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A computational iterative technique for a kind of nonlinear higher-order singular Emden-Fowler type equations

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Abstract

This paper examines the solutions of nonlinear higher-order singular Emden-Fowler type equations arising in various physical models. Generally, it becomes difficult to obtain the solution near the point of singularity. To overcome this problem, an iterative technique is introduced, that depends on the variational iteration method (VIM) and the homotopy perturbation method (HPM). Such a technique generates the solution in terms of a series, which is highly practical from computing perspective. An equivalent recursive integral representation (involving Lagrange's multiplier) for the higher-order nonlinear singular Emden-Fowler type (SEFT) equations with initial conditions (ICs) is established with the support of the variational iteration method (VIM). Making use of the concept of homotopy, a system of integral equations is established, which helps to deal with nonlinearity. Some numerical examples are studied through the proposed iterative technique to show the applicability and efficiency of the technique.

Keywords. Emden-Fowler equations, Variational iteration method, Convergence analysis, Lagrange's multiplier, Homotopy perturbation method, Initial value problems.

2010 Mathematics Subject Classification. 65L05, 65H20, 65L20.

1. Introduction

The second-order nonlinear singular differential equation identify below as the Lane-Emden-Fowler equation

$$\frac{1}{x^p}\left(\frac{d}{dx}\left(x^p\frac{du}{dx}\right)\right)+g(x,u)=0, \tag{1.1}$$

where $g(x,u)$ and p are a real valued continuous function and shape factor (a real number), respectively. Various physical phenomena emerging in the field of mathematical sciences, astrophysics, fluid mechanics and chemical physics can be modelled by using the Lane-Emden-Fowler differential equation [3, 4, 7, 9, 14, 15, 19, 21, 24, 41]. Astrophysicists J. H. Lane [15] and R. Emden [8] studied this equation. J.H. Lane initially presented the Poisson's equation as a dimensionless form that characterising the equilibrium density distribution within a self-gravitating polytrophic isothermal gas sphere and R. Emden later modified it to define polytropes in hydrostatic equilibrium (see [2] and references there in). Earlier hypotheses about gaseous dynamics in astrophysics are the basis for the study of the Lane-Emden-Fowler differential equation. Investigating the equilibrium structure of the mass of gaseous spherical clouds was the central issue in the study of stellar structure, assuming that convective equilibrium exists in the gaseous cloud.

The Lane-Emden-Fowler Equation (1.1) with suitable initial conditions is used to classify a wide range of physical processes, e.g., in chemical reacting system, fluid mechanics, population evolution, relativistic mechanics and in pattern formation [20, 42]. The form of such an equation for $g(x,u)=u^n$ describes the way a spherical cloud of gas reacts thermally (which is moving due to the molecules' mutual attraction) and where n stands for polytropic index, is popularly known as the Lane-Emden equation for $n=1.5$ and $n=2.5$. Moreover, it also appears in modeling of a

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cluster of galaxies, self-gravitating gas clouds, cases of radiatively cooling and in the treatment of a phase transition in critical adsorption [5, 19]. Meanwhile, the Emden-Fowler equation for $g(x,u) = e^u$ describes the non-dimensional density distribution in an isothermal gas sphere. There have been a number of physical models computed using the Lane-Emden equations, such as stellar structure [3], thermionic currents [24], thermal explosions (in different shape vessels) [4] and in the understanding of the thermal asymmetries of human heads [7].

The generalized form of the Lane-Emden-Fowler equation

$$x^{-p}\frac{d^m}{dx^m}\left(x^p\frac{d^k}{dx^k}\right)u + g(x,u) = 0, \tag{1.2}$$

where $p > 0$ is stated as the shape factor and the order of differential operators are defined by m and k. The major challenge in solving the Lane-Emden-Fowler equation is the singularity that occurs at $x = 0$, which prevents the convergence of direct implementation of analytical and numerical techniques. As a result, it always encourage the researchers to find solutions for such type of problems (see [1, 27, 28, 33]). In order to find the exact and approximate solutions to singular Emden-Fowler type (SEFT) Equations (1.2), a number of helpful techniques were employed, like, the Adomian decomposition method (ADM) [40], homotopy perturbation methods [23, 27], the rational Legendre pseudospectral technique [22], Haar wavelet collocation method [29, 30], variational iteration methods [35], non-standard finite difference method [32] and Bernstein collocation method [26], etc.

The purpose of this paper is to obtain the approximate solutions for the third and fourth-order nonlinear SEFT equations. Over the past few years, numerous analytical and numerical techniques have been developed for solving higher-order nonlinear SEFT equations. For instance, the Adomian decomposition method [34, 37], cubic B-spline method [13], reproducing kernel Hilbert space method [6], variational iteration method [38, 39], hybrid block techniques [25], Haar wavelet resolution method [31], quintic B-spline [17] and artificial neural network technique [36]. This study proposes a coupled iterative technique that is a combination of variational iteration method and homotopy perturbation method (VIM and HPM). Making use of the variational iteration method (VIM), we establish an equivalent recursive integral scheme involving Lagrange's multiplier. In order to deal with nonlinearity, a set of Volterra integral equations is built using the homotopy principle. Some numerical examples have been solved to show the proposed method's accuracy and ease for each kind of IVPs. The proposed coupled iterative technique provides the solution in series form.

The rest work is framed as follows: In section 2, the formulation of higher-order singular Emden-Fowler type equations is presented. A recursive scheme with the help of VIM for the nonlinear third and fourth-orders SEFT equations is discussed in section 3. The section 4 provides the basic concept of homotopy, which helps us to deal with the nonlinearity. The development of a coupled iterative technique using VIM and HPM for solving the higher-order SEFT equation is given in section 5. The sufficient conditions for the convergence of the proposed iterative technique are documented in section 6. In the section 7, numerical illustrations are presented to check the applicability and reliability of the technique. Lastly, a conclusion is provided for the proposed work in the last section.

2. Formulation of higher-order equations: SEFT

In this section, the nonlinear SEFT equations of higher-order specifically third and fourth-order are formulated. One can also derived the higher-order type equations in similar manner. The generalized Lane-Emden-Fowler equation with appropriate initial conditions is read as

$$x^{-p}\frac{d^m}{dx^m}\left(x^p\frac{d^k}{dx^k}\right)u + g(x,u(x)) = 0, \tag{2.1}$$

subject to initial conditions (ICs)

$$u(0) = u_0, \qquad u'(0) = u''(0) = u'''(0) = u^{m+k-1}(0) = 0, \tag{2.2}$$

where $' = \frac{d}{dx}$. The following subsections illustrate the derivation of different kinds of third and fourth-orders Emden-Fowler type equations.

