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A highly accurate numerical technique for solving variable-order fractional Burgers-Huxley equation

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Abstract

In this article, we present a highly accurate technique for the numerical solution of the variable-order time-fractional Burgers-Huxley equation. The original equation is first discretized in the temporal and spatial directions. The third-order weighted-shifted Grünwald-Letnikov and the fourth-order compact finite difference methods are used. We then formulate a nonlinear system of algebraic equations using the fully discretized version of the problem. The derived nonlinear system is solved by utilizing an iterative algorithm. The analysis of solvability, stability, and convergence of the method is also addressed. The method achieves a convergence rate of four in the spatial direction and three in the temporal direction. Moreover, it is a low-cost computational method and easy to implement. Finally, various illustrative examples are solved to verify the accuracy of the proposed method.

Keywords. Variable-order fractional derivative, The Burgers-Huxley equation, Highly accurate method.
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1. Introduction

Fractional calculus (FC) is among the most fashionable tools in modeling phenomena featuring persistent memory effect [8]. Fractional-order integrals and derivatives generalize integer-order counterparts [1]. They are useful for modeling systems with memory effects and non-locality [3, 19]. In fact, an extension of the classical FC is the variable-order FC in which the orders of fractional integrals and derivatives are functions dependent on space and/or time [4, 22]. The variable-order fractional (VOF) provides a greater degree of freedom for modeling physical phenomena [24]. Moreover, their nonlocal properties improve the computational accuracy and reduce computational cost [18]. Researchers have recently applied VOF operators to reformulate popular problems in physics and engineering, such as diffusion, integro-differential equations, and optimal control [2, 12, 25]. Thus, designing an effective approximate algorithm to solve the VOF problems is desirable. Despite the beneficial properties of VOF operators, obtaining analytical solutions to these problems is highly arduous and, in most cases, impossible. Over the past thirty years, numerous researchers have employed approximate methods, such as wavelet-based spectral methods [6, 21] and finite difference (FD) methods [14, 29], to solve the partial differential equations involving VOF derivatives. In this paper, we extend an accurate numerical method to solve a boundary-value problem governed by the following inhomogeneous VOF Burgers-Huxley equation:

$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w + \lambda\, w w_z - w_{zz} = f(w,z,t), \quad a < z < b,\ 0 < t \le T, \tag{1.1} $$

where $w = w(z,t)$ is an unknown function, f is a Lipschitz continuous function with respect to w, and λ is a given positive parameter. Moreover, w_z and w_{zz} denote the first- and second-order derivatives of function w with respect to z, respectively. The term ${}_0\mathcal{D}_t^{\alpha(z,t)} w$ signifies the Caputo-type VOF derivative of order $\alpha(z,t)$ of w defined by

$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w(z,t) := \frac{1}{\Gamma(1-\alpha(z,t))} \int_0^t \frac{\partial w(z,\eta)}{\partial \eta} (t-\eta)^{-\alpha(z,t)} d\eta, \tag{1.2} $$

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where $0 < \alpha(z,t) < 1$, $\overline{\alpha} := \sup\{\alpha(z,t) : (z,t) \in [a,b] \times [0,T]\}$, $\underline{\alpha} := \inf\{\alpha(z,t) : (z,t) \in [a,b] \times [0,T]\}$, and $\Gamma(.)$ is the Gamma function [28]. The problem given by Eq. (1.1) with $\alpha(z,t) = 1$ reduces to the classical Burgers-Huxley equation, which can be modeled more accurately with its VOF version [16]. This equation is utilized for modeling diffusion transport, nerve pulse propagation in nerve fibers, the interaction between reaction mechanisms, convection effects, and motion in liquid crystals [17]. Since Eq. (1.1) is a nonlinear PDE, there is no effective method to derive its exact solution. Recently, many approximate methods have been developed by scientists for solving the constant-order fractional Burgers-Huxley equation. In [7], the residual fractional power series method was applied for solving the time-fractional version of the mentioned equation. In [26], an efficient technique has been proposed for the constant-order time-space fractional Burgers-Huxley equation. In [13], Lie symmetry analysis was applied to the constant-order time-fractional version of Eq. (1.1). Recently, an FD method has been presented for the constant-order time-fractional Burgers-Huxley equation [27]. In [10], an optimization technique based on the generalized polynomials is developed for solving the VOF Burgers equation. In [11], a collocation method based on the Fibonacci polynomials has been proposed for the numerical solution of a variable-order time-space fractional Burgers-Huxley equation. To our knowledge, there is no sufficient study on approximate schemes for the VOF Burgers-Huxley equation (1.1).

The main goal of this manuscript is to propose a highly accurate method based on FD formulas for the numerical solution of the VOF Burgers-Huxley equation (1.1). To this end, a fourth-order compact FD method is first applied to discretize this problem in the spatial direction. Then a third-order weighted shifted Grünwald-Letnikov (SGL) method is used to discretize (1.1) in the temporal direction. Consequently, fully discretizing the original problem yields a sequence of nonlinear algebraic systems. Finally, we implement a first-order nonlinear solver for the derived nonlinear systems. The convergence and stability analysis of the proposed method are rigorously proved. The advantages of the current method are listed as follows:

- This novel method is easy to implement and straightforward.
- The coefficient matrices of the present method are tridiagonal. Therefore, it is an inexpensive computational approach to solve the nonlinear VOF Burgers-Huxley equation.
- The method is unconditionally stable, and it does not show sensitivity to round-off errors.
- The proposed technique converges at a rate of four in the spatial direction and three in the temporal direction.

The structure of this paper is as follows: In section 2, we develop a highly accurate numerical technique for the VOF Burgers-Huxley equation, as well as an analysis of its stability, solvability, and convergence. In section 3, four test examples are illustrated to verify the applicability and accuracy of the presented technique. The conclusion part is addressed in section 4.

2. Description of the numerical method

Here, we will attempt to design a high-order fully discrete method for the inhomogeneous VOF Burgers-Huxley equation (1.1). In the first step, consider a reformulation of Eq. (1.1) as

$$-w_{zz} + \lambda\, ww_z = \Psi, \quad a < z < b,\ 0 < t \le T, \tag{2.1}$$

where $\Psi = f - {}_0\mathcal{D}_t^{\alpha(z,t)} w$. Then, we recall the SGL difference formula [5]

$$\mathcal{A}_{\tau,p}^{\alpha(z,t)} w(z,t) := \frac{1}{\tau^{\alpha(z,t)}} \Big[\sum_{k=0}^{[\frac{\zeta}{\tau}]+p} \omega_k^{(\alpha(z,t))} w(z, \zeta - (k-p)\tau) \Big]_{\zeta=t}, \tag{2.2}$$

in which $\tau > 0$, p is an integer, and the coefficients $\omega_k^{(\alpha(z,t))}$ are computed by a recursive formula

$$\omega_{k+1}^{(\alpha(z,t))} = (1 - \frac{1+\alpha(z,t)}{1+k})\omega_k^{(\alpha(z,t))}, \quad k = 0, 1, 2, \ldots, \tag{2.3}$$

where $\omega_0^{(\alpha(z,t))} = 1$. As shown in [5], if $w(z,0) = 0$, then ${}_0\mathcal{D}_t^{\alpha(z,t)} w(z,t) = \mathcal{A}_{\tau,p}^{\alpha(z,t)} w(z,t) + \mathcal{O}(\tau)$. In [9], the following properties of $\omega_k^{(\alpha(z,t))}$ have been easily obtained

