Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 14, No. 2, 2026, pp. 766-779
DOI:10.22034/cmde.2025.64894.2957

$L^{\ell}-$Asymptotic properties of nonlinear Sturm-Liouville problems

Fatemeh Kiyaee and Seyfollah Mosazadeh*
Department of Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran.

Abstract

In this paper, a nonlinear eigenvalue problem consisting of a nonlinear Sturm-Liouville equation $-y'' - q(x)y = \lambda q^{-1}(x)y^r$ with Dirichlet boundary conditions on the interval $(-1/2, 1/2)$ is investigated, where $\lambda > 0$ is the eigenparameter. We provide a simple scheme to obtain the asymptotic behavior of $L^{\ell}-$bifurcation curve $\lambda = \lambda_{\ell}(\gamma)$ as $\gamma \longrightarrow 0$, where $\gamma = ||y_\lambda||_{\ell}$, $\ell \geq 1$, and y_λ is the solution of Dirichlet problem associated with λ.

Keywords. Nonlinear Sturm-Liouville problem, $L^{\ell}-$bifurcation curve, Asymptotic behavior, Eigenvalue.
2010 Mathematics Subject Classification. 34B15, 34L15, 34F10, 34B24.

1. INTRODUCTION

Linear eigenvalue problems and their associated inverse problems have been investigated by many authors, and many results have been established concerning the asymptotic distribution of eigenvalues, eigenfunctions, norming constants, nodal points, etc (see, for instance, [7, 15, 23, 26–28, 38, 39]).

Nonlinear eigenvalue problems have been one of the main topics in mathematical biology, mathematical physics, engineering, etc. For example, the nonlinear differential equaion

$$-(p(x)y')' + s(x)y' + q(x)y = \lambda f(x, y). \tag{1.1}$$

describes the logistic equation of population dynamics (see [6, 8, 10, 11, 14, 19, 20, 33]). The Equation (1.1) also appears in the propagation of electromagnetic waves in nonlinear media (we refer the reader to [32, 35, 36] and the references therein).

Boundary value problems consisting of nonlinear differential equation $-y'' + f(x, y) = \lambda y$ with various conditions have been investigated by many authors. There is extensive literature that deals with many results for these problems (see, for example, [2, 3, 12, 14, 18, 25, 31, 34, 37]).

There are multiple studies about nonlinear elliptic bifurcation problems consisting of nonlinear Sturm-Liouville Equation (1.1). For the case $p(x) \equiv 1$, $s(x) \equiv q(x) \equiv 0$ and $f(x, y) = f(y) > 0$ for all $y > 0$, Laetsch [24] studied necessary conditions for which (1.1) has positive solutions, and he investigated the behavior of the solutions as λ varies (see also [16]). Later, Bonanno [4] considered (1.1) with Dirichlet conditions $y(0) = y(1) = 0$ for the case $p(x) \equiv 1$, $s(x) \equiv q(x) \equiv 0$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous. He established the existence of the solutions by using variational methods and critical points. Moreover, he and D'Agul [5] investigated the existence of infinitely many solutions to a Neumann boundary value problem for (1.1) with $s \equiv 0$ and regular functions p, q, f, by multiple critical points theorems. For the case $s \equiv 0$, $p \in C^1[0, 1]$, $q \in C[0, 1]$, $p(x) > 0$, $q(x) \geq 0$, $f \in C([0, 1] \times \mathbb{R}^+, (0, +\infty))$, Cheng et al. [9] obtained the global existence results of positive solutions for (1.1) with boundary conditions $ay(0) - bp(0)y'(0) = 0$, $cy(1) + dp(1)y'(1) = 0$, where $a, b, c, d \geq 0$ and $(a + b)(c + d) > 0$. They used the fixed point index theory in cones for this aim. Recently, Kato et al. [21] considered two Dirichlet boundary value problems consisting of (1.1) with $p(x) \equiv 1$, $s(x) \equiv q(x) \equiv 0$, $\lambda > 0$, $f(x, y) = f_1(y) = y^3 + \sin(y^3)/y$ and $f(x, y) = f_2(y) = y + y^r \sin(y^{\ell})$ $(y \geq 0,$

Received: 06 December 2024; Accepted: 22 April 2025.
∗ Corresponding author. Email: s.mosazadeh@kashanu.ac.ir.

$0 \le r < 1$, $1 < \ell \le r+2$). They used the stationary phase method and obtained the asymptotic formulas for the bofurcation parameter $\lambda = \lambda(\gamma)$ as $\gamma \longrightarrow 0$ and $\gamma \longrightarrow \infty$, where $\gamma = ||y_\lambda||_\infty$ is the maximum norm of y_λ, and y_λ is the solution of the problem associated with λ.

In the present paper, we consider the following nonlinear boundary value problem

$$-y'' - q(x)y = \lambda q^{-1}(x)y^r, \quad x \in J := (-1/2, 1/2), \tag{1.2}$$

$$y(-1/2) = 0 = y(1/2), \tag{1.3}$$

$$y(x) > 0, \quad x \in J, \tag{1.4}$$

where r is a positive even integer and $\lambda > 0$ is the spectral parameter. We assume that $q(x) \in C^2(\overline{J})$ satisfies the following conditions:

$$q(x) > 0, \quad q(x) = q(-x), \quad x \in \overline{J}, \tag{1.5}$$

$$q'(x) \ge 0, \quad 0 \le x \le 1/2. \tag{1.6}$$

Although, such nonlinear problems can be studied by using classical methods such as variational method, stationary phase method and bifurcation theory (see also [1, 17, 22, 30, 31] and the references therein), but generally, these and other classical methods cannot applicable to analyze the spectral properties of nonlinear problem (1.2)-(1.4). In this paper, we present a simple scheme which does not depend on the asymptotic expansion of the solution y_λ, to investigate the behavior of $\lambda = \lambda_\ell(\gamma)$ in L^ℓ−framework, as $\gamma \longrightarrow 0$, where $\gamma = ||y_\lambda||_\ell$, $\ell \ge 1$, and y_λ is the solution of (1.2)-(1.4) associated with λ. First, we consider the case $q(x) \equiv 1$. Then, we will consider the general case when (1.5)-(1.6) hold.

2. THE CASE $q(x) \equiv 1$

In this section, we consider the following nonlinear boundary value problem

$$-y''(x) - y(x) = \lambda y^r(x), \quad y(x) > 0, \quad x \in J := (-1/2, 1/2), \tag{2.1}$$

$$y(-1/2) = 0 = y(1/2), \tag{2.2}$$

where $\lambda > 0$ and r is a positive even integer. Let y_λ be the solution of (2.1)-(2.2) associated with λ. In the following proposition, by the standard methods (see [13]), we prove that there exists $\Delta_1 > 0$ such that $\phi(x) := (\lambda - \Delta_1)^{1/(1-r)} \sin(\pi(x+1/2))$ is the supersolution of (2.1)-(2.2).