2.1. **Formulation of SEFT equation: Third-order.** The nonlinear third-order SEFT equations for a variety of shape factors are formulated in this subsection. On the basis of different choices of m and k such that $m+k=3,\ m,k \geq 1$, we have the following two kinds of third-order SEFT equations:

(i) **1st kind third-order SEFT equation:**

$$\left.\begin{aligned} &\frac{d^3u}{dx^3}+\frac{2p}{x}\left(\frac{d^2u}{dx^2}\right)+\frac{p(p-1)}{x^2}\left(\frac{du}{dx}\right)+g(x,u)=0,\\ &\text{subject to ICs}\\ &u(0)=\alpha_0,\quad u'(0)=u''(0)=0, \end{aligned}\right\} \text{For m=2, k=1} \tag{2.3}$$

where α_0 is a constant and the singular point $x=0$ arise two times as x and x^2 with corresponding shape factors $2p$ and $p(p-1)$.

(ii) **2nd kind third-order SEFT equation:**

$$\left.\begin{aligned} &\frac{d^3u}{dx^3}+\frac{p}{x}\left(\frac{d^2u}{dx^2}\right)+g(x,u)=0,\\ &\text{subject to ICs}\\ &u(0)=\alpha_1,\quad u'(0)=u''(0)=0, \end{aligned}\right\} \text{For m=1, k=2} \tag{2.4}$$

where α_1 is a constant. In this case, the singular point present at $x=0$ with shape factor p and all remaining terms are vanished.

2.2. **Formulation of SEFT equation: Fourth-order.** The nonlinear fourth-order SEFT equations for a variety of shape factors are formulated in this subsection. On the basis of different choices of m and k such that $m+k=4,\ m,k \geq 1$, we have the following three kinds of fourth-order SEFT equations:

(i) **1st kind fourth-order SEFT equation:**

$$\left.\begin{aligned} &\frac{d^4u}{dx^4}+\frac{3p}{x}\left(\frac{d^3u}{dx^3}\right)+\frac{3p(p-1)}{x^2}\left(\frac{d^2u}{dx^2}\right)+\frac{p(p-1)(p-2)}{x^3}\left(\frac{du}{dx}\right)+g(x,u)=0,\\ &\text{subject to ICs}\\ &u(0)=\beta_0,\quad u'(0)=u''(0)=u'''(0)=0, \end{aligned}\right\} \text{For m=3, k=1,} \tag{2.5}$$

where β_0 is a constant and the singular point $x=0$ arise thrice as x, x^2 and x^3 with corresponding shape factors $3p$, $3p(p-1)$ and $p(p-1)(p-2)$.

(ii) **2nd kind fourth-order SEFT equation:**

$$\left.\begin{aligned} &\frac{d^4u}{dx^4}+\frac{2p}{x}\left(\frac{d^3u}{dx^3}\right)+\frac{p(p-1)}{x^2}\left(\frac{d^2u}{dx^2}\right)+g(x,u)=0,\\ &\text{subject to ICs}\\ &u(0)=\beta_1,\quad u'(0)=u''(0)=u'''(0)=0, \end{aligned}\right\} \text{For m=2, k=2,} \tag{2.6}$$

where β_1 is a constant and the singular point $x=0$ arise two times as x and x^2 with corresponding shape factors $2p$ and $p(p-1)$.

(iii) **3rd kind fourth-order SEFT equation:**

$$\left.\begin{aligned} &\frac{d^4u}{dx^4}+\frac{p}{x}\left(\frac{d^3u}{dx^3}\right)+g(x,u)=0,\\ &\text{subject to ICs}\\ &u(0)=\beta_2,\quad u'(0)=u''(0)=u'''(0)=0, \end{aligned}\right\} \text{For m=1, k=3,} \tag{2.7}$$

where β_2 is a constant. In this case, the singular point present at $x=0$ with shape factor p and all remaining terms are vanished.

3. Equivalent integral transformation: SEFT

This section provides an equivalent integral recursive transformation for nonlinear third and fourth-order SEFT equations with the help of variational iteration method (for more details, see [11, 38]).

We can rewrite the Emden-Fowler type equation as follow

$$L(u) + N(u) = h(x), \tag{3.1}$$

defining L and N as linear and nonlinear differential operators, while $h(x)$ is an inhomogeneous function. A correction functional (as per VIM) for the aforementioned Equation (3.1) is constructed as

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^{x} \psi(x,t)[Lu_n(t) + N\tilde{u}_n(t) - h(t)]dt, \tag{3.2}$$

where $\psi(x,t)$ is a Lagrange's multiplier and $\tilde{u}_n$ is a restricted variation, i.e., $\delta\tilde{u}_n = 0$ and making Eq. (3.2) stationary allows us to find the optimal value of Lagrange's multiplier.

3.1. **Third-order SEFT equation: VIM.** By using the similar analysis for Eq. (2.3), the recursive integral form (referred as correction functional) is expressed as

$$u_{n+1}(x) = u_n(x) + \int_0^x \psi(x,t)\left[\frac{d^3u_n(t)}{dt^3} + \frac{2p}{t}\left(\frac{d^2u_n(t)}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{du_n(t)}{dt}\right) + \tilde{g}(t,u_n(t))\right]dt. \tag{3.3}$$

Using the approach of [11], we achieve the following stationary conditions

$$\begin{cases} \psi(x,t)|_{t=x} = 0, \\ -\psi'(x,t) + \dfrac{2p}{t}\psi(x,t)\Big|_{t=x} = 0, \\ 1 + \psi''(x,t) - \dfrac{2p}{t}\psi'(x,t) + \dfrac{2p+p(p-1)}{t^2}\psi(x,t)\Big|_{t=x} = 0, \\ -\psi'''(x,t) + \dfrac{2p}{t}\psi''(x,t) - \dfrac{4p+p(p-1)}{t^2}\psi'(x,t) + \dfrac{4p+2p(p-1)}{t^3}\psi(x,t)\Big|_{t=x} = 0. \end{cases} \tag{3.4}$$

The Lagrange's multiplier is calculated using the aforementioned stationary conditions, which is as follows:

$$\psi(x,t) = \begin{cases} -xt + t^2\left(1 - \ln\dfrac{t}{x}\right), \quad for \quad p = 1, \\ -\dfrac{t^3}{x} + t^2\left(1 + \ln\dfrac{t}{x}\right), \quad for \quad p = 2, \\ -\dfrac{t^2}{(p-1)(p-2)} + \dfrac{x^2}{(p-2)}\left(\dfrac{t}{x}\right)^p - \dfrac{x^2}{(p-1)}\left(\dfrac{t}{x}\right)^{p+1}, \quad for \quad p \neq 1, 2. \end{cases} \tag{3.5}$$

From the above discussion, the recursive integral transformation for SEFT equation (2.3) using VIM is as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \psi(x,t)\left[\frac{d^3u_n(t)}{dt^3} + \frac{2p}{t}\left(\frac{d^2u_n(t)}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{du_n(t)}{dt}\right) + g(t,u_n(t))\right]dt, \tag{3.6}$$

where $\psi(x,t)$ is given by Eq. (3.5).