(1) $\omega_k^{(\alpha(z,t))} < 0, \quad k \ge 1,$

(2) $\gamma_0 := 1,\ \gamma_k := \max\{|\omega_k^{(\alpha(z,t))}| : \ \underline{\alpha} \le \alpha(z,t) \le \overline{\alpha}\} \le \overline{\alpha}, \quad k \ge 1,$

(3) $|\omega_{k+1}^{(\alpha(z,t))}| < |\omega_k^{(\alpha(z,t))}| \le \gamma_k, \quad k \ge 0,$

(4) $\sum_{k=0}^{n} |\omega_k^{(\alpha(z,t))}| \le \sum_{k=0}^{n} \gamma_k < 2, \quad \forall\, n \ge 1,$

where $0 < \underline{\alpha} \le \alpha(z,t) \le \overline{\alpha} < 1$, and γ_k is not affected by changes in z and t.

The subsequent lemma from [15] presents a third-order weighted SGL formula designed to approximate the VOF derivative of order $\alpha(z,t)$ of w as defined in (1.2).

Now, we mesh the solution area $\Omega := [a,b] \times [0,T]$. For positive integers M and N, let $(z_i, t_n) := (a + ih, n\tau)$ with $0 \le n \le N+1, 0 \le i \le M+1$, be a uniform mesh on Ω of $h := (b-a)/(M+1)$ and $\tau := T/(N+1)$. For every i and n, we also set $\alpha_i^n := \alpha(z_i, t_n)$.

Lemma 2.1. *[15] Let $0 < \alpha_i^n < 1$ and $\frac{\partial^m w(z,t)}{\partial t^m}|_{t=0} = 0$, for $m = 0,1,2,3$; then we deduce*

$$ {}_0\mathcal{D}_t^{\alpha_i^n} w(z_i, t_n) = \tau^{-\alpha_i^n} \sum_{k=0}^{n} g_k^{(\alpha_i^n)} w(z_i, t_n - k\tau) + \mathcal{O}(\tau^3), \quad as \quad \tau \to 0, \tag{2.4} $$

uniformly for $n \ge 1$, where

$$ \begin{cases} g_0^{(\alpha_i^n)} = \rho_0^{(\alpha_i^n)} \omega_0^{(\alpha_i^n)}, \\ g_1^{(\alpha_i^n)} = \rho_0^{(\alpha_i^n)} \omega_1^{(\alpha_i^n)} + \rho_1^{(\alpha_i^n)} \omega_0^{(\alpha_i^n)}, \\ g_k^{(\alpha_i^n)} = \rho_0^{(\alpha_i^n)} \omega_k^{(\alpha_i^n)} + \rho_1^{(\alpha_i^n)} \omega_{k-1}^{(\alpha_i^n)} + \rho_2^{(\alpha_i^n)} w_{k-2}^{(\alpha_i^n)}, \quad k = 2, 3, \dots, \end{cases} \tag{2.5} $$

and $\rho_0^{(\alpha_i^n)} = 1 + \frac{17}{24}\alpha_i^n + \frac{1}{8}(\alpha_i^n)^2,\ \rho_1^{(\alpha_i^n)} = -\frac{11}{12}\alpha_i^n - \frac{1}{4}(\alpha_i^n)^2,\ \rho_2^{(\alpha_i^n)} = \frac{5}{24}\alpha_i^n + \frac{1}{8}(\alpha_i^n)^2.$

Lemma 2.2. *[15] For $0 < \alpha_i^n < 1$, the coefficients $\{g_k^{(\alpha_i^n)}\}$ satisfy in*

(1) $1 < g_0^{(\alpha_i^n)} < 1 + \frac{5}{6}\overline{\alpha}, \quad g_1^{(\alpha_i^n)} < 0,\ g_2^{(\alpha_i^n)} > 0, \quad g_k^{(\alpha_i^n)} < 0, \quad \forall k \ge 3,$

(2) $\sum_{k=0}^{\infty} g_k^{(\alpha_i^n)} = 0,$ *and* $|g_k^{(\alpha_i^n)}| < \frac{10}{3}\overline{\alpha}, \quad \forall\, k \ge 1,$

(3) $-\sum_{k=1}^{n} g_k^{(\alpha_i^n)} < g_0^{(\alpha_i^n)} + g_2^{(\alpha_i^n)}, \quad \forall\, n \ge 1.$

Using the second and third properties, we can also deduce that

$$ |g_1^{(\alpha(z,t))}| \le (1 + \frac{11}{4}\overline{\alpha})\gamma_0, \quad \text{and} \quad |g_k^{(\alpha(z,t))}| \le (1 + \frac{11}{4}\overline{\alpha})|\omega_{k-2}^{(\alpha(z,t))}| \le (1 + \frac{11}{4}\overline{\alpha})\gamma_{k-2}, \quad k \ge 2. \tag{2.6} $$

Below, we formulate a fourth-order compact FD method for the space discretization of nonlinear differential equation (2.1) that yields a system of time-dependent VOF equations. Then, to discretize the derived system in the temporal direction, we will apply the third-order weighted SGL method (2.4).

The spatial discretization of Eq. (2.1). A semi-discrete form of the nonlinear differential equation (2.1) at z_i can be formulated as

$$ (-\frac{1}{h^2} + \frac{\lambda}{4h}\delta_z w_i - \frac{\lambda^2}{12} w_i w_i)\delta_z^2 w_i + (\frac{\lambda}{h} w_i - \frac{\lambda^2}{12} w_i \delta_z w_i)\delta_z w_i = (1 - \frac{\lambda h}{12} w_i \delta_z + \frac{1}{12}\delta_z^2)\Psi_i + R_i(h)\ , \tag{2.7} $$

where $w_i := w(z_i, .)$, $\Psi_i := \Psi(z_i, .)$, and the difference operators δ_z and δ_z^2 are define as

$$ \delta_z w_i := \frac{1}{2}(w_{i+1} - w_{i-1}), \qquad \delta_z^2 w_i := w_{i+1} - 2w_i + w_{i-1}. $$

Lemma 2.3. *Let the solution $w(z,.)$ be a six times continuously differentiable function of z. Then, the error term $R_i(h)$ of the formula (2.7) for Eq. (2.1) is of order four, i.e., $R_i(h) = \mathcal{O}(h^4)$.*

Proof. Using the Taylor expansion of $w_{i\pm1}$ at point z_i, the value of Eq. (2.1) can be calculated as

$$ -\frac{1}{h^2}\delta_z^2 w_i + \frac{\lambda}{h}\ w_i \delta_z w_i + \frac{h^2}{12}\zeta_i = \Psi_i + \frac{h^4}{360}(w_{zzzzzz} - 3\lambda w w_{zzzzz})|_{z=\eta_i}, \tag{2.8} $$

where $\zeta = 2\lambda w w_{zzz} - w_{zzzz}$ and $z_{i-1} < \eta_i < z_{i+1}$. After calculating the derivatives of Eq. (2.1) with respect to z, we have

$$\begin{cases} w_{zzz} = \lambda(w w_{zz} + (w_z)^2) - \Psi_z, \\ w_{zzzz} = \lambda(w w_{zzz} + 3 w_z w_{zz}) - \Psi_{zz} = ((\lambda\ w)^2 + 3\lambda w_z) w_{zz} + (\lambda\ w_z)^2 w - \lambda\ \Psi_z w - \Psi_{zz}. \end{cases}$$