Proposition 2.1. *There exists a constant Δ_1 such that for $\lambda > \Delta_1$ and $x \in J$,*

$$y_\lambda(x) \le (\lambda - \Delta_1)^{1/(1-r)} \sin(\pi(x+1/2)).$$

Proof. Choose $\Delta_1 > \frac{\pi^2-1}{\pi^2-2}$. We show that $\phi(x)$ satisfies

$$\begin{cases} -\phi''(x) - \phi(x) \ge \lambda\phi^r(x), & x \in J, \\ \phi(-1/2) \ge 0, & \phi(1/2) \ge 0. \end{cases}$$

Since $r > 1$, $\sin(\pi(x+1/2)) \ge \sin^r(\pi(x+1/2))$ for $0 \le x \le 1/2$. Thus, we have for $\lambda > \Delta_1$,

$$\frac{(\lambda - \Delta_1)(\pi^2 - 1)}{\lambda} \sin(\pi(x+1/2)) \ge \sin^r(\pi(x+1/2)).$$

Therefore,

$$\begin{aligned} -\phi''(x) - \phi(x) &= \pi^2(\lambda - \Delta_1)^{1/(1-r)} \sin(\pi(x+1/2)) - (\lambda - \Delta_1)^{1/(1-r)} \sin(\pi(x+1/2)) \\ &\ge \lambda(\lambda - \Delta_1)^{r/(1-r)} \sin^r(\pi(x+1/2)) \\ &= \lambda\phi^r(x). \end{aligned}$$

From $\phi(-1/2) = \phi(1/2) = 0$, the proof is complete. □

Similarly, one can prove that there exists $\Delta_2 > 0$ such that for each $\lambda > \Delta_2$, $\psi(x) = -(\lambda - \Delta_2)^{1/(1-r)}$ is the subsolution of (2.1)-(2.2). This together with Proposition 2.1 yields that for $x \in J$,

$$(\lambda - \Delta_2)^{1/(1-r)} - o(1) \leq y_\lambda(x) \leq (\lambda - \Delta_1)^{1/(1-r)}, \quad \lambda \longrightarrow \infty. \tag{2.3}$$

Moreover, for $x \in [0, 1/2]$ and sufficiently large λ, $y_\lambda(x) = y_\lambda(-x)$, and hence, $||y_\lambda||_\infty = y_\lambda(0)$.

Multiplying (2.1) by $y'_\lambda(x)$ and then integrating from 0 to x, we obtain

$$\Big(y'_\lambda(x)\Big)^2 = \|y_\lambda\|^2_\infty - y^2_\lambda(x) - \frac{2\lambda}{r+1}\Big(\|y_\lambda\|^{r+1}_\infty - y^{r+1}_\lambda(x)\Big).$$

Hence, we get for $r > 1$,

$$|y'_\lambda(x)| = \sqrt{\|y_\lambda\|^2_\infty - y^2_\lambda(x) - \frac{2\lambda}{r+1}(\|y_\lambda\|^{r+1}_\infty - y^{r+1}_\lambda(x))}.$$

Since $y'_\lambda(x) \geq 0$ for $x \in (-1/2, 0)$, we can write

$$\begin{aligned}
T &:= \|y_\lambda\|^\ell_\infty - \|y_\lambda\|^\ell_\ell \\
&= 2\int_{-1/2}^{0} \Big(\|y_\lambda\|^\ell_\infty - y^\ell_\lambda(x)\Big) \frac{y'_\lambda(x)}{\sqrt{(\|y_\lambda\|^2_\infty - y^2_\lambda) - \frac{2\lambda}{r+1}(\|y_\lambda\|^{r+1}_\infty - y^{r+1}_\lambda)}} \mathrm{d}x \\
&= 2\frac{\|y_\lambda\|^{\ell-(r+1)/2}_\infty}{\sqrt{\lambda}} \int_{-1/2}^{0} \frac{\Big(1 - \frac{y^\ell_\lambda(x)}{\|y_\lambda\|^\ell_\infty}\Big) y'_\lambda(x)}{\sqrt{1 - \frac{y^2_\lambda(x)}{\|y_\lambda\|^2_\infty} - \frac{2}{r+1}\Big(1 - \frac{y^{r+1}_\lambda(x)}{\|y_\lambda\|^{r+1}_\infty}\Big)}} \mathrm{d}x.
\end{aligned}$$

Put $\frac{y_\lambda(x)}{\|y_\lambda\|_\infty} = t$. So, we get

$$T = 2\frac{\|y_\lambda\|^{\ell+1-(r+1)/2}_\infty}{\sqrt{\lambda}} \int_0^1 \frac{1 - t^\ell}{\sqrt{1 - t^2 - \frac{2}{r+1}(1 - t^{r+1})}} \mathrm{d}t. \tag{2.4}$$

Let

$$\begin{aligned}
F(t) &:= (1 - t^2) - \frac{2}{r+1}(1 - t^{r+1}), \\
T_\lambda(t) &:= \|y_\lambda\|^{r-1}_\infty \Big(1 - t^2 - \frac{2}{r+1}(1 - t^{r+1})\Big) \\
&= \|y_\lambda\|^{r-1}_\infty F(t).
\end{aligned}$$

As a result, from (2.4) we obtain

$$T = \frac{\|y_\lambda\|^\ell_\infty}{\sqrt{\lambda}}(D^*_{r,\ell} + Y^*_\lambda),$$

where

$$\begin{aligned}
D^*_{r,\ell} &= 2\int_0^1 \frac{(1 - t^\ell)}{\sqrt{F(t)}} \mathrm{d}t, \\
Y^*_\lambda &= \frac{2(1 - \|y_\lambda\|^{(r-1)/2}_\infty)}{\|y_\lambda\|^{(r-1)/2}_\infty} \int_0^1 \frac{1 - t^\ell}{\sqrt{F(t)}} \mathrm{d}t.
\end{aligned}$$

This yields that for a sufficiently small $\delta > 0$,

$$\frac{\|y_\lambda\|^\ell_\infty}{\sqrt{\lambda}}\Big(D^*_{r,\ell} + (\sqrt{\lambda} - 1)C^*_1\Big) \leq \|y_\lambda\|^\ell_\infty - \|y_\lambda\|^\ell_\ell \leq \frac{\|y_\lambda\|^\ell_\infty}{\sqrt{\lambda}}\Big(D^*_{r,\ell} + (\sqrt{\lambda} - 1)C^*_2\Big),$$

where

$$C_1^* = 2\int_0^1 \frac{(1-t^\ell)}{\sqrt{1-t^2}}\mathrm{d}t,$$
$$C_2^* = 2\int_0^1 \frac{(1-t^\ell)}{\sqrt{1-t^2-\frac{2\delta}{r+1}(1-t^{r+1})}}\mathrm{d}t.$$

Consequently,

$$\|y_\lambda\|_\infty^\ell C_1^*\Big(1+\frac{D_{r,\ell}^*-C_1^*}{C_1^*\sqrt{\lambda}}\Big) \le \|y_\lambda\|_\infty^\ell - \|y_\lambda\|_\ell^\ell \le \|y_\lambda\|_\infty^\ell C_2^*\Big(1+\frac{D_{r,\ell}^*-C_2^*}{C_2^*\sqrt{\lambda}}\Big). \tag{2.5}$$

We know from [29] that $||y_\lambda||_\infty \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Now, we obtain the asymptotic behavior of $\lambda = \lambda_\ell(\gamma)$, $\ell \ge 1$. In the following theorems, we prove the main results of this section.