Remark 3.1. (i): A similar integral transformation can be established for the 2nd kind third-order SEFT equation (2.4).

(ii): Similarly, one can calculate the Lagrange's multiplier for SEFT equation (2.4) as

$$\psi(x,t)=\begin{cases} -t^2+xt\left(1+\ln\dfrac{t}{x}\right), & for \quad p=1,\\ -xt+t^2\left(1-\ln\dfrac{t}{x}\right), & for \quad p=2,\\ \dfrac{t^2}{(p-2)}-\dfrac{x^2}{(p-1)(p-2)}\left(\dfrac{t}{x}\right)^p-\dfrac{xt}{(p-1)}, & for \quad p\neq 1,2. \end{cases} \tag{3.7}$$

3.2. **Fourth-order SEFT equation: VIM.** For the 1st kind fourth-order SEFT (2.5), the recursive integral form (referred as correction functional) is expressed as

$$u_{n+1}\quad(x)=u_n(x)+\int_0^x\psi(x,t)\left[\frac{d^4u_n}{dt^4}+\frac{3p}{t}\left(\frac{d^3u_n}{dt^3}\right)+\frac{3p(p-1)}{t^2}\left(\frac{d^2u_n}{dt^2}\right)+\frac{p(p-1)(p-2)}{t^3}\left(\frac{du_n}{dt}\right)+\tilde{g}(t,u_n)\right]dt. \tag{3.8}$$

Using the approach of [11], we achieve the following stationary conditions

$$\begin{cases} \psi(x,t)|_{t=x}=0,\\ -\psi'(x,t)+\dfrac{3p}{t}\psi(x,t)\Big|_{t=x}=0,\\ \psi''(x,t)-\dfrac{3p}{t}\psi'(x,t)+\dfrac{3p^2}{t^2}\psi(x,t)\Big|_{t=x}=0,\\ 1-\psi'''(x,t)+\dfrac{3p}{t}\psi''(x,t)-\dfrac{3p(p+1)}{t^2}\psi'(x,t)+\dfrac{p(p+1)(p+2)}{t^3}\psi(x,t)\Big|_{t=x}=0.\\ \psi^{(iv)}(x,t)-\dfrac{3p}{t}\psi'''(x,t)+\dfrac{3p(p+2)}{t^2}\psi''(x,t)-\dfrac{p(p+1)(p+8)}{t^3}\psi'(x,t)+\dfrac{3p(p+1)(p+2)}{t^4}\psi(x,t)\Big|_{t=x}=0. \end{cases} \tag{3.9}$$

The Lagrange's multiplier is calculated using the aforementioned stationary conditions, which is as follows:

$$\psi(x,t)=\begin{cases} -\dfrac{1}{4}x^2t+xt^2+\dfrac{1}{2}t^3\left(\ln\dfrac{t}{x}-\dfrac{3}{2}\right), & for \quad p=1,\\ -\dfrac{1}{2}xt^2+\dfrac{1}{2}\left(\dfrac{t^4}{x}\right)-t^3\left(\ln\dfrac{t}{x}\right), & for \quad p=2,\\ -\dfrac{t^4}{x}+\dfrac{1}{4}\left(\dfrac{t^5}{x^2}\right)+\dfrac{1}{2}t^3\left(\ln\dfrac{t}{x}+\dfrac{3}{2}\right), & for \quad p=3,\\ \dfrac{x^3}{2(p-3)}\left(\dfrac{t}{x}\right)^p-\dfrac{t^3}{(p-1)(p-2)(p-3)}-\dfrac{x^3}{(p-2)}\left(\dfrac{t}{x}\right)^{p+1}+\dfrac{x^3}{2(p-1)}\left(\dfrac{t}{x}\right)^{p+2}, for\, p\neq 1,2,3. \end{cases} \tag{3.10}$$

From the above discussion, the recursive integral transformation for SEFT Equation (2.5) using VIM is as follows

$$\begin{aligned} u_{n+1}(x)=u_n(x)+\int_0^x\psi(x,t)\Bigg[&\frac{d^4u_n}{dt^4}+\frac{3p}{t}\left(\frac{d^3u_n}{dt^3}\right)\\ &+\frac{3p(p-1)}{t^2}\left(\frac{d^2u_n}{dt^2}\right)+\frac{p(p-1)(p-2)}{t^3}\left(\frac{du_n}{dt}\right)+g(t,u_n)\Bigg]dt. \end{aligned} \tag{3.11}$$

where $\psi(x,t)$ is given by Eq. (3.10).

Remark 3.2. **(i):** Similar integral transformations can be established for the 2nd and 3rd kinds of fourth-order SEFT equation.

(ii): The Lagrange's multiplier for the 2nd kind fourth-order SEFT Equation (2.6) is calculated as

$$\psi(x,t) = \begin{cases} -\frac{1}{2}x^2t + \frac{1}{2}t^3 - xt^2\left(\ln\frac{t}{x}\right), & for \quad p = 1, \\ -2t^3 + 2xt^2 + (xt^2 + t^3)\left(\ln\frac{t}{x}\right), & for \quad p = 2, \\ -\frac{1}{2}xt^2 + \frac{1}{2}\left(\frac{t^4}{x}\right) - t^3\left(\ln\frac{t}{x}\right), & for \quad p = 3, \\ \frac{-x^3}{(p-2)(p-3)}\left(\frac{t}{x}\right)^p + \frac{t^3}{(p-2)(p-3)} + \frac{x^3}{(p-1)(p-2)}\left(\frac{t}{x}\right)^{p+1} - \frac{xt^2}{(p-2)(p-1)}, & for\, p \neq 1,2,3. \end{cases} \tag{3.12}$$

(iii): The Lagrange's multiplier for the 3rd kind fourth-order SEFT Equation (2.7) is calculated as

$$\psi(x,t) = \begin{cases} -xt^2 + \frac{3}{4}x^2t + \frac{1}{4}t^3 + \frac{1}{2}x^2t\left(\ln\frac{t}{x}\right), & for \quad p = 1, \\ -\frac{1}{2}x^2t + \frac{1}{2}t^3 - xt^2\left(\ln\frac{t}{x}\right), & for \quad p = 2, \\ -\frac{1}{4}x^2t + xt^2 + \frac{1}{2}t^3\left(\ln\frac{t}{x} - \frac{3}{2}\right), & for \quad p = 3, \\ -\frac{t^3}{2(p-3)} + \frac{x^3}{(p-1)(p-2)(p-3)}\left(\frac{t}{x}\right)^p - \frac{x^2t}{2(p-1)} + \frac{xt^2}{(p-2)}, & for \quad p \neq 1,2,3. \end{cases} \tag{3.13}$$

4. Homotopy Perturbation Method (HPM): SEFT equation

This section deals with the basic concept of homotopy. A system of Volterra integral equations is constructed using the homotopy notion to address the nonlinearity. Homotopy is an important part of differential topology, and using the property of homotopy, any nonlinear problem can be transformed into a number of linear problems (see [10, 16, 18]).