This leads to

$$\zeta = 2\lambda w w_{zzz} - w_{zzzz} = (3\lambda w_z - (\lambda\ w)^2) w_{zz} - (\lambda\ w_z)^2 w + \lambda\ w \Psi_z - \Psi_{zz}.$$

Using operators δ_z and δ_z^2, we can derive

$$\begin{aligned} \zeta_i = \zeta(z_i, .) &= (3\lambda h\ \delta_z w_i - \frac{\lambda^2}{h^2} w_i w_i)\delta_z^2 w_i - \frac{\lambda^2}{h^2} w_i (\delta_z w_i)^2 + \frac{\lambda}{h} w_i \delta_z \Psi_i - \frac{1}{h^2}\delta_z^2 \Psi_i \\ &- \frac{\lambda h^2}{36}(6\lambda w^2 w_{zzzz} + (\lambda w w_{zzz} - 3 w_{zzzz}) w_{zzz} h^2)|_{z=\theta_i}, \end{aligned} \tag{2.9}$$

where $z_{i-1} < \theta_i < z_{i+1}$. By substituting relation (2.9) into (2.8), relation (2.7) can be concluded for

$$R_i(h) = \max_{0 \le z \le L} \{ | \frac{1}{360}(w_{zzzzzz} - 3\lambda w w_{zzzzz}) + \frac{\lambda^2}{144} w^2 w_{zzzz} | \} h^4. \tag{2.10}$$

□

Full discretization of Eq. (2.1). Here, to develop a high-order discretization technique for Eq. (1.1), the discrete forms (2.4) and (2.7) are implemented in temporal and spatial directions, respectively. Consider the following fully discrete method for solving the initial-boundary value problem (1.1), which determines approximations w_i^n to the values $w(z_i, t_n)$ from the relation

$$(-\frac{1}{h^2} + \frac{\lambda}{4h}\delta_z w_i^n - \frac{\lambda^2}{12} w_i^n w_i^n)\delta_z^2 w_i^n + (\frac{\lambda}{h} w_i^n - \frac{\lambda^2}{12} w_i^n \delta_z w_i^n)\delta_z w_i^n = (1 - \frac{\lambda h}{12} w_i^n \delta_z + \frac{1}{12}\delta_z^2)\Psi_i^n, \tag{2.11}$$

for $n = 1, 2, \ldots, N+1$, and $i = 1, 2, \ldots, M$, where

$$\Psi_i^n = f_i^n - \tau^{-\alpha_i^n} \sum_{k=0}^{n} g_k^{(\alpha_i^n)} w_i^{n-k}, \tag{2.12}$$

and $f_i^n := f(w_i^n, z_i, t_n)$. Using the homogenous initial and boundary conditions, we have

$$w_i^0 = 0, \quad 0 \le i \le M+1, \quad \text{and} \quad w_0^n = w_{M+1}^n = 0, \quad 0 \le n \le N+1.$$

Consider the matrix forms of grid functions as $\mathbf{W}^n := [w_1^n, w_2^n, \ldots, w_M^n]^T$, $\mathbf{F}^n := f(\mathbf{W}^n, t_n, \mathbf{z})$, and $\mathbf{z} := [z_1, z_2, \ldots, z_M]^T$; then, we present a matrix form of Eqs. (2.11) as

$$\begin{cases} \mathbf{W}^0 = [0, 0, \ldots, 0]^T, \\ ((-\frac{1}{h^2}\mathbf{I} + \frac{\lambda}{4h}\mathbf{J}_1 \mathbf{W}^n - \frac{\lambda^2}{12}(\mathbf{W}^n)^2)\mathbf{J}_2 + (\frac{\lambda}{h}\mathbf{W}^n - \frac{\lambda^2}{12}\mathbf{W}^n \mathbf{J}_1 \mathbf{W}^n)\mathbf{J}_1)\mathbf{W}^n \\ = (\mathbf{I} - \frac{\lambda h}{12}\mathbf{W}^n \mathbf{J}_1 + \frac{1}{12}\mathbf{J}_2)\big(\mathbf{F}^n - \sum_{k=0}^{n} \mathbf{diag}(\mathbf{g}_{\tau,k}^{(\alpha^n)})\mathbf{W}^{n-k}\big), \quad n = 1, \ldots, N+1, \end{cases} \tag{2.13}$$

where $\mathbf{g}_{\tau,k}^{(\alpha^n)} := [\tau^{-\alpha_1^n} g_k^{(\alpha_1^n)}, \tau^{-\alpha_2^n} g_k^{(\alpha_2^n)}, \ldots, \tau^{-\alpha_M^n} g_k^{(\alpha_M^n)}]^T$, $\mathbf{J}_1 := \mathbf{tridiag}(-\frac{1}{2}, 0, \frac{1}{2})$, $\mathbf{J}_2 := \mathbf{tridiag}(1, -2, 1)$, and $\mathbf{I}$ denotes an identity matrix. As it is demonstrated that for every $1 \le n \le N+1$, the implicit numerical method (2.13) forms a system of nonlinear algebraic equations. Therefore, a simple and fast nonlinear solver is needed to solve these nonlinear systems. A linearized form of Eq. (2.13) can be developed as follows:

$$\begin{cases} \mathbf{W}^0 = [0,0,\ldots,0]^T, \\ \text{For s} = 0,1,\ldots,\text{r}, \\ \quad \mathbf{U}^{[0]} = \mathbf{W}^{n-1}, \\ \quad ((-\frac{1}{h^2}\mathbf{I} + \frac{\lambda}{4h}\mathbf{J}_1\mathbf{U}^{[s]} - \frac{\lambda^2}{12}(\mathbf{U}^{[s]})^2)\mathbf{J}_2 + (\frac{\lambda}{h}\mathbf{U}^{[s]} - \frac{\lambda^2}{12}\mathbf{U}^{[s]}\mathbf{J}_1\mathbf{U}^{[s]})\mathbf{J}_1)\mathbf{U}^{[s+1]} \\ \quad\quad = (\mathbf{I} - \dfrac{\lambda h}{12}\mathbf{U}^{[s]}\mathbf{J}_1 + \dfrac{1}{12}\mathbf{J}_2)\left[\mathbf{F}(\mathbf{U}^{[s]},\mathbf{U}^{[s+1]}) - \mathbf{diag}(\mathbf{g}_{\tau,0}^{(\alpha^n)})\mathbf{U}^{[s+1]} - \displaystyle\sum_{k=1}^{n}\mathbf{diag}(\mathbf{g}_{\tau,k}^{(\alpha^n)})\mathbf{W}^{n-k}\right], \\ \mathbf{W}^n = \mathbf{U}^{[r+1]}, \quad n = 1,\ldots,N+1, \end{cases} \tag{2.14}$$

where $\mathbf{F}^n(\mathbf{U}^{[s]},\mathbf{U}^{[s+1]})$ is a first-order linearization form of the inhomogeneous term $\mathbf{F}^n = f(\mathbf{W}^n, t_n, \mathbf{z})$; for instance, if $f(w,z,t) = w(1-w)(w-\gamma)$, then we set $\mathbf{F}^n(\mathbf{U}^{[s]},\mathbf{U}^{[s+1]}) = \mathbf{diag}((1-\mathbf{U}^{[s]})(\mathbf{U}^{[s]}-\gamma))\mathbf{U}^{[s+1]}$. The symbol $\mathbf{U}^{[s]}$ represents the approximate solution that is found in the s-th iteration of (2.14). The number of iterations r can be determined such that the inequality $\|\mathbf{U}^{[r+1]} - \mathbf{U}^{[r]}\| < \varepsilon$ is attained.

