436	16.684
[51]	2.98534	4.2900513	5.5947621	8.5801026	–

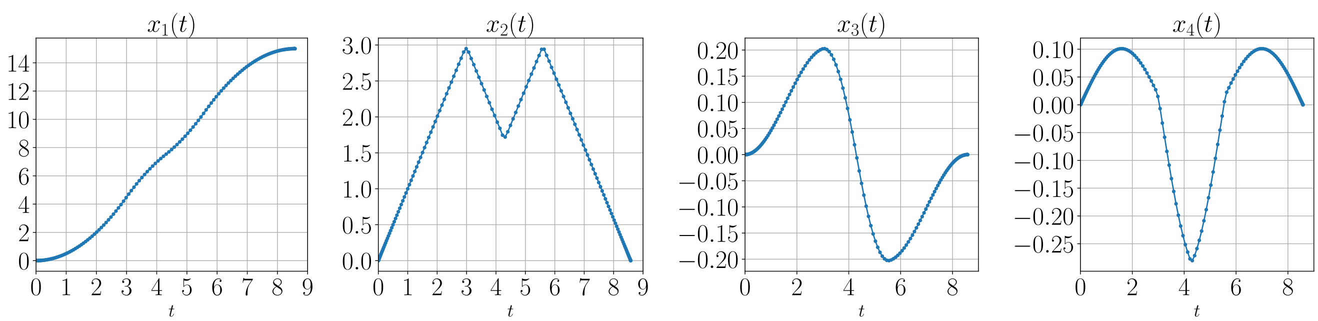

FIGURE 4. The state functions obtained using the proposed method with parameters $\beta = 400$ and $n = 150$.

$$
\begin{aligned}
\text{subject to:}\quad & \dot{x}_1(t) = x_2(t), && x_1(0) = \frac{9}{2}, \quad x_1(t_f) = \frac{9}{2},\\
& \dot{x}_2(t) = \frac{u_1(t)}{5}, && x_2(0) = 0, \quad x_2(t_f) = 0,\\
& \dot{x}_3(t) = x_4(t), && x_3(0) = 0, \quad x_3(t_f) = \frac{2\pi}{3},\\
& \dot{x}_4(t) = \frac{3u_2(t)}{((5 - x_1(t))^3 + x_1^3(t))\sin^2(x_5(t))}, && x_4(0) = 0, \quad x_4(t_f) = 0,
\end{aligned}
$$

$$\dot{x}_5(t) = x_6(t), \qquad x_5(0) = \frac{\pi}{4}, \quad x_5(t_f) = \frac{\pi}{4},$$
$$\dot{x}_6(t) = \frac{3u_3(t)}{((5 - x_1(t))^3 + x_1^3(t))}, \qquad x_6(0) = 0, \quad x_6(t_f) = 0,$$
$$-1 \le u_i(t) \le 1, \quad i = 1, 2, 3.$$

The robot arm problem exhibits a bang-bang control structure for all three control components, with two switching points for u_1, one for u_2, and two for u_3. The number of switching points, as well as the structure of the three controls, can be determined using the proposed method; however, the process is omitted here for brevity.

By applying the proposed method to the robot arm problem with $\beta = 400$ and $n = 150$, the obtained states and control functions are plotted in Figure 5. To evaluate the method's ability to capture switching times, the switching points obtained using the proposed method with $\beta = 400$ and $n = 100, 150$, and 200 are compared with the results from [47] in Table 2. As observed, our results match those of [47] with at least four correct significant digits.

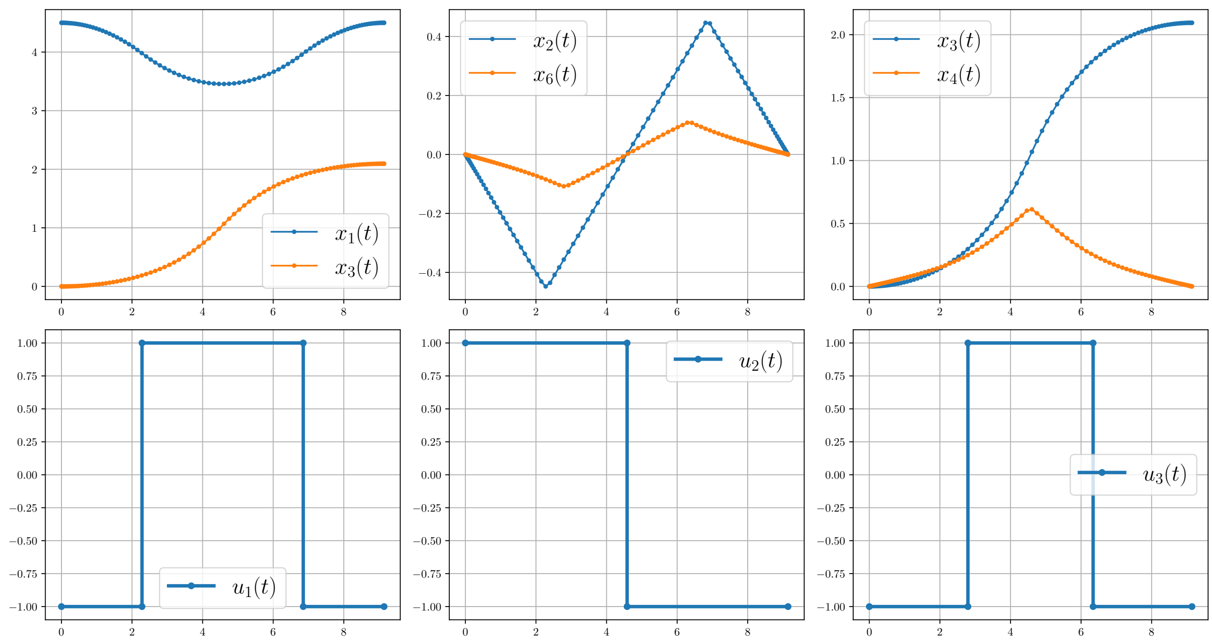

FIGURE 5. The state and control functions obtained using the proposed method with $\beta = 400$ and $n = 100$.

TABLE 2. The switching points and final time obtained using the proposed method with $\beta = 400$ for various values of n, compared with the results from [47] for the robot arm problem.

n	s_{11}	s_{12}	s_{21}	s_{31}	s_{32}	t_f
100	2.282646	6.851677	4.582725	2.793798	6.341290	9.142325
150	2.286757	6.857798	4.573584	2.797828	6.347573	9.142074
200	2.285411	6.856439	4.571556	2.795871	6.345702	9.142050
[47]	2.285228	6.855684	4.570456	2.796043	6.344869	9.140912

We note that our method is significantly simpler than the approach presented in [47] and does not require complex algorithms for mesh refinement. Despite its simplicity, our method achieves comparable results.

6. Conclusion

This study presents an improved Legendre-Gauss-Radau pseudospectral method that incorporates sigmoid-based parameterization for control functions in bang-bang optimal control problems. The proposed approach resolves the limitations of traditional methods in managing piecewise constant control profiles. The method reduces decision variables, leverages sigmoid functions for flexibility, and improves the detection of both the number and location of switching points. As part of future research, the proposed parameterization of the control function could be applied to other well-established methods, such as direct and indirect shooting methods. Additionally, this framework could be extended to more complex control systems, including singular optimal control problems.

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