We examine the following general differential equation as

$$L(u) + N(u) - g(r) = 0, \quad r \in \Omega, \tag{4.1}$$

$$\beta\left(u, \frac{\partial u}{\partial \xi}\right) = 0, \quad r \in \Upsilon, \tag{4.2}$$

where, L and N stands for linear and nonlinear differential operators, respectively. The boundary operator is symbolized by β and the analytical function is defined by $g(r)$, while Υ represents the boundary of the domain Ω.

Now a homotopy has been constructed by using the homotopy concept for Eq. (4.1) (for more details, see [12]), which gives,

$$\phi(w,r,q) = (1-q)[L(w) - L(u_0)] + q[L(w) + N(w) - g(r)] = 0, \quad r \in \Omega, \tag{4.3}$$

or equivalently

$$\phi(w,r,q) = L(w) - L(u_0) + qL(u_0) + q[N(w) - g(r)] = 0, \quad r \in \Omega, \tag{4.4}$$

where $u_0(x)$ is an initial estimate satisfying (4.2), while $q \in [0,1]$ is called an embedding parameter. From Eq. (4.4), we obtain

$$\phi(w,r,0) = L(w) - L(u_0) = 0, \tag{4.5}$$

$$\phi(w,r,1) = L(w) + N(w) - g(r) = 0, \tag{4.6}$$

whenever q changes from 0 to 1. This process is referred as deformation and $L(w) - L(u_0)$ and $L(w) + N(w) - g(r)$ are stated to as homotopic. Now, the solution of Eq. (4.4) is constructed as a power series of q (for q as a small parameter), i.e.,

$$w = w_0 + qw_1 + q^2w_2 + \ldots. \tag{4.7}$$

The approximate solution (in terms of series of $w_i's$) of Eq. (4.1) at $q = 1$ can be expressed as follows

$$u(r) = \lim_{q\to 1} w = w_0 + w_1 + w_2 + \dots. \tag{4.8}$$

Additionally, by using He's polynomials, the nonlinearity present in Eq. (4.1) can be expressed as below

$$N(w) = \sum_{i=0}^{\infty} q^i H_i = H_0 + qH_1 + q^2 H_2 + \dots, \tag{4.9}$$

where He's polynomials is denoted by $H_n's$ and is given by

$$H_n(w_0, w_1, w_2, \dots, w_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} N\left(\sum_{i=0}^{n} q^i w_i\right)\Bigg|_{q=0}, \text{ where } \quad n = 0, 1, 2, \dots. \tag{4.10}$$

By combining the values from Equations (4.7) and (4.9) into Eq. (4.4) and equalized various power of q, we get

$$\begin{cases} q^0 : L(w_0) - L(u_0) = 0, \\ q^1 : L(w_1) + L(u_0) + H_0 - g(r) = 0, \\ q^2 : L(w_2) + H_1 = 0, \\ q^3 : L(w_3) + H_2 = 0, \\ \vdots \\ q^{n+1} : L(w_{n+1}) + H_n = 0, \\ \vdots \end{cases} \tag{4.11}$$

Now, by making use of the above system (4.11), the solution for the nonlinear differential equation (4.1) is obtained in series form i.e. $\sum_{i=0}^{\infty} w_i$.

Note: The above concept of homotopy is followed by the third and fourth-order nonlinear SEFT equations and a equivalent system of equations is derived for each kind of differential equations discussed in Section 2.

5. Coupled iterative scheme: SEFT equation

A coupled iterative scheme for nonlinear third-order (or similar for fourth-order) SEFT Equation (2.3) is derived in this section. We construct a homotopy for the recursive integral transformation of SEFT equation obtained in section 3 and generate a series solution for the problem.

From Eq. (3.6), we have the following iterative scheme for SEFT Equation (2.3)

$$u_{n+1}(x) = u_n(x) + \int_0^x \psi(x,t)\left[\frac{d^3u_n(t)}{dt^3} + \frac{2p}{t}\left(\frac{d^2u_n(t)}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{du_n(t)}{dt}\right) + N(u_n(t))\right]dt, \tag{5.1}$$

where $N(u_n)$ is the nonlinear term present in the differential equation. We construct a homotopy (making use of the same steps as we did in the previous section) for Eq. (5.1) as

$$\phi(w,x,q) = (1-q)[u_0 - w] + q\int_0^x \psi(x,t)\left[\frac{d^3w_n}{dt^3} + \frac{2p}{t}\left(\frac{d^2w_n}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{dw_n}{dt}\right) + N(w_n)\right]dt, \tag{5.2}$$

where q is an embedding parameter, while $u_0(x)$ is an initial estimate that meets the ICs for Eq. (2.3). From Eq. (5.2), we get

$$\phi(w,x,0) = w - u_0 = 0, \tag{5.3}$$

$$\phi(w,x,1) = \int_0^x \psi(x,t)\left[\frac{d^3w_n}{dt^3} + \frac{2p}{t}\left(\frac{d^2w_n}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{dw_n}{dt}\right) + N(w_n)\right]dt = 0. \tag{5.4}$$

Now, we represent the $w(x,q)$ in terms of embedding parameter (q) as power series, i.e.,

$$w(x,q)=\sum_{i=0}^{\infty} q^i w_i. \tag{5.5}$$

Also, decompose the nonlinear term using He's polynomials as

$$N(w)=\sum_{i=0}^{\infty} q^i H_i. \tag{5.6}$$

By merging the values from Equations (5.5) and (5.6) into Eq. (5.2) and equalizing the coefficient of similar power of q, we get

$$\begin{cases} q^0: w_0 = u_0, \\ q^1: w_1 = w_0 - u_0 + \int_0^x \psi(x,t)\left[\frac{d^3w_0}{dt^3}+\frac{2p}{t}\left(\frac{d^2w_0}{dt^2}\right)+\frac{p(p-1)}{t^2}\left(\frac{dw_0}{dt}\right)+H_0\right]dt, \\ q^2: w_2 = w_1 + \int_0^x \psi(x,t)\left[\frac{d^3w_1}{dt^3}+\frac{2p}{t}\left(\frac{d^2w_1}{dt^2}\right)+\frac{p(p-1)}{t^2}\left(\frac{dw_1}{dt}\right)+H_1\right]dt, \\ q^3: w_3 = w_2 + \int_0^x \psi(x,t)\left[\frac{d^3w_2}{dt^3}+\frac{2p}{t}\left(\frac{d^2w_2}{dt^2}\right)+\frac{p(p-1)}{t^2}\left(\frac{dw_2}{dt}\right)+H_2\right]dt, \\ \vdots \\ q^{n+1}: w_{n+1} = w_n + \int_0^x \psi(x,t)\left[\frac{d^3w_n}{dt^3}+\frac{2p}{t}\left(\frac{d^2w_n}{dt^2}\right)+\frac{p(p-1)}{t^2}\left(\frac{dw_n}{dt}\right)+H_n\right]dt, \\ \vdots \end{cases} \tag{5.7}$$