2.1. **The analysis of solvability, stability, and convergence of the present method.** The analysis of solvability, stability, and convergence of the present iterative method (2.14) for the VOF Burgers-Huxley equation (1.1) will be stated by the following lemma and theorems. For matrix $\mathbf{A}$, let $\|\mathbf{A}\|$ be its L_2 norm. As it is concluded in [23], the eigenvalues of $\mathbf{J}_1$ and $\mathbf{J}_2$ are respectively derived as

$$\gamma_s^{\mathbf{J}_1} := \mathbf{i}\cos(\frac{s\pi}{M+1}), \quad \gamma_s^{\mathbf{J}_2} := -4\sin^2(\frac{s\pi}{2(M+1)}), \tag{2.15}$$

where $s = 1,2,\ldots,M$, and $\mathbf{i} = \sqrt{-1}$.

Lemma 2.4. *(Solvability) For every integer number n, the system of Equations (2.14) has a unique solution.*

Proof. Let $q = \max_{i,n}|w_i^n|$, thus we can give the following form of Eqs. (2.14) by

$$A^n\mathbf{W}^n = -\sum_{k=1}^{n}\beta_k^n\mathbf{W}^{n-k} + \beta_0\mathbf{F}^n, \tag{2.16}$$

where $\beta_0 = (\mathbf{I} - \frac{\lambda h}{12}q\mathbf{J}_1 + \frac{1}{12}\mathbf{J}_2)$ and $\beta_k^n = \beta_0\mathbf{diag}(\mathbf{g}_{\tau,k}^{(\alpha^n)})$. Moreover, the tridiagonal matrix A^n is given by

$$A^n := \begin{bmatrix} d_1^n & c_2^n & & & \\ b_1^n & d_2^n & c_3^n & & \\ & \ddots & \ddots & \ddots & \\ & & & & c_M^n \\ & & & b_{M-1}^n & d_M^n \end{bmatrix},$$

where its lower, main, and upper diagonal elements are the following expressions, respectively

$$\begin{cases} b_k^n &= -(\frac{1}{h} + \frac{\lambda}{4}q)^2 - \frac{\lambda^2}{48}q^2 + (\frac{1}{12} - \frac{\lambda h}{24}q)\tau^{-\alpha_k^n}g_0^{(\alpha_k^n)}, \quad k = 1,\ldots,M-1, \\ d_k^n &= \frac{2}{h^2} + \frac{\lambda^2}{6}q^2 + \frac{5}{6}\tau^{-\alpha_k^n}g_0^{(\alpha_k^n)}, \quad k = 1,\ldots,M, \\ c_k^n &= -(\frac{1}{h} - \frac{\lambda}{4}q)^2 - \frac{\lambda^2}{48}q^2 + (\frac{1}{12} + \frac{\lambda h}{24}q)\tau^{-\alpha_k^n}g_0^{(\alpha_k^n)}, \quad k = 2,\ldots,M, \end{cases}$$

Using Lemma 2.2, we have $1 < g_0^{(\alpha_k^n)} < 2$. Then, for small values of step sizes h and τ, the elements b_k^n and c_k^n are negative numbers. Therefore, we have

$$|b_k^n| + |c_k^n| = -(b_k^n + c_k^n) = \frac{2}{h^2} + \frac{\lambda^2}{6}q^2 + \frac{1}{6}\tau^{-\alpha_k^n}g_0^{(\alpha_k^n)} < d_k^n = |d_k^n|, \qquad 2 \le k \le M-1.$$

It is easy to get that $|d_k^n| > |b_k^n| + |c_k^n|$ for $1 \le k \le M$ if we set $b_M^n = c_1^n = 0$. This yields that the matrix A^n is diagonally dominant. Thus, this matrix is invertible. Consequently, for every integer number n, the system (2.14) has a unique solution. □

To establish the unconditionally stability and convergence analysis of the proposed method, we first recall Lemma 1.4.2 in [20] as follows.

Lemma 2.5. *(Gronwall's inequality) Assume that k_n is a non-negative sequence, and that the sequence ϕ_n satisfies*

$$\phi_0 \leq s_0, \qquad \phi_n \leq s_0 + \sum_{i=0}^{n-1} k_i \phi_i, \quad n \geq 1,$$

where $s_0 \geq 0$. Then ϕ_n satisfies $\phi_n \leq s_0 \exp(\sum_{i=0}^{n-1} k_i)$ for $n \geq 1$.

Theorem 2.6. *(Unconditionally stability) The iterative method (2.14) is unconditionally stable to solve the VOF equation (1.1).*

Proof. Let $\tilde{\mathbf{W}}^n$ holds in below

$$\tilde{\mathbf{W}}^0 = \Phi^0, \tag{2.17a}$$

$$A^n \tilde{\mathbf{W}}^n = -\sum_{k=1}^{n} \beta_k^n \tilde{\mathbf{W}}^{n-k} + \beta_0 \tilde{\mathbf{F}}^n, \qquad 1 \leq n \leq N+1, \tag{2.17b}$$

in which $\Phi^0 := [\phi_1, \phi_2, \ldots, \phi_M]^T$ and $\tilde{\mathbf{F}}^n = f(\tilde{\mathbf{W}}^n, t_n, \mathbf{z})$. Now, we define $\mathbf{E}^n := \tilde{\mathbf{W}}^n - \mathbf{W}^n - \Phi^0$; subtract (2.17b) from (2.16), a roundoff error equation is concluded as

$$A^n \mathbf{E}^n = -\sum_{k=1}^{n} \beta_k^n \mathbf{E}^{n-k} - (\sum_{k=1}^{n} \beta_k^n + A^n)\Phi^0 + \beta_0(\tilde{\mathbf{F}}^n - \mathbf{F}^n), \quad n \geq 1, \tag{2.18}$$

where $\mathbf{E}^0 := [0, \ldots, 0]^T$. Applying (2.15) , we have

$$\|\beta_0\| = \max_s \sqrt{(1 - \frac{1}{3}\sin^2(\frac{s\pi}{2(M+1)}))^2 + \frac{\lambda^2 h^2}{144} q^2 \cos^2(\frac{s\pi}{M+1})} \leq C_0, \tag{2.19}$$

where $C_0 = \sqrt{\frac{4}{9} + \frac{\lambda^2 h^2}{144} q^2}$. Let γ_s^n be the sth eigenvalue of A^n. Thanks to the Gershgorin Circle Theorem, we can achieve that

$$|\gamma_s^n| \geq \frac{2}{3} \tau^{-\alpha_s^n} g_0^{(\alpha_s^n)}, \qquad s = 1, 2, \ldots, M. \tag{2.20}$$