Theorem 2.2. *As $\gamma \longrightarrow 0$, the following inequality holds:*

$$\lambda_\ell(\gamma) \ge d_1\gamma^{1-r} + \frac{d_2}{\sqrt{C}}\gamma^{(1-r)/2} + \frac{d_3}{C} + O(\gamma^{(r-1)/2}), \tag{2.6}$$

where C is a positive constant, and

$$d_1 = 1 - C_1C_2^* + \frac{C_2C_2^{*2}}{2!} - \frac{C_3C_2^{*3}}{3!} + \cdots,$$
$$d_2 = (D_{r,\ell}^* - C_2^*)\Big(-C_1 + \frac{C_2C_2^*}{1!} - \frac{C_3C_2^{*2}}{2!} + \cdots\Big),$$
$$d_3 = (D_{r,\ell}^* - C_2^*)^2\Big(\frac{C_2}{2!} - \frac{C_3}{2!} + \cdots\Big),$$

and

$$C_1 = \frac{r-1}{\ell},$$
$$C_2 = (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1),$$
$$C_3 = (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1)(\frac{r-1}{\ell}-2),$$
$$\vdots$$

Proof. According to (2.5) we have

$$\|y_\lambda\|_\infty^\ell - \|y_\lambda\|_\ell^\ell \le \|y_\lambda\|_\infty^\ell C_2^*\Big(1+\frac{D_{r,\ell}^*-C_2^*}{C_2^*\sqrt{\lambda}}\Big).$$

Hence,

$$\|y_\lambda\|_\infty^\ell\Big(1 - C_2^*(1+\frac{D_{r,\ell}^*-C_2^*}{C_2^*\sqrt{\lambda}})\Big) \le \gamma^\ell.$$

Therefore,

$$\lambda_\ell(\gamma) \ge \gamma^{1-r}\Big(1 - C_2^*(1+\frac{D_{r,\ell}^*-C_2^*}{C_2^*\sqrt{\lambda}})\Big)^{(r-1)/\ell}. \tag{2.7}$$

By the Taylor expansion for the function $f(t) = (1-t)^{(r-1)/\ell}$, we get

$$f(t) = 1 - (\frac{r-1}{\ell})t + (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1)\frac{t^2}{2!} - (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1)(\frac{r-1}{\ell}-2)\frac{t^3}{3!} + \cdots. \tag{2.8}$$

For convenience in calculations, set

$$\begin{aligned} C_1 &= (\frac{r-1}{\ell}), \\ C_2 &= (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1), \\ C_3 &= (\frac{r-1}{\ell})(\frac{r-1}{\ell}-1)(\frac{r-1}{\ell}-2), \\ &\vdots \end{aligned} \tag{2.9}$$

Put $t = C_2^*(1 + \frac{D^*_{r,\ell} - C_2^*}{C_2^* \sqrt{\lambda}})$. Thus, we have

$$f(t) = d_1 + \frac{1}{\sqrt{\lambda}} d_2 + \frac{1}{\lambda} d_3 + O\Big(\frac{1}{\lambda\sqrt{\lambda}}\Big),$$

where the coefficients d_1, d_2 and d_3 were defined in the theorem. Since there exists a positive constant C such that $\lambda \sim C\gamma^{1-r}$ as $\gamma \longrightarrow 0$, for $t = C_2^*(1 + \frac{D^*_{r,\ell} - C_2^*}{C_2^* \sqrt{\lambda}})$ we obtain

$$f(t) = d_1 + \frac{d_2}{\sqrt{C}\gamma^{(1-r)/2}} + \frac{d_3}{C\gamma^{1-r}} + O\Big(\frac{1}{\gamma^{3(1-r)/2}}\Big). \tag{2.10}$$

Substituting (2.10) into (2.7), we arrive at (2.6). The proof is complete. □

Theorem 2.3. *As $\gamma \to 0$, the following inequality holds:*

$$\lambda_\ell(\gamma) \le d_1^* \gamma^{1-r} + \frac{d_2^*}{\sqrt{C}} \gamma^{(1-r)/2} + \frac{d_3^*}{C} + O(\gamma^{(r-1)/2}), \tag{2.11}$$

where

$$\begin{aligned} d_1^* &= 1 - C_1 C_1^* + \frac{C_2 C_1^{*2}}{2!} - \frac{C_3 C_1^{*3}}{3!} + \cdots, \\ d_2^* &= (D^*_{r,\ell} - C_1^*)\Big(- C_1 + \frac{C_2 C_1^*}{1!} - \frac{C_3 C_1^{*2}}{2!} + \cdots \Big), \\ d_3^* &= (D^*_{r,\ell} - C_1^*)^2 \Big(\frac{C_2}{2!} - \frac{C_3}{2!} + \cdots \Big). \end{aligned}$$

Proof. From (2.5) we have

$$\|y_\lambda\|_\infty^\ell - \|y_\lambda\|_\ell^\ell \ge \|y_\lambda\|_\infty^\ell C_1^* \Big(1 + \frac{D^*_{r,\ell} - C_2^*}{C_1^* \sqrt{\lambda}}\Big).$$

This yields

$$\|y_\lambda\|_\infty^\ell \Big(1 - C_1^*(1 + \frac{D^*_{r,\ell} - C_1^*}{C_1^* \sqrt{\lambda}})\Big) \ge \gamma^\ell.$$

Hence, we get

$$\lambda \le \gamma^{1-r} \Big(1 - C_1^*(1 + \frac{D^*_{r,\ell} - C_1^*}{C_1^* \sqrt{\lambda}})\Big)^{(r-1)/\ell}. \tag{2.12}$$

We use the Taylor expansion (2.8) of the function $f(t) = (1-t)^{(r-1)/\ell}$. For $t = C_1^*(1 + \frac{D^*_{r,\ell} - C_1^*}{C_1^* \sqrt{\lambda}})$ we see that

$$f(t) = d_1^* + \frac{d_2^*}{\sqrt{C}\gamma^{(1-r)/2}} + \frac{d_3^*}{C\gamma^{1-r}} + O\Big(\frac{1}{\gamma^{3(1-r)/2}}\Big), \tag{2.13}$$

where the coefficients d_1, d_2 and d_3 were defined in the theorem. Substituting (2.13) into (2.12) we arrive at (2.11). The proof is complete. □