Lastly, we solve the above system of Equation (5.7) to obtain the series solution of Emden-Fowler type Equation (2.3) as $u=\lim_{n\to\infty} u_n = \sum_{i=0}^{\infty} w_i$.

Remark 5.1. A similar finding can be established for each kind of fourth-order SEFT equations as well as for 2nd kind of third-order SEFT equations.

6. Convergence Analysis

This section includes the convergence analysis of the proposed technique. We prove the existence and uniqueness of the solution of nonlinear third-order SEFT Equation (2.3) and demonstrates the efficiency and reliability of the proposed method. Additionally, we demonstrate that the sequence $u_n(x)$ converges to the solution of SEFT problem (2.3).

Theorem 6.1. *The nonlinear SEFT problem (2.3) subject to initial conditions has a unique solution, under the following assumptions*

(a) the norm

$$\|u\| = \max_{x\in[0,1]} |u(x)|, \qquad u \in \mathbb{X};$$

(b) $\exists$ a Lipschitz constant $0 \le k_0 < 3p^2+9p+6$, such that $\forall\ g(x,v_1), g(x,v_2) \in D$,

$$|g(x,v_1)-g(x,v_2)| \le k_0|v_1-v_2|; \tag{6.1}$$

where $\mathbb{X}=C[0,1]$ is a Banach space and $D=\{(x,v)\in[0,1]\times\mathbb{R}\}$.

Proof. Suppose that u_1 and u_2 such that $u_1 \neq u_2$ are two solutions of a nonlinear SEFT equation (2.3), so both of them will follow the Eq. (5.4)

$$\phi(u_1,x,1) = \int_0^x \psi(x,t)\left(\frac{d^3u_1}{dt^3} + \frac{2p}{t}\left(\frac{d^2u_1}{dt^2}\right) + \frac{p(p-1)}{t^2}\left(\frac{du_1}{dt}\right) + g(t,u_1(t))\right)dt = 0. \tag{6.2}$$

Making use of integration by parts three times and using the stationary conditions, we get

$$u_1(x) = w_0 - \int_0^x \psi(x,t)g(t,u_1)dt. \tag{6.3}$$

Similarly,

$$u_2(x) = w_0 - \int_0^x \psi(x,t)g(t,u_2)dt. \tag{6.4}$$

From Equations (6.3)-(6.4), we have

$$|u_1 - u_2| = \left|\int_0^x \psi(x,t)(g(t,u_1) - g(t,u_2))dt\right|,$$

$$\max_{x\in[0,1]} |u_1 - u_2| = \max_{x\in[0,1]} \left|\int_0^x \psi(x,t)\left(g(t,u_1) - g(t,u_2)\right)dt\right|,$$

$$= \max_{t\in[0,1]} |g(t,u_1) - g(t,u_2)| \max_{x\in[0,1]} \left|\int_0^x \psi(x,t)dt\right|,$$

$$\|u_1 - u_2\| \leq \frac{k_0}{3p^2+9p+6}\|u_1 - u_2\|.$$

Thus, we have

$$\|u_1 - u_2\| \leq \alpha \|u_1 - u_2\|,$$

where $\alpha = \frac{k_0}{3p^2+9p+6} < 1$. This proves the theorem.

Theorem 6.2. *The nonlinear SEFT Equation (2.4) subject to ICs has a unique solution, under the following assumptions*

(i) the norm

$$\|u\| = \max_{x\in[0,1]} |u(x)|, \qquad u \in \mathbb{X};$$

(ii) $\exists$ a Lipschitz constant $0 \leq k_0 < 6p+6$, such that $\forall\ g(x,v_1), g(x,v_2) \in D$

$$|g(x,v_1) - g(x,v_2)| \leq k_0|v_1 - v_2|; \tag{6.5}$$

where $\mathbb{X} = C[0,1]$ is a Banach space and $D = \{(x,v) \in [0,1]\times\mathbb{R}\}$.

Proof. The proof follows the same procedure as in Theorem 6.1.

6.1. **Convergence analysis: SEFT equation.** In this subsection, we show the convergence of the proposed technique for nonlinear SEFT Equation (2.3). Using Eq. (3.6) and stationary conditions (3.4), we determine

$$u_{n+1}(x) = u_n(x) + \int_0^x \left[\left(-\psi'(x,t) + \frac{2p\psi(x,t)}{t}\right)\frac{d^2u_n}{dt^2} + \frac{p(p-1)}{t^2}\left(\frac{du_n}{dt}\right) + g(t,u_n)\right]dt. \tag{6.6}$$

Consequently, we could compose

$$u_n(x) = u_{n-1}(x) + \int_0^x \left[\left(-\psi'(x,t) + \frac{2p\psi(x,t)}{t}\right)\frac{d^2u_{n-1}}{dt^2} + \frac{p(p-1)}{t^2}\left(\frac{du_{n-1}}{dt}\right) + g(t,u_{n-1})\right]dt. \tag{6.7}$$

Furthermore,

$$|u_{n+1}-u_n| = \left|\int_0^x \psi(x,t)(g(t,u_n)-g(t,u_{n-1}))dt\right|, \tag{6.8}$$

or

$$\max_{x\in[0,1]}|u_{n+1}-u_n| = \max_{x\in[0,1]}\left|\int_0^1 \psi(x,t)(g(t,u_n)-g(t,u_{n-1}))dt\right|, \tag{6.9}$$

$$\leq \max_{t\in[0,1]}|g(t,u_n)-g(t,u_{n-1})| \max_{x\in[0,1]}\left|\int_0^1 \psi(x,t)dt\right|. \tag{6.10}$$

Given that $g(x,u)$ meets the Lipschitz condition, we obtain

$$\| u_{n+1}-u_n \| \leq k_0 \max_{t\in[0,1]}|u_n-u_{n-1}| \max_{x\in[0,1]}\left|\int_0^1 \psi(x,t)dt\right|, \tag{6.11}$$

$$= \frac{k_0}{3p^2+9p+6} \| u_n - u_{n-1} \|, \tag{6.12}$$

$$\leq \alpha \| u_n - u_{n-1} \|, \tag{6.13}$$

where $\alpha = \frac{k_0}{3p^2+9p+6} < 1$.