From Lemma 2.2, we then have

$$\|(A^n)^{-1} \beta_k^n\| = \frac{3}{2} \|\beta_0\| |g_k^{(\alpha_s^n)}| \leq \frac{3}{2} C_0 |g_k^{(\alpha_s^n)}|, \qquad s = 1, 2, \ldots, M. \tag{2.21}$$

From inequality (2.6), we can write

$$\sum_{k=0}^{n} |g_k^{(\alpha_i^n)}| \leq (1 + \frac{11}{4}\overline{\alpha})(2\gamma_0 + \sum_{k=2}^{n} \gamma_{k-2}) \leq (4 + 11\overline{\alpha}), \quad \forall n \geq 1. \tag{2.22}$$

Since f is a Lipschitz continuous function with respect to w, then there exists a positive constant $L := \max |\frac{\partial f}{\partial w}|$ such that $|f(\tilde{w}, z, t) - f(w, z, t)| \leq L|\tilde{w} - w|$. Now by applying Eqs. (2.21)–(2.22) to the difference equations (2.18) and using the properties of $\omega_k^{(\alpha(z,t))}$, it yields

$$\begin{aligned}
(1 - \frac{3}{2} C_0 L \tau^{\overline{\alpha}}) \|\mathbf{E}^n\| &\leq \sum_{k=1}^{n} \|(A^n)^{-1} \beta_k^n\| \|\mathbf{E}^{n-k}\| + \|\Phi^0\| \sum_{k=0}^{n-1} \|(A^n)^{-1} \beta_k^n\|) + (1 + \frac{3}{2} C_0 L T^{\overline{\alpha}}) \|\Phi^0\| \\
&\leq \frac{3}{2} C_0 \sum_{k=1}^{n} |g_k^{(\alpha_s^n)}| \|\mathbf{E}^{n-k}\| + \frac{3}{2} C_0 \|\Phi^0\| \sum_{k=0}^{n} |g_k^{(\alpha_s^n)}| + (1 + \frac{3}{2} C_0 L T^{\overline{\alpha}}) \|\Phi^0\|
\end{aligned}$$

$$\leq \frac{3}{8}(4+11\overline{\alpha})C_0(\gamma_0\|\mathbf{E}^{n-1}\| + \sum_{k=2}^{n}\gamma_{k-2}\|\mathbf{E}^{n-k}\|) + \frac{3}{2}C_0(4+11\overline{\alpha})\|\Phi^0\| + (1+\frac{3}{2}C_0LT^{\overline{\alpha}})\|\Phi^0\|$$

$$\leq \frac{3}{8}(4+11\overline{\alpha})C_0\sum_{k=0}^{n-1}\tilde{\gamma}_k\ \|\mathbf{E}^k\| + (\frac{3}{2}C_0(4+11\overline{\alpha}+LT^{\overline{\alpha}})+1)\|\Phi^0\|, \tag{2.23}$$

where $\tilde{\gamma}_k = \gamma_{n-k-2}$ for $k = 0,\dots,n-2$, and $\tilde{\gamma}_{n-1} = \gamma_0$. Equation (2.23) leads to

$$\|\mathbf{E}^n\| \leq \frac{3(4+11\overline{\alpha})C_0}{8-12C_0L\tau^{\overline{\alpha}}}\sum_{k=0}^{n-1}\tilde{\gamma}_k\|\mathbf{E}^k\| + \frac{(3C_0(4+11\overline{\alpha}+LT^{\overline{\alpha}})+2)}{2-3C_0L\tau^{\overline{\alpha}}}\|\Phi^0\|.$$

Let $3C_0L\tau^{\overline{\alpha}} \neq 2$. Since coefficients $\tilde{\gamma}_k$ are independent of n and $\sum_{k=0}^{n-1}\tilde{\gamma}_k < 2$, then using the discrete Gronwall's inequality given by Lemma 2.5, there exists a positive constant C_1 such that

$$\|\mathbf{E}^n\| \leq C_1\|\Phi^0\|, \qquad n = 1,\dots N+1, \tag{2.24}$$

in which

$$C_1 = \frac{(3C_0(4+11\overline{\alpha}+LT^{\overline{\alpha}})+2)}{2-3C_0L\tau^{\overline{\alpha}}}\exp\big(\frac{3(4+11\overline{\alpha})C_0}{8-12C_0L\tau^{\overline{\alpha}}}\sum_{k=0}^{n-1}\tilde{\gamma}_k\big).$$

Therefore, the unconditional stability of the iterative method (2.14) is established, as required. □

Theorem 2.7. *(Convergence analysis) Let $w \in \mathcal{C}^{6,4}_{x,t}(\Omega)$ be the solution of the Burgers-Huxley equation (1.1) satisfying in $\frac{\partial^k w(z,t)}{\partial t^k}|_{t=0} = 0$ for $k = 0,1,2,3$; then the solution of the iterative method (2.14) converges to w as $h,\tau \to 0$. Furthermore, its convergence rate is of the order $\mathcal{O}(\tau^3+h^4)$.*

Proof. Define $e_i^n := w(z_i,t_n) - w_i^n$ where $0 \leq n \leq N+1,\ 0 \leq i \leq M+1$. Since f is a Lipschitz continuous function with respect to w, then there exists a positive constant $L = \max|\frac{\partial f}{\partial w}|$ such that $|f(w_i^n,z_i,t_n) - f(w(z_i,t_n),z_i,t_n)| \leq L|e_i^n|$. Let $q = \max_{i,n}\{|w_i^n|,|w(z_i,t_n)|\}$. Then, by subtracting (2.7) from (2.16), we obtain the error equation

$$A^n\mathbf{e}^n = -\sum_{k=1}^{n}\beta_k^n\mathbf{e}^{n-k} + \beta_0(f(\mathbf{w}^n,t_n,\mathbf{z}) - f(\mathbf{W}^n,t_n,\mathbf{z})) + \mathbf{R}^n(h,\tau), \quad n \geq 1, \tag{2.25}$$

where $\mathbf{w}^n = [w(z_1,t_n),\dots,w(z_M,t_n)]^T$, $\mathbf{e}^0 := [0,\dots,0]^T$, $\mathbf{e}^n := [e_1^n,\dots,e_M^n]^T$ and $\mathbf{R}^n := [R_1^n,\dots,R_M^n]^T$. From Lemmas 2.1 and 2.3, there exists a constant C_2 such that $\|\mathbf{R}^n(h,\tau)\| \leq C_2(\tau^3+h^4)$. Hence, applying the discrete Gronwall's inequality given by Lemma 2.5, Lemma 2.2, Eqs. (2.19)–(2.22), and Eq. (2.6) to (2.25) deduces