3. The general case

In this section, we consider the main problem (1.2)-(1.6), where $\lambda > 0$ and r is a positive even integer. Put $w_\lambda(x) = Q^{-\alpha}(x)y_\lambda(x)$, where $Q(x) = (q(x))^{1/(\alpha(r-1))}$ and $\alpha < 0$ is a constant. Then, from (1.2) we obtain

$$\begin{aligned}
&-w_\lambda''(x) - 2\alpha\frac{Q'(x)}{Q(x)}w_\lambda'(x) - \Big(\alpha\frac{Q''(x)}{Q(x)} + \alpha(\alpha-1)\Big(\frac{Q'(x)}{Q(x)}\Big)^2 + Q^{\alpha(r-1)}(x)\Big)w_\lambda(x) = \lambda w_\lambda^r(x), \quad x \in J, \qquad (3.1)\\
&w(-1/2) = 0 = w(1/2),\\
&w_\lambda(x) > 0, \quad w_\lambda(x) = w_\lambda(-x), \quad x \in J,\\
&w_\lambda'(x) \le 0, \quad x \in [0, 1/2], \quad ||w_\lambda||_\infty = w_\lambda(0).
\end{aligned}$$

Let $r > 1$ and $\ell \ge 1$ be fixed constants such that $\ell - r \ge 1$. We define the condition
(Q^*): The function $q(x)$ satisfies (1.5) -(1.6), and

$$\frac{Q''(x)}{Q(x)} + (\alpha-1)\Big(\frac{Q'(x)}{Q(x)}\Big)^2 + \frac{1}{\alpha}Q^{\alpha(r-1)}(x) < 0.$$

Acting in the same way as in the previous section (see also [12]), one can obtain that there exist positive numbers Δ_3 and Δ_4 such that for $x \in J$ and sufficiently large λ,

$$(\lambda - \Delta_3)^{1/(1-r)} - o(1) \le \|w_\lambda\|_\infty < (\lambda - \Delta_4)^{1/(1-r)}. \qquad (3.2)$$

On the other hand, it follows from (3.2) that

$$\Big|\frac{w_\lambda(x)}{\|w_\lambda\|_\infty} - 1\Big| = O\Big(\frac{1}{\lambda}\Big), \quad \lambda \longrightarrow \infty.$$

According to (3.1), we get

$$\begin{aligned}
\Big(w_\lambda'(x)\Big)^2 = &-4\alpha\int_0^x \frac{Q'(t)}{Q(t)}\Big(w_\lambda'(t)\Big)^2\mathrm{d}t - 2\int_0^t \Big(\alpha\frac{Q''(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2\\
&+ Q^{\alpha(r-1)}(t)\Big)w_\lambda(t)w_\lambda'(t)\mathrm{d}t - 2\lambda\int_0^x w_\lambda^r(t)w_\lambda'(t)\mathrm{d}t.
\end{aligned}$$

Consequently,

$$-w_\lambda^{'}(x) = \sqrt{M(w_\lambda(x)) + N_\lambda(x) + P_\lambda(x)},$$

where

$$\begin{aligned}
M(w_\lambda(x)) &= \frac{2\lambda}{r+1}\Big(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x)\Big),\\
N_\lambda(x) &= -2\int_0^x \Big(\alpha\frac{Q^{''}(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big)w_\lambda(t)w_\lambda^{'}(t)\mathrm{d}t,\\
P_\lambda(x) &= -4\alpha\int_0^x \frac{Q^{'}(t)}{Q(t)}\Big(w_\lambda'(t)\Big)^2\mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-w_\lambda^{'}(x) &\le \sqrt{M(w_\lambda(x)) + N_\lambda(x) + \Delta_3(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x))}\\
&= \sqrt{\frac{2\lambda + \Delta_3(r+1)}{r+1}(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x)) + N_\lambda(x)}.
\end{aligned}$$

Put

$$\begin{aligned}
&\Delta_5 := -\frac{\Delta_4(r+1)}{2}, \quad \mu := \lambda - \Delta_5,\\
&M_{0,\lambda}(w_\lambda(x)) := \frac{2\mu}{r+1}(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x)).
\end{aligned}$$

Then, we have

$$-w'_\lambda(x) = \sqrt{M(w_\lambda(x)) + N_\lambda(x) + P_\lambda(x)} \le \sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}.$$

Therefore, we obtain

$$\begin{aligned}
\|w_\lambda\|_\infty^\ell - \|w_\lambda\|_\ell^\ell &= 2\int_0^{1/2} (\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) \frac{-w_\lambda^{'}(x)}{\sqrt{M(w_\lambda(x)) + N_\lambda(x) + P_\lambda(x)}}\mathrm{d}x && (3.3)\\
&\ge 2\int_0^{1/2} (\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) \frac{-w_\lambda^{'}(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}}\mathrm{d}x \\
&= 2\int_0^{1/2} (\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) \frac{-w_\lambda^{'}(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x))}}\mathrm{d}x \\
&+ 2\int_0^{1/2} (\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) \Big(\frac{-w_\lambda^{'}(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}} + \frac{w_\lambda^{'}(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x))}} \Big)\mathrm{d}x \\
&=: H + H^*. && (3.4)
\end{aligned}$$

Set $t = \frac{w_\lambda(x)}{\|w_\lambda\|_\infty}$. Then, for sufficiently large λ, we have

$$\begin{aligned}
H &= 2\int_0^{1/2} \Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big) \frac{-w_\lambda^{'}(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x))}}\mathrm{d}x \\
&= 2\frac{\|w_\lambda\|_\infty^q}{\sqrt{\mu}} \int_0^1 \frac{(1-t^\ell)}{\|w_\lambda\|_\infty^{(r-1)/2}\sqrt{\frac{2}{r+1}(1-t^{r+1})}}\mathrm{d}t \\
&=: \frac{\|w_\lambda\|_\infty^\ell}{\sqrt{\mu}}(D_{r,\ell} + Y_\lambda),
\end{aligned}$$

where

$$\begin{aligned}
D_{r,\ell} &= \sqrt{2(r+1)}\int_0^1 \frac{1-t^\ell}{\sqrt{1-t^{r+1}}}\mathrm{d}t, \\
Y_\lambda &= 2\int_0^1 \frac{(1-\|w_\lambda\|_\infty^{(r-1)/2})(1-t^\ell)}{\|w_\lambda\|_\infty^{(r-1)/2}\sqrt{\frac{2}{r+1}(1-t^{r+1})}}\mathrm{d}t \\
&= \sqrt{2(r+1)}\frac{(1-\|w_\lambda\|_\infty^{(r-1)/2})}{\|w_\lambda\|_\infty^{(r-1)/2}}\int_0^1 \frac{1-t^\ell}{\sqrt{1-t^{r+1}}}\mathrm{d}t.
\end{aligned}$$