Hence, we obtain

$$\| u_{n+1}-u_n \| \leq \alpha \| u_n - u_{n-1} \| . \tag{6.14}$$

□

Remark 6.3. The similar analysis is followed for the 2nd kind of third-order nonlinear singular Emden-Fowler Equation (2.4).

Theorem 6.4. *Suppose $w_n(x), u(x) \in \mathbb{X}$. Additionally, we assume that $\| w_0 \| < \infty$, then there holds $\| w_{n+1} \| \leq \alpha \| w_n \|$, $\alpha < 1$, where $n = 0, 1, 2, \ldots$, and the sequence $u_n (= \sum_{i=0}^{n} w_i)$ approaches to the solution of nonlinear singular Emden-Fowler type problem.*

Proof. The sequence $\{u_n\}$ can be expressed into partial sum as follows

$$\begin{cases} u_1 = w_0 + w_1, \\ u_2 = w_0 + w_1 + w_2, \\ \vdots \\ u_n = w_0 + w_1 + w_2 + \ldots + w_n, \\ \vdots \end{cases} \tag{6.15}$$

From Eq. (6.15), we can write

$$w_{n+1} = u_{n+1} - u_n, \quad n = 1, 2, 3, \ldots.$$

By use of Eq. (6.14), we get

$$\| w_{n+1} \| = \| u_{n+1} - u_n \| \leq \alpha \| u_n - u_{n-1} \| = \alpha \| w_n \| .$$

Hence, we have

$$\| u_{n+1} - u_n \| = \| w_{n+1} \| \leq \alpha \| w_n \| \leq \alpha^2 \| w_{n-1} \| \leq \cdots \leq \alpha^{n+1} \| w_0 \| .$$

Now, we use Cauchy criterion to show the convergence of sequence $\{u_n\}$ as

$$\begin{aligned} \| u_n - u_m \| &= \| (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \cdots + (u_{m+1} - u_m) \| \\ &\leq \| (u_n - u_{n-1}) \| + \| (u_{n-1} - u_{n-2}) \| + \cdots + \| (u_{m+1} - u_m) \| \end{aligned}$$

$$\leq \alpha^n \parallel w_0 \parallel + \alpha^{n-1} \parallel w_0 \parallel + \cdots + \alpha^{m+1} \parallel w_0 \parallel$$
$$\leq \alpha^{m+1} \left[1 + \alpha + \alpha^2 + \cdots + \alpha^{n-(m+1)} \right] \parallel w_0 \parallel$$
$$\leq \frac{\alpha^{m+1}(1-\alpha^{n-m})}{1-\alpha} \parallel w_0 \parallel .$$

Since $0 < \alpha < 1$ and $\parallel w_0 \parallel < \infty$, which gives

$$\parallel u_n - u_m \parallel \leq \frac{\alpha^{m+1}}{1-\alpha} \parallel w_0 \parallel . \tag{6.16}$$

Taking the limit as $m \to \infty$, we have $\parallel u_n - u_m \parallel \to 0$. Therefore, $\{u_n\}$ is a Cauchy sequence in the Banach space $\mathbb{X}$ and as a result the series $\sum_{i=0}^{n} w_i$ is convergent. □

Similarly, we can establish the convergence analysis for fourth-order SEFT equation.

7. Numerical Illustrations

This section documents the accuracy and applicability of the proposed coupled iterative technique. We tested the proposed technique on some numerical examples of third and fourth-order nonlinear SEFT equations of each kind with suitable initial conditions (ICs).

Example 7.1. Consider the following 1st kind nonlinear third-order SEFT equation with shape factor $p = 3$ as

$$\frac{d^3u}{dx^3} + \frac{6}{x}\left(\frac{d^2u}{dx^2}\right) + \frac{6}{x^2}\left(\frac{du}{dx}\right) - 6(10 + 2x^3 + x^6)e^{-3u} = 0, \tag{7.1}$$

subject to ICs

$$u(0) = 0, \quad u'(0) = u''(0) = 0,$$

where $g(x, u) = -6(10 + 2x^3 + x^6)e^{-3u}$. The Lagrange's multiplier for the problem (7.1) is calculated by (3.5), which is defined as

$$\psi(x,t) = \frac{-t^2}{2} - \frac{1}{2}\left(\frac{t^4}{x^2}\right) + \frac{t^3}{x}. \tag{7.2}$$

Choosing an initial approximation as $u_0(x) = 0$ that meets the initial conditions. Applying the system of Equations (5.7), we obtain the values of $w_1, w_2, w_3, \ldots,$ as

$$\begin{cases} w_0 = 0, \\ w_1 = x^3 + \dfrac{x^6}{28} + \dfrac{x^9}{165}, \\ w_2 = -\dfrac{15x^6}{28} - \dfrac{3x^9}{70} - \dfrac{523x^{12}}{56056} - \dfrac{13x^{15}}{61600} + \ldots, \\ w_3 = \dfrac{57x^9}{154} + \dfrac{9x^{12}}{196} + \dfrac{402069x^{15}}{34652800} + \dfrac{15237x^{18}}{26626600} + \ldots, \\ w_4 = -\dfrac{2295x^{12}}{8008} - \dfrac{140751x^{15}}{2932160} - \dfrac{195616251x^{18}}{14484870400} - \dfrac{23440386933x^{21}}{22446117293600} + \ldots, \\ \vdots \end{cases} \tag{7.3}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \frac{x^{15}}{5} + \ldots, \tag{7.4}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = \ln(1 + x^3)$, which also meets with the exact solution.