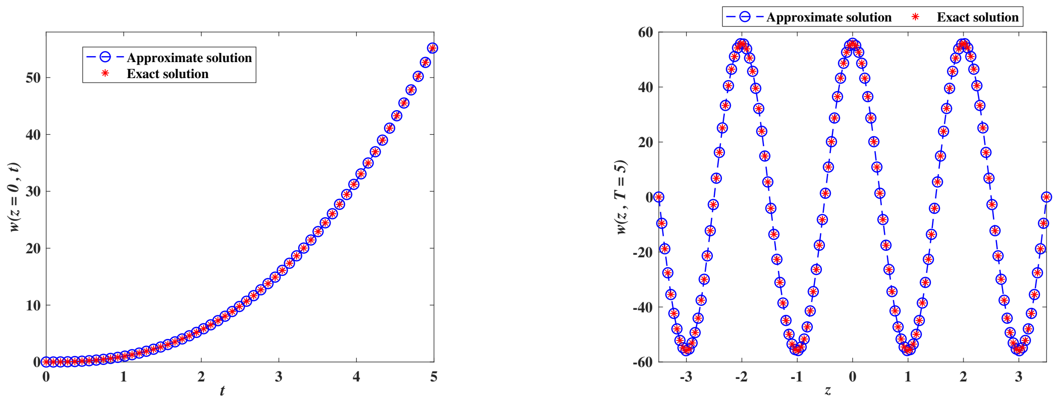
$$(1-\frac{3}{2}C_0L\tau^{\overline{\alpha}})\|\mathbf{e}^n\| = \sum_{k=1}^{n}\|(A^n)^{-1}\beta_k^n\|\|\mathbf{e}^{n-k}\| + C_2(\tau^3+h^4)$$

$$\leq \frac{3}{8}(4+11\overline{\alpha})C_0(\gamma_0\|\mathbf{e}^{n-1}\| + \sum_{k=2}^{n}\gamma_{k-2}\|\mathbf{e}^{n-k}\|) + C_2(\tau^3+h^4)$$

$$\leq C_2(\tau^3+h^4)\exp(\frac{3}{8}C_0(4+11\overline{\alpha})\sum_{k=0}^{n-1}\gamma_k)$$

$$\leq C_2\exp(\frac{3}{4}C_0(4+11\overline{\alpha}))(\tau^3+h^4), \qquad n \geq 1. \tag{2.26}$$

This completes the proof. □



(A) Plots of solutions versus time direction. (B) Plots of solutions versus space direction.

FIGURE 1. The comparison of the exact and approximate solutions to Example 3.1 after 20 iterations is made when $\lambda = 0.1, \beta(z,t) = \frac{5}{2}, \alpha(z,t) = \frac{1}{80}(8 - z + t^3)$, and $[-\frac{7}{2}, -\frac{7}{2}] \times [0, 5]$ is divided into 256×1620 cells.

TABLE 1. The temporal convergence rate of the method (2.14) after $r = 20$ iterations to solve Example 3.1 with $\lambda = 1$, the exact solution $w(z,t) = t^{\beta(z,t)} \cos(\pi z)$, $M \times N$ cells, and $M \approx \sqrt[4]{N^3}$.

		$\alpha(z,t) = \frac{1}{30}(\exp(zt) + \cos(zt))$		$\alpha(z,t) = \frac{1}{5}(1 - (zt)^4)$		$\alpha(z,t) = \frac{1}{80}(8 - z + t^3)$	
$\beta(z,t)$	N	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate
	20	3.0330e-05	-	3.0488e-05	-	3.0379e-05	-
$\frac{5}{2}$	80	5.6363e-07	2.88	5.6542e-07	2.88	5.6411e-07	2.88
	320	8.8011e-09	3.00	8.8763e-09	3.00	8.9997e-09	2.99
	20	3.1701e-05	-	3.3044e-05	-	3.2131e-05	-
$\pi + z\sin(z)$	80	5.8859e-07	2.88	6.1036e-07	2.88	5.9529e-07	2.88
	320	9.3604e-09	2.99	9.7011e-09	2.99	9.4662e-09	2.99
	20	3.4485e-05	-	3.4353e-05	-	3.4445e-05	-
$2\cosh(zt)$	80	6.4728e-07	2.87	1.2903e-06	2.37	6.4657e-07	2.87
	320	1.1713e-08	2.89	2.6390e-08	2.81	1.1876e-08	2.88
	20	3.0610e-05	-	3.1413e-05	-	3.0865e-05	-
$2 + \alpha(z,1)$	40	5.6745e-07	2.88	5.7888e-07	2.88	5.7075e-07	2.88
	320	8.8049e-09	3.00	8.9329e-09	3.00	8.8904e-09	3.00

3. NUMERICAL RESULTS

In this section, numerical results of the iterative method (2.14) to solve three types of the inhomogeneous VOF Burgers-Huxley equation are illustrated. The provided numerical results verify the theoretical results of Theorems 2.6 and 2.7. All simulations were performed by applying the MATLAB software on an Intel ® Core$^{\text{TM}}$ i7-4702MQ machine with 8 GB of RAM and a 2.20 GHz processor. The convergence rate of the present iterative method is evaluated by

$$\text{Rate} := \log_{(\frac{m_2}{m_1})}\left(\frac{\varepsilon_1}{\varepsilon_2}\right),$$

where ε_1 and ε_2 are the maximum absolute errors (MAEs) of the present method using m_1 and m_2 cells in the time/space direction, respectively.

Table 2. The spatial convergence rate of the method (2.14) after $r = 20$ iterations to solve Example 3.1 with $\lambda = 1$, the exact solution $w(z,t) = t^{\beta(z,t)}\ \cos(\pi z)$, $M \times N$ cells, and $N \approx \sqrt[3]{M^4}$.

		$\alpha(z,t) = \frac{1}{30}(\exp(zt) + \cos(zt))$		$\alpha(z,t) = \frac{1}{5}(1-(zt)^4)$		$\alpha(z,t) = \frac{1}{80}(8 - z + t^3)$	
$\beta(z,t)$	M	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate
	8	7.4866e-05	-	7.5052e-05	-	7.5052e-05	-
$\frac{5}{2}$	32	2.8636e-07	4.02	2.8738e-07	4.01	2.8738e-07	4.01
	128	1.1171e-09	4.00	1.1168e-09	4.00	1.1298e-09	4.00
	8	7.6131e-05	-	7.8739e-05	-	7.6902e-05	-
$\pi + z\sin(z)$	32	2.9774e-07	4.00	3.0815e-07	4.00	3.0099e-07	4.00
	128	1.1628e-09	4.00	1.2042e-09	4.00	1.1757e-09	4.00
	8	8.5197e-05	-	9.4983e-05	-	8.5122e-05	-
$2\cosh(z)$	32	3.2795e-07	4.01	4.2059e-07	3.91	3.2757e-07	4.01
	128	1.2810e-09	4.00	1.8003e-09	3.93	1.3011e-09	3.99
	8	7.5321e-05	-	7.5706e-05	-	7.5706e-05	-
$2 + \alpha(z,1)$	32	2.8837e-07	4.01	2.9011e-07	4.01	2.9011e-07	4.01
	128	1.1247e-09	4.00	1.1314e-09	4.00	1.1314e-09	4.00

Table 3. The comparison of MAEs obtained by the present iterative method and the algorithm in [10] to solve Example 3.1 at $[0,1] \times [0,1]$ with the exact solution $w(z,t) = t^2 \cos(\pi z)$.