Thus, we obtain

$$H = D_{r,\ell}\|w_\lambda\|_\infty^\ell\Big(1 + \frac{1}{\sqrt{\mu}}\Big) + O(\|w_\lambda\|_\infty^{\ell+(r-1)/2}). \tag{3.5}$$

Note that, for each $r, \ell > 1$, the integral $\int_0^1 \frac{1-t^\ell}{\sqrt{1-t^{r+1}}}\mathrm{d}t$ is convergent (see Appendix). On the other hand, for an arbitrary fixed number $0 < \varepsilon \ll 1$, we have

$$\begin{aligned}
H^* &= 2\int_0^{1/2} (\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) \Big\{ \frac{w_\lambda^{'}(x) N_\lambda(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}} && (3.6)\\
&\times \frac{1}{\sqrt{M_{0,\lambda}(w_\lambda(x))}\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big)} \Big\}\mathrm{d}x \\
&= 2\int_0^{1/2-\varepsilon} + 2\int_{1/2-\varepsilon}^{1} =: H_1^* + H_2^*.
\end{aligned}$$

In order to obtain the asymptotic of H^*, we first prove several auxiliary lemmas for H_1^* and H_2^*.

Lemma 3.1. *Let $\ell - r \geq 1$. Then,*

$$H_1^* = O(\|w_\lambda\|^{\ell+(r-1)/2}),$$

as $\lambda \longrightarrow \infty$.

Proof. First, for $0 \leq x \leq 1/2$, we have

$$\begin{aligned} N_\lambda(x) &= -2\int_0^x \Big(\alpha\frac{Q''(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big) w_\lambda(t) w_\lambda'(t)\mathrm{d}t \\ &\leq 2L_1\int_0^t w_\lambda(t)(-w_\lambda'(t))\mathrm{d}t \leq L_1\|w_\lambda\|^{1-r}, \end{aligned} \tag{3.7}$$

for sufficiently large λ, where

$$L_1 = \max_{0\leq t\leq 1/2}\Big(\alpha\frac{Q''(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big). \tag{3.8}$$

Since $-H_1^* \geq 0$, we get

$$-H_1^* = |H_1^*| = 2\int_0^{1/2-\varepsilon} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda(x)\Big)N_\lambda(x)w_\lambda'(x)}{\sqrt{M_{0,\lambda}(w_\lambda(x))}\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big)}\mathrm{d}x.$$

Moreover,

$$\begin{aligned} &\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} > \sqrt{M_{0,\lambda}(w_\lambda(x))}, \\ &\sqrt{M_{0,\lambda}(w_\lambda(x))}\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big) > 2\Big(M_{0,\lambda}(w_\lambda(x))\Big)^{3/2}. \end{aligned}$$

These together with (3.7) yield

$$\begin{aligned} -H_1^* &\leq \int_0^{1/2-\varepsilon} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big)N_\lambda(x)(-v_\lambda'(x))}{\Big(M_{0,\lambda}(w_\lambda(x))\Big)^{3/2}}\mathrm{d}x \\ &\leq \int_0^{1/2-\varepsilon} \|w_\lambda\|_\infty^\ell \frac{\Big(1 - \frac{w_\lambda^\ell(x)}{\|w_\lambda\|_\infty^\ell}\Big)L_1\|w_\lambda\|_\infty^2(-w_\lambda'(x))}{\Big(\frac{2\mu}{r+1}(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x))\Big)^{3/2}}\mathrm{d}x. \end{aligned} \tag{3.9}$$

Put $t = \frac{w_\lambda(x)}{\|w_\lambda\|_\infty}$. Then,

$$\begin{aligned} |H_1^*| &\leq (\frac{r+1}{2})^{3/2}L_1\|w_\lambda\|_\infty^\ell \int_{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}}^1 \frac{1-t^\ell}{(1-t^{r+1})^{3/2}}\mathrm{d}t \\ &\leq (\frac{r+1}{2})^{3/2}\ell L_1\|w_\lambda\|_\infty^\ell \int_{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}}^1 \frac{\mathrm{d}t}{(1-t)^{1/2}} \\ &= \frac{2}{3}(\frac{r+1}{2})^{3/2}\ell L_1\|w_\lambda\|_\infty^\ell O(\lambda^{-1/2}). \end{aligned}$$

Therefore, we arrive at the assertion of the lemma. ☐

Lemma 3.2. *Let $\ell - r \geq 1$. Then,*

$$-H_2^* \leq 2\ell C\|w_\lambda\|_\infty^\ell + O(\|w_\lambda\|_\infty^{\ell+(r-1)/2}),$$

as $\lambda \longrightarrow \infty$, where $C = L_1(\frac{r+1}{2})^{-3/2}$.

Proof. For $0 \leq x \leq 1/2$, we have

$$N_\lambda(x) = -2\int_0^x \Big(\alpha\frac{Q^{''}(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big)w_\lambda(t)w_\lambda^{'}(t)\mathrm{d}t \leq L_1\|w_\lambda\|_\infty^2,$$

for sufficiently large λ, where L_1 is defined by (3.8). Also,

$$-H_2^* \leq 2\int_{1/2-\varepsilon}^{1/2} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big)N_\lambda(x)(-w_\lambda'(x))}{\sqrt{M_{0,\lambda}(w_\lambda(x))}\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big)}\mathrm{d}x.$$

Since

$$\frac{1}{\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}} \leq \frac{1}{\sqrt{M_{0,\lambda}(w_\lambda(x))}},$$

we obtain

$$\begin{aligned}
-H_2^* &\leq \int_{1/2-\varepsilon}^{1/2} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big)N_\lambda(x)(-w_\lambda'(x))}{\Big(M_{0,\lambda}(w_\lambda(x))\Big)^{3/2}}\mathrm{d}x \\
&\leq \int_{1/2-\varepsilon}^{1/2} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big)L_1\|w_\lambda\|_\infty^2(-w_\lambda'(x))}{\Big(\frac{2\mu}{r+1}(\|w_\lambda\|_\infty^{r+1} - w_\lambda^{r+1}(x))\Big)^{3/2}}\mathrm{d}x \\
&\leq L_1(\frac{r+1}{2})^{-3/2}\mu^{-3/2}\|w_\lambda\|_\infty^{\ell+2-3(r+1)/2}\int_{1/2-\varepsilon}^{1/2} \frac{\Big(1-\frac{w_\lambda^\ell(x)}{\|w_\lambda\|_\infty^\ell}\Big)(-w_\lambda'(x))}{\Big(1-\frac{w_\lambda^{r+1}(x)}{\|w_\lambda\|_\infty^{r+1}}\Big)^{3/2}}\mathrm{d}x.
\end{aligned}$$