Example 7.2. Consider the following 2nd kind nonlinear third-order SEFT equation with shape factor $p = 1$ as

$$\begin{aligned} &\frac{d^3u}{dx^3} + \frac{1}{x}\left(\frac{d^2u}{dx^2}\right) + 4x(9 + 22x^4 + x^8)e^{-3u} = 0, \\ &\text{subject to ICs} \\ &u(0) = 0, \quad u'(0) = u''(0) = 0, \end{aligned} \tag{7.5}$$

where $g(x, u) = 4x(9 + 22x^4 + x^8)e^{-3u}$. The Lagrange's multiplier for problem (7.5) is calculated by Eq. (3.7), which is defined as

$$\psi(x, t) = -t^2 + xt\left(1 + \ln\frac{t}{x}\right). \tag{7.6}$$

Choosing an initial approximation as $u_0(x) = 0$ that meets the initial conditions. Applying the system of equations (5.7), we obtain the values of $w_1, w_2, w_3, \ldots,$ as

$$\begin{cases} w_0 = 0, \\ w_1 = -x^4 - \dfrac{11x^8}{49} - \dfrac{x^{12}}{363}, \\ w_2 = -\dfrac{27x^8}{98} - \dfrac{107x^{12}}{539} - \dfrac{5893x^{16}}{296450} - \dfrac{461x^{20}}{972895} + \ldots, \\ w_3 = -\dfrac{783x^{12}}{5929} - \dfrac{43x^{16}}{275} - \dfrac{477024132x^{20}}{13109760125} - \dfrac{378413341x^{24}}{126092056475} + \ldots, \\ w_4 = -\dfrac{43731x^{16}}{592900} - \dfrac{141862761x^{20}}{1191796375} - \dfrac{649558629403x^{24}}{13870126212250} - \dfrac{2496846903909956x^{28}}{334649280166133625} + \ldots, \\ \vdots \end{cases} \tag{7.7}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = -x^4 - \frac{x^8}{2} - \frac{x^{12}}{3} - \frac{x^{16}}{4} - \frac{x^{20}}{5} + \ldots, \tag{7.8}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = \ln(1 - x^4)$, which also meets with the exact solution.

Example 7.3. Consider the following 1st kind nonlinear third-order SEFT equation with shape factor $p = 1$ as

$$\begin{aligned} &\frac{d^3u}{dx^3} + \frac{2}{x}\left(\frac{d^2u}{dx^2}\right) - \frac{9}{8}(8 + x^6)u^{-5} = 0, \\ &\text{subject to ICs} \\ &u(0) = 1, \quad u'(0) = u''(0) = 0, \end{aligned} \tag{7.9}$$

where $g(x, u) = -\frac{9}{8}(8 + x^6)u^{-5}$. The Lagrange's multiplier for problem (7.9) is calculated by Eq. (3.5), which is defined as

$$\psi(x, t) = -xt + \left(1 - \ln\frac{t}{x}\right)t^2. \tag{7.10}$$

Choosing an initial approximation as $u_0(x) = 1$ that meets the initial conditions. Applying the system of equations (5.7), we obtain the values of $w_1, w_2, w_3, \dots,$ as

$$\begin{cases} w_0 = 1, \\ w_1 = \dfrac{x^3}{2} + \dfrac{x^9}{576}, \\ w_2 = -\dfrac{x^6}{8} - \dfrac{185x^{12}}{101376} - \dfrac{5x^{18}}{2820096}, \\ w_3 = \dfrac{35x^9}{576} + \dfrac{11801x^{15}}{7096320} + \dfrac{110431x^{21}}{24320507904} + \dfrac{305x^{27}}{95025954816}, \\ w_4 = -\dfrac{3775x^{12}}{101376} - \dfrac{1287271x^{18}}{868589568} - \dfrac{34273759x^{24}}{4474973454336} - \dfrac{53853419x^{30}}{4097519171665920} + \dots, \\ \vdots \end{cases} \tag{7.11}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \dots, \tag{7.12}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = \sqrt{1 + x^3}$, which also meets with the exact solution.

Example 7.4. Consider the following 2nd kind linear third-order SEFT equation with shape factor $p = 2$ as

$$\begin{aligned} &\frac{d^3u}{dx^3} + \frac{2}{x}\left(\frac{d^2u}{dx^2}\right) - \frac{25x^2}{8}(16 + 52x^2 + 7x^{10})u^7 = 0, \\ &\text{subject to ICs} \\ &u(0) = 1, \quad u'(0) = u''(0) = 0, \end{aligned} \tag{7.13}$$

where $g(x,u) = -\frac{25x^2}{8}(16 + 52x^2 + 7x^{10})u^7$. The Lagrange's multiplier for problem (7.13) is calculated by (3.7) as

$$\psi(x,t) = -xt + \left(1 - \ln\frac{t}{x}\right)t^2. \tag{7.14}$$

Choosing an initial approximation as $u_0(x) = 1$ that meets the initial conditions. Applying the system of equations (5.7), we obtain the values of $w_1, w_2, w_3, \dots,$ as

$$\begin{cases} w_0 = 1, \\ w_1 = \dfrac{x^5}{2} + \dfrac{13x^{10}}{72} + \dfrac{x^{15}}{144}, \\ w_2 = \dfrac{7x^{10}}{36} + \dfrac{65x^{15}}{324} + \dfrac{91x^{20}}{2432} + \dfrac{91x^{25}}{38400} + \dots, \\ w_3 = \dfrac{17x^{15}}{162} + \dfrac{455x^{20}}{2592} + \dfrac{822311x^{25}}{11819520} + \dfrac{3627533x^{30}}{321343200} + \dots, \\ w_4 = \dfrac{5957x^{20}}{98496} + \dfrac{107653x^{25}}{777600} + \dfrac{551358157x^{30}}{6169789440} + \dfrac{6962482397x^{35}}{271927756800} + \dots, \\ \vdots \end{cases} \tag{7.15}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = 1 + \frac{x^5}{2} + \frac{3x^{10}}{8} + \frac{5x^{15}}{16} + \frac{35x^{20}}{128} + \dots, \tag{7.16}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = \frac{1}{\sqrt{1-x^5}}$, which also meets with the exact solution.

Example 7.5. Consider the following 1st kind nonlinear fourth-order SEFT equation with shape factor $p = 4$ as

$$\frac{d^4u}{dx^4} + \frac{12}{x}\left(\frac{d^3u}{dx^3}\right) + \frac{36}{x^2}\left(\frac{d^2u}{dx^2}\right) + \frac{24}{x^3}\left(\frac{du}{dx}\right) + 60(7 - 18x^4 + 3x^8)u^9 = 0, \tag{7.17}$$

subject to ICs

$$u(0) = 1, \quad u'(0) = u''(0) = u'''(0) = 0,$$

where $g(x, u) = 60(7 - 18x^4 + 3x^8)u^9$. The Lagrange's multiplier for problem (7.17) is calculated by Eq. (3.10), which is defined as

$$\psi(x, t) = -\frac{1}{6}t^3 + \frac{1}{2}\left(\frac{t^4}{x}\right) - \frac{1}{2}\left(\frac{t^5}{x^2}\right) + \frac{1}{6}\left(\frac{t^6}{x^3}\right). \tag{7.18}$$

Choosing an initial approximation as $u_0(x) = 1$ that meets the initial conditions. Applying the system of equations (5.7), we obtain the values of $w_1, w_2, w_3, \ldots,$ as

$$\begin{cases} w_0 = 1, \\ w_1 = -\dfrac{x^4}{2} + \dfrac{3x^8}{22} - \dfrac{x^{12}}{182}, \\ w_2 = \dfrac{21x^8}{88} - \dfrac{657x^{12}}{4004} + \dfrac{8565x^{16}}{369512} + \ldots, \\ w_3 = -\dfrac{327x^{12}}{2288} + \dfrac{116955x^{16}}{739024} - \dfrac{277268148x^{20}}{6298655363} + \ldots, \\ w_4 = \dfrac{543935x^{16}}{5912192} - \dfrac{309316185x^{20}}{2214911776} + \dfrac{686706167261x^{24}}{11229602704320} + \ldots, \\ \vdots \end{cases} \tag{7.19}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = 1 - \frac{t^4}{2} + \frac{3t^8}{8} - \frac{5t^{12}}{16} + \frac{35t^{16}}{128} + \ldots, \tag{7.20}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = \frac{1}{\sqrt{1+x^4}}$, which also meets with the exact solution.