Order of fractional derivative $\alpha(z,t)$	Method in [10] (with $m_1 = 5,\ m_2 = 3$)	Present iterative method (with $M = 128, N \approx \sqrt[3]{M^4}$)
$\frac{1}{30}(\exp(zt) + \cos(zt))$	2.9625e-06	4.7326e-09
$\frac{1}{80}(8 - z + t^3)$	1.3854e-06	8.6052e-09

Example 3.1. Consider the VOF Burgers-Huxley equation (1.1) as follows:

$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w + \lambda w w_z - w_{zz} = g(z,t), \quad (z,t) \in \Omega, \tag{3.1}$$

in which $g(z,t)$ is calculated by applying the exact solution $w(z,t) = t^{\beta(z,t)} \cos(\pi z)$.

The maximum absolute error (MAE) and the temporal/spatial convergence rate of the proposed iterative method (2.14) for solving Example 3.1 with $\lambda = 1$ are presented in Tables 1 and 2 when $\Omega = [-\frac{1}{2}, \frac{1}{2}] \times [0,1]$ is divided into $M \times N$ cells, and the VO function $\alpha(z,t)$ is selected as

$$\alpha(z,t) = \frac{1}{30}(\exp(zt) + \cos(zt)),\ \frac{1}{5}(1 - (zt)^4),\ \frac{1}{80}(8 - z + t^3). \tag{3.2}$$

The obtained numerical results in Tables 1 and 2 illustrate that this new method is of order $\mathcal{O}(\tau^3 + h^4)$, and they confirm the concluded results of Theorem 2.7. As Tables 1 and 2 show the present method is unconditionally stable for different cases of the exact solution $w(z,t) = t^{\beta(z,t)}\ \cos(\pi z)$ where $\beta(z,t) = \frac{5}{2}, \pi + z\sin(z), 2\cosh(zt), 2 + \alpha(z,t)$. Although these cases of exact solutions have some singular behavior at $t = 0$, the present method is effective and successful in deriving the approximate solution of Example 3.1.

A comparison of the exact and approximate solutions to Example 3.1 is also made in Figure 1 when $\lambda = 0.1$, $\beta(z,t) = \frac{5}{2}$, $\alpha(z,t) = \frac{1}{80}(8 - z + t^3)$, and $[-\frac{7}{2}, -\frac{7}{2}] \times [0,5]$ is divided into 256×1620 cells. Part (A) shows solutions at $z = 0$ and $0 \le t \le 5$, and part (B) demonstrates them at $t = 0$ and $-\frac{7}{2} \le z \le \frac{7}{2}$. To derive these plots, the proposed iterative method is converged after 20 iterations and 197.02 seconds, and its maximum absolute error is 2.2432×10^{-4}. Hence, this figure illustrates that the present method is unconditionally stable and is not affected by round-off errors in the large computational domain $\Omega = [-\frac{7}{2}, -\frac{7}{2}] \times [0,5]$. Moreover, the MAEs obtained by the present iterative method and the algorithm in [10] for solving Example 3.1 for two different values of $\alpha(z,t)$ are compared in Table 3. This table shows that the iterative method (2.14) is accurate and very effective compared to the algorithm given by [10].

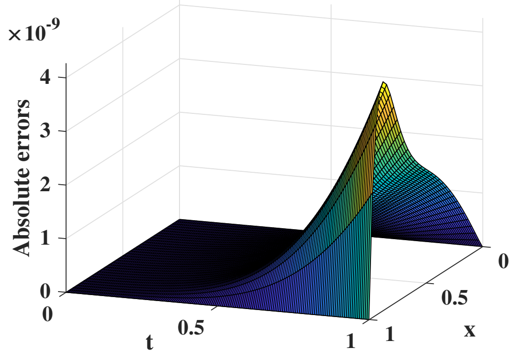


FIGURE 2. The absolute error of the present iterative technique after $r = 30$ iterations to solve Example 3.2 with $\alpha(z,t) = \frac{20\exp(zt)-12}{20\exp(zt)-10}$ and 256×1620 cells.

TABLE 4. The temporal convergence rate of the iterative technique (2.14) after $r = 30$ iterations to solve Example 3.2 with $M \times N$ cells and $M \approx \sqrt[4]{N^3}$.

	$\alpha(z,t) = \frac{1}{30}(\exp(zt)+\cos(zt))$		$\alpha(z,t) = \frac{1}{5}(1-(zt)^4)$		$\alpha(z,t) = \frac{20\exp(zt)-12}{20\exp(zt)-10}$		
N	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate	CPU time
10	1.1100e-02	-	1.1100e-02	-	1.0500e-02	-	0.03
20	1.7000e-03	2.69	1.7000e-03	2.69	1.7000e-03	2.67	0.05
40	2.9193e-04	2.57	2.9180e-04	2.57	2.7688e-04	2.58	0.08
80	3.7046e-05	2.99	3.7032e-05	2.54	3.5228e-05	2.97	0.21
160	4.8858e-06	2.93	4.8836e-06	2.92	4.6225e-06	2.93	1.10
320	6.0372e-07	3.02	6.0346e-07	3.02	5.7241e-07	3.01	4.30
640	7.5234e-08	3.00	7.5203e-08	3.00	7.1239e-08	3.00	25.22

TABLE 5. The spatial convergence rate of the iterative technique (2.14) after $r = 30$ iterations to solve Example 3.2 with $M \times N$ cells and $N \approx \sqrt[3]{M^4}$.

	$\alpha(z,t) = \frac{1}{30}(\exp(zt)+\cos(zt))$		$\alpha(z,t) = \frac{1}{5}(1-(zt)^4)$		$\alpha(z,t) = \frac{20\exp(zt)-12}{20\exp(zt)-10}$		
M	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate	CPU time
4	3.5830e-02	-	3.5795e-02	-	3.3292e-02	-	0.01
8	4.0641e-03	3.14	4.0625e-03	3.13	3.8724e-03	3.01	0.02
16	2.9193e-04	3.80	2.9180e-04	3.80	2.7688e-04	3.80	0.08
32	1.9006e-05	3.94	1.8998e-05	3.94	1.8001e-05	3.94	0.31
64	1.2005e-06	3.98	1.2000e-06	3.98	1.1367e-06	3.99	2.74
128	6.6503e-08	4.17	7.5203e-08	4.00	7.1239e-08	4.00	26.12
256	4.1701e-09	4.00	4.6821e-09	4.00	4.4503e-09	4.00	300.05

Example 3.2. Consider the following inhomogeneous VOF Burgers-Huxley equation:

$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w + \frac{\pi}{2} w w_z - w_{zz} = w(1-w)(w-\frac{3}{4}) + g(z,t), \quad (z,t) \in (0,1)^2, \tag{3.3}$$

where the function $g := g(z,t)$ is calculated using the exact solution $w(z,t) = 30\sqrt[3]{t^7}\,(\sqrt{z^{11}} - z^6)$.