Put $t = \frac{w_\lambda(x)}{\|w_\lambda\|_\infty}$. Hence,

$$\begin{aligned}
-H_2^* &\leq L_1(\frac{r+1}{2})^{-3/2}\mu^{-3/2}\|w_\lambda\|_\infty^{\ell+3-3(r+1)/2}\int_0^{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}} \frac{1-t^\ell}{(1-t^{r+1})^{3/2}}\mathrm{d}t \\
&\leq \ell C\|w_\lambda\|_\infty^\ell \int_0^{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}} \frac{\mathrm{d}t}{(1-t)^{1/2}} \\
&\leq 2\ell C\|w_\lambda\|_\infty^\ell + O(\|w_\lambda\|_\infty^{\ell+(r-1)/2}),
\end{aligned}$$

where $C = L_1(\frac{r+1}{2})^{-3/2}$. The proof is complete. □

Lemma 3.3. *Let $\ell - r \geq 1$. Then,*

$$-H_2^* \geq 2\ell C_{H_2^*}\|w_\lambda\|_\infty^{\ell+r-1} + O(\|w_\lambda\|_\infty^{\ell+3(r-1)/2}),$$

as $\lambda \longrightarrow \infty$, where $C_{H_2^*}$ is a constant which depends on L_2 and

$$L_2 = \min_{0\leq t\leq 1/2}\Big(\alpha\frac{Q^{''}(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big).$$

Proof. We know that $M_{0,\lambda}(w_\lambda(t))$ and $N_\lambda(x)$ both are positive, and for sufficiently large λ, $N_\lambda(x) < M_{0,\lambda}(w_\lambda(t))$. Thus,

$$\sqrt{M_{0,\lambda}(w_\lambda(x))}\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)}\,\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big) \leq 6\Big(M_{0,\lambda}(w_\lambda(x))\Big)^{3/2}.$$

Moreover, there is a constant C_{L_2} such that

$$N_\lambda(x) \geq -2\int_{1/2-\varepsilon}^{x} \Big(\alpha\frac{Q^{''}(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big) w_\lambda(t) w_\lambda^{'}(t)\mathrm{d}t \geq L_2 C_{L_2} \|w_\lambda\|_\infty^{r+1},$$

where

$$L_2 = \min_{0\leq t\leq 1/2} \Big(\alpha\frac{Q^{''}(t)}{Q(t)} + \alpha(\alpha-1)\Big(\frac{Q'(t)}{Q(t)}\Big)^2 + Q^{\alpha(r-1)}(t)\Big).$$

Therefore, we have

$$\begin{aligned}
-H_2^* &= 2\int_{1/2-\varepsilon}^{1/2} \frac{\Big(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)\Big) N_\lambda(x)(-w_\lambda'(x))}{\sqrt{M_{0,\lambda}(w_\lambda(x))}\Big(\sqrt{M_{0,\lambda}(w_\lambda(x)) + N_\lambda(x)} + \sqrt{M_{0,\lambda}(w_\lambda(x))}\Big)}\mathrm{d}x \\
&\geq \int_{1/2-\varepsilon}^{1/2} \frac{(\|w_\lambda\|_\infty^\ell - w_\lambda^\ell(x)) N_\lambda(x)(-v_\lambda'(x))}{3(M_{0,\lambda}(w_\lambda(x)))^{3/2}}\mathrm{d}x \\
&\geq L_2 C_{L_2} (\frac{r+1}{2})^{-3/2} \mu^{-3/2} \|w_\lambda\|_\infty^{\ell-(r+1)/2} \int_{1/2-\varepsilon}^{1/2} \frac{\Big(1-\frac{w_\lambda(x)^\ell}{\|w_\lambda\|_\infty^\ell}\Big)(-w_\lambda'(x))}{\Big(1-\frac{w_\lambda^{r+1}(x)}{\|w_\lambda\|_\infty^{r+1}}\Big)^{3/2}}\mathrm{d}x \\
&\geq C_{H_2^*} \|w_\lambda\|_\infty^{\ell+r-1} \int_0^{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}} \frac{1-t^\ell}{(1-t^{r+1})^{3/2}}\mathrm{d}t \\
&\geq \ell C_{H_2^*} \|w_\lambda\|_\infty^{\ell+r-1} \int_0^{\frac{w_\lambda(1/2-\varepsilon)}{\|w_\lambda\|_\infty}} \frac{\mathrm{d}t}{(1-t)^{1/2}} \\
&= \ell C_{H_2^*} \|w_\lambda\|_\infty^{\ell+r-1} \Big(2 + O\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big) \\
&\geq 2\ell C_{H_2^*} \|w_\lambda\|_\infty^{\ell+r-1} + O(\|w_\lambda\|_\infty^{\ell+3(r-1)/2}),
\end{aligned}$$

where $C_{H_2^*} = L_2 C_{L_2}(\frac{r+1}{2})^{-3/2}$. We arrive at the assertion of the lemma. □

From (3.6) and Lemmas 3.1, 3.2, and 3.3, as $\lambda \longrightarrow \infty$, we obtain

$$-2\ell C_{H_2^*} \|w_\lambda\|_\infty^\ell + O(\|w_\lambda\|_\infty^{\ell+(r-1)/2}) \leq H^* \leq O(\|w_\lambda\|_\infty^{\ell+(r-1)/2}). \tag{3.10}$$

According to (3.5) and (3.10), we obtain the following corollary.

Corollary 3.4. *Let r be a positive even integer and $\ell - r \geq 1$. Then,*

$$H + H^* \geq \|w_\lambda\|_\infty^\ell \Big(\frac{D_{r,\ell}}{\sqrt{\mu}} - 2\ell C_{H_2^*} + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2})\Big),$$

$$H + H^* \leq \|w_\lambda\|_\infty^\ell \Big(\frac{D_{r,\ell}}{\sqrt{\mu}} + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2})\Big),$$

as $\lambda \longrightarrow \infty$.

Now, we can prove the main result of this section in the following theorem.