Example 7.6. Consider the following 2nd kind nonlinear fourth-order SEFT equation with shape factor $p = 5$ as

$$\frac{d^4u}{dx^4} + \frac{10}{x}\left(\frac{d^3u}{dx^3}\right) + \frac{20}{x^2}\left(\frac{d^2u}{dx^2}\right) + 21(5x^4 - 6)u^{-15} = 0, \tag{7.21}$$

subject to ICs

$$u(0) = 1, \quad u'(0) = u''(0) = u'''(0) = 0,$$

where $g(x, u) = 21(5x^4 - 6)u^{-15}$. The Lagrange's multiplier for problem (7.21) is calculated by Eq. (3.12), which is defined as

$$\psi(x, t) = -\frac{1}{12}xt^2 + \frac{1}{6}t^3 - \frac{1}{6}\left(\frac{t^5}{x^2}\right) + \frac{1}{12}\left(\frac{t^6}{x^3}\right). \tag{7.22}$$

Choosing an initial approximation as $u_0(x) = 1$ that meets the initial conditions. Applying the system of Equations (5.7), we obtain the values of $w_1, w_2, w_3, \ldots,$ as

$$\begin{cases} w_0 = 1, \\ w_1 = \dfrac{x^4}{4} - \dfrac{3x^8}{176}, \\ w_2 = -\dfrac{27x^8}{352} + \dfrac{119x^{12}}{7744} - \dfrac{35x^{16}}{107008}, \\ w_3 = \dfrac{609x^{12}}{15488} - \dfrac{45871x^{16}}{3531264} + \dfrac{16079049x^{20}}{22633048064} - \dfrac{546175x^{24}}{50680700928} + \ldots, \\ w_4 = -\dfrac{342965x^{16}}{14125056} + \dfrac{248941203x^{20}}{22633048064} - \dfrac{1542142685825x^{24}}{1461732776165376} + \dfrac{145446588293x^{28}}{3940323135750144} + \ldots, \\ \vdots \end{cases} \tag{7.23}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = 1 + \frac{x^4}{4} - \frac{3x^8}{32} + \frac{7x^{12}}{128} - \frac{77x^{16}}{2048} + \ldots, \tag{7.24}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = (1 + x^4)^{\frac{1}{4}}$, which also meets with the exact solution.

Example 7.7. Consider the following 3rd kind nonlinear fourth-order SEFT equation with shape factor $p = 4$ as

$$\frac{d^4u}{dx^4} + \frac{4}{x}\left(\frac{d^3u}{dx^3}\right) + 32(15 - 129x^4 + 49x^8 + x^{12})e^u = 0,$$

subject to ICs

$$u(0) = 0, \quad u'(0) = u''(0) = u'''(0) = 0, \tag{7.25}$$

where $g(x, u) = 32(15 - 129x^4 + 49x^8 + x^{12})e^u$. The Lagrange's multiplier for problem (7.25) is calculated by Eq. (3.13), which is defined as

$$\psi(x, t) = \frac{1}{2}xt^2 - \frac{1}{6}x^2t - \frac{1}{2}t^3 + \frac{1}{6}\left(\frac{t^4}{x}\right). \tag{7.26}$$

Choosing an initial approximation as $u_0(x) = 0$ that meets the initial conditions. Applying the system of Equations (5.7), we obtain the values of $w_1, w_2, w_3, \ldots,$ as

$$\begin{cases} w_0 = 0, \\ w_1 = -4x^4 + \dfrac{86x^8}{63} - \dfrac{196x^{12}}{2145} - \dfrac{x^{16}}{1785}, \\ w_2 = \dfrac{40x^8}{63} - \dfrac{45064x^{12}}{45045} + \dfrac{1121518x^{16}}{5360355} - \dfrac{228712252x^{20}}{13749310575} + \ldots, \\ w_3 = -\dfrac{2176x^{12}}{9009} + \dfrac{103888x^{16}}{153153} - \dfrac{82279938128x^{20}}{288735522075} + \dfrac{924613331144x^{24}}{17392877877375} + \ldots, \\ w_4 = \dfrac{40384x^{16}}{357357} - \dfrac{25495405888x^{20}}{57747104415} + \dfrac{111193516505728x^{24}}{365250435424875} - \dfrac{6526443926378554016x^{28}}{67665492115854407775} + \ldots, \\ \vdots \end{cases} \tag{7.27}$$

Thus, the series solution is calculated by using $u = \sum_{i=0}^{n} w_i$

$$u(x) = -4x^4 + 2x^8 - \frac{4x^{12}}{3} + x^{16} - \frac{4x^{20}}{5} + \ldots, \tag{7.28}$$

and the closed form solution of the aforementioned equation for $n \to \infty$ is $u(x) = -4\log(1+x^4)$, which also meets with the exact solution.

8. Conclusion

This work deals with the solutions of the higher-order nonlinear SEFT equations with suitable initial conditions. The proposed coupled iterative scheme is developed by combining the concepts of VIM and HPM. With the assistance of the variational iteration method, an analogous recursive integral transform including Lagrange's multipliers is constructed for the nonlinear higher-order singular differential equation with suitable initial conditions. Some numerical examples are used to show the accuracy and easiness for the proposed method. To handle the nonlinearity, and the singularity (at point $x = 0$) the notion of homotopy is taken into account. On the basis of the proposed study, we have drawn the following conclusions:

- A coupled iterative technique is established in terms of Lagrange's multipliers for a class of nonlinear higher order Lane-Emden initial value problems.
- In contrast to standard Emden-Fowler equations having unique shape factors, we discussed the cases with different shape factors and found the proposed study yields effective results for all shape factors $p > 0$.
- The proposed iterative technique adequately tackles the numerical problems containing the singular point at more than one place.
- The findings show that the proposed study produces an effective and reliable approach that can be used to obtain exact series solutions to the nonlinear higher-order SEFT equations.
- The proposed iterative technique is found to be easy and fast and it can be applied to other types of nonlinear singular differential equations.

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