The MAE, temporal/spatial convergence rate, and elapsed CPU time (in seconds) of the proposed iterative technique (2.14) after $r = 30$ iterations for the numerical solution of the inhomogeneous VOF Burgers-Huxley equation (3.3)

Table 6. The temporal convergence rate of the iterative technique (2.14) after $r = 40$ iterations to solve Example 3.3 with $M \times N$ cells and $M \approx \sqrt[4]{N^3}$.

	$\alpha(z,t) = 0.5$		$\alpha(z,t) = \frac{1}{4}(1-z^4)$		$\alpha(z,t) = \sqrt{3}(\cosh(z)-1)$		
N	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate	CPU time
10	4.0624e-01	-	5.2281e-01	-	5.0135e-01	-	0.04
20	1.7000e-03	2.53	7.9784e-02	2.71	7.7465e-02	2.69	0.06
40	9.8610e-03	2.84	1.1178e-02	2.84	1.0773e-02	2.85	0.10
80	1.2934e-03	2.94	1.4625e-03	2.93	1.4119e-03	2.94	0.28
160	1.7184e-04	2.91	1.9548e-04	2.90	1.8749e-04	2.92	1.36
320	2.1095e-05	3.03	2.4937e-05	2.97	2.3014e-05	3.03	6.10
640	2.6230e-06	3.01	3.1219e-06	3.00	2.8624e-06	3.00	37.02

Table 7. The spatial convergence rate of the iterative technique (2.14) after $r = 40$ iterations to solve Example 3.3 with $M \times N$ cells and $N \approx \sqrt[3]{M^4}$.

	$\alpha(z,t) = 0.5$		$\alpha(z,t) = \frac{1}{4}(1-z^4)$		$\alpha(z,t) = \sqrt{3}(\cosh(z)-1)$		
M	Maximum Error	Rate	Maximum Error	Rate	Maximum Error	Rate	CPU time
16	9.8610e-03	-	1.1178e-02	-	1.0773e-02	-	0.10
32	6.7037e-04	3.88	7.5541e-04	3.89	7.3123e-04	3.88	0.41
64	4.1891e-05	4.00	4.8604e-05	3.96	4.5697e-05	4.00	3.64
128	2.6230e-06	4.00	3.0566e-06	3.99	2.8624e-06	4.00	37.12
256	1.6404e-07	4.00	1.8999e-07	4.00	1.7878e-07	4.00	410.05

are presented in Tables 4 and 5. These tables demonstrate that the convergence rate of this method is $\mathcal{O}(\tau^3 + h^4)$ for different values of the VOF order function $\alpha(z,t)$. Moreover, the absolute error of the present technique after $r = 30$ iterations on a grid with 300×2000 cells for solving Example 3.2 with $\alpha(z,t) = \frac{20\exp(zt)-12}{20\exp(zt)-10}$ is plotted by Figure 2. This figure demonstrates that the present method is not affected by round-off errors. Also, it is unconditionally stable when the number of cells is increased. Consequently, these numerical results follow the analytical results of Theorems 2.6 and 2.7.

Example 3.3. Consider the inhomogeneous VOF Burgers-Huxley equation (1.1) as

$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w + 10ww_z - w_{zz} = \frac{1}{3}w(1-w^2) + g(z,t), \quad (z,t) \in (-1,1) \times (0,1], \tag{3.4}$$

where the function $g(z,t)$ is calculated using the exact solution $w(z,t) = (1-z^6)\exp(1-z^2)t^{\sqrt{3}+\alpha(z,t)}$.

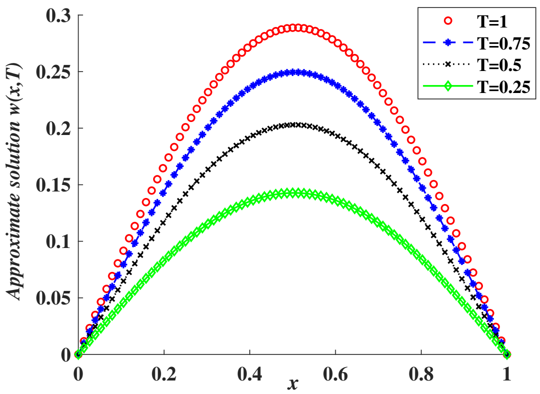
For solving the inhomogeneous VOF Burgers-Huxley equation (3.5), the present iterative method is converged after $r = 40$ iterations. Numerical results are provided in Tables 6 and 7. These two tables show that this method is not sensitive to round-off error and has third- and fourth-order accuracy in temporal and spatial directions, respectively. Moreover, the approximate solution follows the initial singularity of the exact solution. Thus, they follow the theoretical results of Theorems 2.6 and 2.7.

Example 3.4. Consider the following VOF Burgers-Huxley equation with homogeneous initial-boundary conditions:

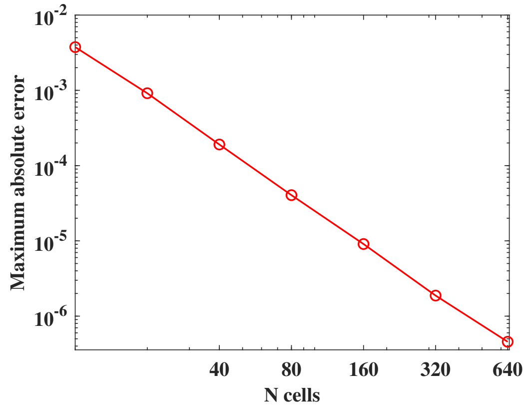
$$ {}_0\mathcal{D}_t^{\alpha(z,t)} w + ww_z - w_{zz} = w(1-w)(w-\frac{1}{4}) + \pi\sqrt{t}\sin(\pi x), \quad (z,t) \in \Omega = (0,1) \times (0,1]. \tag{3.5}$$

The exact solution of this problem is unknown.

Due to the unavailability of an exact solution for the underlying problem, we utilize a reference solution obtained by the present technique after 25 iterations on a grid with 300×2000 cells. The plots of the approximate solution and the log-log plot of MAEs deriving the present technique after 25 iterations for Example 3.4 with $\alpha(z,t) = \frac{1+\sin^2(100xt)}{3}$ are depicted by Figure 3. Here, we divide Ω into $M \times N$ cells, where M and N are two positive integers that satisfy $N^3 = M^4$. It indicates that decreasing mesh sizes leads to more accurate results.



(A) $w(x,T)$ versus x at final times $T = 0.25, 0.5, 0.75$, and 1.



(B) The log-log plot of MAEs versus N cells.

FIGURE 3. Results of the present technique after 25 iterations for Example 3.4 with $\alpha(z,t) = \frac{1+\sin^2(100xt)}{3}$.

4. Conclusion

In this work, we have presented a simple and efficient iterative technique for the numerical solution of the VOF Burgers-Huxley equation. The idea is to use a fourth-order compact FD formula and a third-order weighted SGL method to discretize the VOF Burgers-Huxley equation in spatial and temporal directions, respectively. Thus, the VOF Burgers-Huxley problem reduces to a nonlinear algebraic system requiring a nonlinear solver to derive its solution. The presented iterative method is tested on four problems governed by the VOF Burgers-Huxley equation. The numerical results validate the theoretical analysis and demonstrate the method's effectiveness. In future works, we will use the presented strategy for nonlinear problems containing the time-space VOF derivatives.

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