Theorem 3.5. *Let r be a positive even integer and $\ell \geq 1$ be a real constant such that $\ell - r \geq 1$. Then, as $\gamma \longrightarrow 0$, the following inequalities hold:*

$$\lambda_\ell(\gamma) \leq E_1\gamma^{1-r} + \frac{E_2}{\sqrt{d_1^*}}\gamma^{(1-r)/2} - \frac{E_2 d_2^*}{2d_1^*\sqrt{C d_1^*}} + \frac{E_3}{d_1^*} + \Delta_5 + O(1), \tag{3.11}$$

$$\lambda_\ell(\gamma) \geq d_1\gamma^{1-r} + \frac{d_2}{\sqrt{C}}\gamma^{(1-r)/2} + \frac{d_3}{C} + \Delta_3 + O(\gamma^{(r-1)/2}), \tag{3.12}$$

where $C = L_1(\frac{r+1}{2})^{-3/2}$*, the coefficients* d_i *and* d_i^**,* $i = 1, 2, 3$*, were defined in Theorems 2.2 and 2.3, respectively, and*

$$E_1 = 1 - C_1(D_1 + D_{r,\ell}) + \frac{C_2}{2!}(D_1^2 + D_{r,\ell}^2 + 2D_1D_{r,\ell}) - \frac{C_3}{3!}(D_1^3 + D_{r,\ell}^3 + 3D_1^2D_{r,\ell} + 3D_1D_{r,\ell}^2) + \cdots , \tag{3.13}$$

$$E_2 = -C_1D_{r,\ell} + \frac{C_2}{2!}2D_1D_{r,\ell} - \frac{C_3}{3!}(3D_{r,\ell}D_1^2 + 3D_{r,\ell}^3) + \cdots , \tag{3.14}$$

$$E_3 = \frac{C_2}{2!}D_{r,\ell}^2 - \frac{C_3}{3!}(3D_{r,\ell}^3 + 3D_{r,\ell}^2D_1) + \cdots , \tag{3.15}$$

where $D_1 = -2\ell C_{H_2^*}$ *and the coefficients* C_1*,* C_2*,* $\cdots$*, were defined in (2.9).*

Proof. We first prove (3.11). It follows from (3.3) and Corollary 3.4 that

$$\|w_\lambda\|_\infty^\ell - \|w_\lambda\|_\ell^\ell \geq \|w_\lambda\|_\infty^\ell\Big(\frac{D_{r,\ell}}{\sqrt{\mu}} + D_1 + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2})\Big),$$

where $D_1 = -2\ell C_{H_2^*}$. Hence, we have

$$\|w_\lambda\|_\infty^{1-r} \leq \gamma^{1-r}\left\{1 - \Big(\frac{D_{r,\ell}}{\sqrt{\mu}} + D_1 + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2})\Big)\right\}^{(r-1)/\ell}. \tag{3.16}$$

From this inequality, as $\gamma \to 0$, we obtain

$$\mu \leq d_1^*\gamma^{1-r} + \frac{d_2^*}{\sqrt{C}}\gamma^{(1-r)/2} + O(1) =: h_\gamma. \tag{3.17}$$

Since $\mu = \lambda - \Delta_5$, the inequality (3.16) yields

$$\lambda - \Delta_5 \leq \gamma^{1-r}\left\{1 - \Big(\frac{D_{r,\ell}}{\sqrt{h_\gamma}} + D_1 + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2})\Big)\right\}^{(r-1)/\ell}.$$

Applying now the Taylor expansion to the function $f(t) = (1-t)^{(r-1)/\ell}$, for

$$t = \frac{D_{r,\ell}}{\sqrt{h_\gamma}} + D_1 + D_{r,\ell} + O(\|w_\lambda\|_\infty^{(r-1)/2}),$$

and sufficiently large λ, we get

$$\lambda - \Delta_5 \leq \gamma^{1-r}\Big(E_1 + \frac{E_2}{\sqrt{h_\gamma}} + \frac{E_3}{h_\gamma} + O(h_\gamma^{-3/2})\Big), \tag{3.18}$$

where the coefficients E_1, E_2, E_3 are defined as in the hypothesis of the theorem, and the coefficients C_1, C_2, $\cdots$, were defined in (2.9). Since, from (3.17),

$$h_\gamma = d_1^*\gamma^{1-r}\Big(1 + \frac{d_2^*}{d_1^*\sqrt{C}}\gamma^{(r-1)/2} + O(\gamma^{r-1})\Big),$$

we have

$$h_\gamma^{-1/2} = \frac{\gamma^{(r-1)/2}}{\sqrt{d_1^*}}\Big(1 - \frac{d_2^*}{2d_1^*\sqrt{C}}\gamma^{(r-1)/2} + \frac{3d_2^{*2}}{8d_1^{*2}C}\gamma^{r-1} - \frac{5d_2^{*3}}{16d_1^{*3}C\sqrt{C}}\gamma^{3(r-1)/2} + O(1)\Big), \tag{3.19}$$

and

$$h_\gamma^{-1} = \frac{\gamma^{r-1}}{d_1^*}\Big(1 - \frac{d_2^*}{d_1^*\sqrt{C}}\gamma^{(r-1)/2} + \frac{d_2^{*2}}{d_1^{*2}C}\gamma^{r-1} - \frac{d_2^{*3}}{d_1^{*3}C\sqrt{C}}\gamma^{3(r-1)/2} + O(1)\Big). \tag{3.20}$$

Substituting (3.19)-(3.20) into (3.18), we arrive at (3.11). By a same way, we obtain (3.12). The proof is complete. □

4. CONCLUSION

In the present research, a nonlinear eigenvalue problem consisting of a nonlinear Sturm-Liouville equation with a nonlinear term $\lambda q^{-1}(x)y^r(x)$, together with Dirichlet boundary conditions on a symmetric interval, was investigated. First, in the case $q(x) \equiv 1$, by using the supersolution of the problem, we obtained the lower and upper bounds for the solution $y_\lambda(x)$ associated with λ as $\lambda \to \infty$, and the asymptotic bounds of the spectral parameter λ were presented. Then, in the general case, we obtained the asymptotic formulas for the bifurcation parameter $\lambda = \lambda_\ell(\gamma)$ in $L^\ell-$framework, as $\gamma \longrightarrow 0$, and $L^\ell-$asymptotic properties of the nonlinear Sturm-Liouville problem were investigated.

5. APPENDIX

For each arbitrary real numbers $r, \ell > 1$, there exist positive integers m and n such that

$$m \leq r < m+1, \quad n \leq \ell < n+1.$$

Hence, we get

$$\begin{aligned}
0 &< \int_0^1 \frac{1-t^\ell}{\sqrt{1-t^{r+1}}}\mathrm{d}t \\
&\leq \int_0^1 \frac{1-t^{n+1}}{\sqrt{1-t^{m+1}}}\mathrm{d}t \\
&= \int_0^1 \frac{\sqrt{1-t}(1+t^2+t^3+\cdots+t^n)}{\sqrt{1+t^2+t^3+\cdots+t^m}}\mathrm{d}t \\
&\leq \int_0^1 (1+t^2+t^3+\cdots+t^n)\mathrm{d}t, \qquad \leq n \leq \ell.
\end{aligned}$$

Thus, for each $r, \ell > 1$, the integral $\int_0^1 \frac{1-t^\ell}{\sqrt{1-t^{r+1}}}\mathrm{d}t$ is convergent.

ACKNOWLEDGMENT

The authors wishes to express gratitude to Professor Angelo Bernardo Mingarelli for helpful discussions during the preparation of the manuscript. This research is partially supported by the University of Kashan under grant number 1312068/1.

REFERENCES

[1] G. A. Afrouzi, A. Hadjian, and V. D. Radulescu, *A variational approach of Sturm-Liouville problems with the nonlinearity depending on the derivative*, Boundary Value Problems, *81* (2015), 1–17.

[2] Z. S. Aliyev and G. M. Mamedova, *Some global results for nonlinear Sturm-Liouville problems with spectral parameter in the boundary condition*, Annales Polonici Math., *115* (2015), 75–87.

[3] H. Berestycki, *Le nombre de solutions de certains problemes semi-lineaires elliptiques*, J. Func. Anal., *40* (1981), 1–29.

[4] G. Bonanno, *Existence of three solutions for a two point boundary value problem*, Appl. Math. Lett., *13* (2000), 53–57.

[5] G. Bonanno and G. D. Agul, *A Neumann boundary value problem for the Sturm-Liouville equation*, Appl. Math. Comput., *208* (2009), 318–327.

[6] A. Bongers, H. P. Heinz, and T. Kupper, *Existence and bifurcation theorems for nonlinear elliptic eigenvalue problems on unbounded domains*, J. Diff. Equ., *47* (1983), 327–357.

[7] S. A. Buterin and C. T. Shieh, *Incomplete inverse spectral and nodal problems for differential pencils*, Results Math., *62* (2012), 167–179.

[8] J. Chabrowski, *On nonlinear eigenvalue problems*, Forum Math., *4* (1992), 359–375.

[9] X. Cheng and G. Dai, *Positive solutions of sub-superlinear Sturm-Liouville problems*, Appl. Math. Comput., *261* (2015), 351–359.

[10] R. Chiappinelli, *On spectral asymptotics and bifurcation for elliptic operators with odd superlinear term*, Nonlinear Analysis TMA, *13* (1989), 871–878.

[11] R. Chiappinelli, *Constrained critical points and eigenvalue approximation for semilinear elliptic operators*, Forum Math., *11* (1999), 459–481.

[12] J. Deuel and P. Hess, *A criterion for the existence of solutions of non-linear elliptic boundary value problems*, Proc. Royal Soc. Edinburgh, *74*(A) (1976), 49–54.

[13] P. Drabek, *Methods of Nonlinear Analysis: Applications to Differential Equations*, 2nd ed., Birkhäuser, Basel, 2013.

[14] J. M. Fraile, J. Lopez-Gomez , and J. C. Sabina de Lis, *On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems*, J. Diff. Equ., *123* (1995), 180–212.

[15] G. Freiling and V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publishers Inc., New York, 2001.

[16] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys., *68* (1979), 209–243.

[17] P. Girg, F. Rocab, and S. Villegas, *Semilinear Sturm-Liouville problem with periodic nonlinearity*, Nonlinear Analysis, *61* (2005), 1157–1178.

[18] M. Heid and H. P. Heinz, *Nonlinear eigenvalue problems admitting eigenfunctions with known geometric properties*, Topological Meth. Nonlinear Analysis, *13* (1999), 17–51.

[19] H. P. Heinz, *Free Ljusternik-Schnirelman theory and the bifurcation diagrams of certain singular nonlinear problems*, J. Diff. Equ., *66* (1987), 263–300.

[20] M. Holzmann and H. Kielhofer, *Uniqueness of global positive solution branches of nonlinear elliptic problems*, Mathematische Annalen, *300* (1994), 221–241.

[21] K. Katoa and T. Shibata, *Simple proof of stationary phase method and application to oscillatory bifurcation problems*, Nonlinear Analysis, *190* (2020), 1–13.

[22] J. B. Keller and S. Antman, *Bifurcation theory and nonlinear eigenvalueproblems*, W.A. Benjamin, Inc., New York, 1969.

[23] H. Koyunbakan, *The inverse nodal problem for a differential operator with an eigenvalue in the boundary condition*, Appl. Math. Lett., *21* (2008), 1301–1305.

[24] T. Laetsch, *The number of solutions of a nonlinear two point boundary value problem*, Indiana Univ. Math. J., *20* (1970), 1–13.

[25] G. M. Mamedova, *Local and global bifurcation for some nonlinearizable eigenvalue problems*, Proc. Institute Math. Mech., *40* (2014), 45–51.

[26] J. R. McLaughlin, *Inverse spectral theory using nodal points as data, a uniqueness result*, J. Diff. Equ., *73* (1988), 354–362.

[27] S. Mosazadeh, *Energy levels of a physical system and eigenvalues of an operator with a singular potential*, Reports Math. Phys., *82* (2018), 137–148.

[28] S. Mosazadeh, *Local stability of half inverse problems with boundary conditions dependent on the spectral parameter*, J. Math. Anal. Appl., *531* (2024) 127908.

[29] S. Mosazadeh, *Asymptotic behavior of eigenvalues and eigenfunctions of some nonlinear Sturm-Liouville problems*, Under review.

[30] A. Ozbekler, *Sturmian theory for second order differential equations with mixed nonlinearities*, Appl. Math. Comput., *259* (2015), 379–389.

[31] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Func. Anal., *7* (1971), 487–513.

[32] Y. R. Shen, *The Principles of Nonlinear Optics*, John Wiley & Sons, New York, 1984.

[33] T. Shibata, *Global behavior of the branch of positive solutions to a logistic equation of population dynamics*, Proc. Amer. Math. Soc., *136* (2008), 2547–2554.

[34] T. Shibata, *Direct and inverse bifurcation problems for non-autonomous logistic equations*, Elect. J. Diff. Equ., *2013* (2013), 1–14.

[35] Y. G. Smirnov and D. V. Valovik, *Guided electromagnetic waves propagating in a plane dielectric waveguide with nonlinear permittivity*, Phys. Rev. A, *91* (2015), 1–6.

[36] E. Y. Smolkin and D. V. Valovik, *Guided electromagnetic waves propagating in a two-layer cylindrical dielectric waveguide with inhomogeneous nonlinear permittivity*, Adv. Math. Phys., *2015* (2015), 1–11.

[37] D. V. Valovik, *The spectral properties of some nonlinear operators of Sturm-Liouville type*, Sbornik: Mathematics, *208* (2017), 1282–1297.

[38] Y. P. Wang and V. Yurko, *On the inverse nodal problems for discontinuous Sturm-Liouville operators*, J. Diff. Equ., *260* (2016), 4086–4109.

[39] R. Zhang, N. P. Bondarenko, and C. F. Yang, *Solvability of an inverse problem for discontinuous Sturm-Liouville operators*, Math. Meth. Appl. Sci., *44* (2021), 124–139.