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φ-Caputo fractional integro-differential equations with nonlocal conditions via noncompactness measure

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Abstract

This article aims to prove the existence and uniqueness of solutions to fractional integro-differential equations involving the φ-Caputo fractional derivative with nonlocal conditions. The results are obtained using the measure of noncompactness, probability density functions, Mönch's fixed point theorem, and semigroup theory. To demonstrate the applicability and effectiveness of the results, an illustrative example is presented.

Keywords. ϕ-Caputo fractional derivative, Semigroup theory, Measure of noncompactness, Mönch's fixed point theorem, Integro-differential equations, Nonlocal conditions.

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1. Introduction

In the current era, fractional calculus is experiencing a surge in interest, finding applications in fields as diverse as physics, electrochemistry, biophysics, viscoelasticity, control theory, biomedicine, and signal processing. Fractional differential equations have emerged as valuable tools for modeling several phenomena in different fields of engineering and science. Recent years have witnessed significant developments in fractional order differential equations, as demonstrated by the work of Kilbas et al. [9], Podlubny [15], Zhou [19], as well as related papers such as [1, 6, 14, 16], and [5, 8, 17, 18].

Malar et al. [11] have explored the existence of solutions to abstract impulsive integro-differential problems with noncompactness measures

$$\begin{cases} v'(\vartheta) = Av(\vartheta) + f\left(\vartheta, v(\vartheta), \displaystyle\int_0^{\vartheta} \varrho(\vartheta,s)h(\vartheta,s,v(s))ds\right), \vartheta \in (s_i, \vartheta_{i+1}], i = 0,1,2,...,m, \\ v(\vartheta) = g_i(\vartheta, v(\vartheta)), \quad \vartheta \in (\vartheta_i, s_i], i = 0,1,2,...,m, \\ v(0) = v_0 + \Phi(v), \end{cases}$$

where $0 = \vartheta_0 = s_0 < \vartheta_1 \leq s_1 \leq \vartheta_2 < ... < \vartheta_m \leq s_m \leq \vartheta_{m+1} = a$ are pre-fixed numbers, A generates a C_0-semigroup of linear bounded operator $\{T(\vartheta), \vartheta \geq 0\}$ defined on a Banach space $(\mathbb{X}, \|.\|)$. $\Phi : \mathbb{X} \to \mathbb{X}$, $v_0 \in \mathbb{X}$, $g_i \in \mathcal{C}((\vartheta_i, s_i] \times \mathbb{X}, \mathbb{X})$, for every $i = 1,2,...,m$ and $f : [0,a] \times \mathbb{X}^2 \to \mathbb{X}$ is a function, $h \in \mathcal{C}(\Delta \times \mathbb{X}, \mathbb{R}^+)$ where $\Delta = \{(\vartheta, s) \ / \ \vartheta, s \in [0,a], \vartheta \geq s\}$.

In this study, we investigate the existence of solutions for nonlinear evolution equations

$$\begin{cases} {}^C D_{0^+}^{\alpha,\varphi} v(\vartheta) = Av(\vartheta) + f\left(\vartheta, v(\vartheta), \displaystyle\int_0^{\vartheta} \varrho(\vartheta,s)h(\vartheta,s,v(s))ds\right), \vartheta \in \mathcal{U}, \\ v(0) = v_0 + \Phi(v), \end{cases} \tag{1.1}$$

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where $\mathcal{U}=[0,a]$, ${}^{C}D_{0^+}^{\alpha,\varphi}$ is the φ-Caputo fractional derivative of order $\alpha\in(0,1)$, A generates an analytical semigroup of linear bounded operators $\{T(\vartheta),\vartheta\geq 0\}$ on a Banach space $(\mathbb{X},\|.\|)$. $v_0\in\mathbb{X}$, $f:\mathcal{U}\times\mathbb{X}^2\to\mathbb{X}$ and $\Phi:\mathbb{X}\to\mathbb{X}$ are functions, $h\in\mathcal{C}(\Delta\times\mathbb{X},\mathbb{R}^+)$ with $\Delta=\{(\vartheta,s)\ /\ \vartheta,s\in\mathcal{U},\ \vartheta\geq s\}$.

The structure of this paper is organized as follows: In section 2, we recall some notation and several well-known results related to fractional calculus. In section 3, we present our main findings regarding the existence of solutions to the fractional problem. In section 4 provides an example to illustrate the practical applications of the abstract results.

2. Preliminaries

In this part, we present several definitions, properties, and notations that will be used consistently throughout this paper.

$\mathcal{C}(\mathcal{U},\mathbb{X})$ is the Banach space of all continuous functions v with the norm is defined as follows $\|v\|_\infty=\sup\limits_{\vartheta\in\mathcal{U}}\|v(\vartheta)\|$, where $\mathcal{U}$ is a bounded interval.

Additionally, we denote by $B_r=\big\{v\in\mathcal{C}(\mathcal{U},\mathbb{X}):\ \|v\|_\infty\leq r\big\}$ and $\Delta=\big\{(\vartheta,s):\ \vartheta,s\in\mathcal{U},\vartheta\geq s\big\}$.

Definition 2.1. [2] Let $\alpha>0$, $\varphi\in\mathcal{C}^n(\mathcal{U},\mathbb{R})$ with $\varphi'(\vartheta)>0$, for every $\vartheta\in\mathcal{U}$ and $g\in L^1(\mathcal{U},\mathbb{R})$. The φ-Riemann-Liouville fractional integral of order α of g is defined by

$$I_{0^+}^{\alpha,\varphi}g(\vartheta)=\frac{1}{\Gamma(\alpha)}\int_0^\vartheta\varphi'(s)(\varphi(\vartheta)-\varphi(s))^{\alpha-1}g(s)ds. \tag{2.1}$$

Remark 2.2. It's worth noting that when $\varphi(\vartheta)=\log(\vartheta)$ or $\varphi(\vartheta)=\vartheta$, Equation (2.1) simplifies to the Hadamard and Riemann-Liouville fractional integrals, respectively.

Definition 2.3. [2] Let $\alpha>0$, $\varphi\in\mathcal{C}^n(\mathcal{U},\mathbb{R})$ with $\varphi'(\vartheta)>0$, for every $\vartheta\in\mathcal{U}$ and $g\in C^{n-1}(\mathcal{U},\mathbb{R})$. The φ-Caputo fractional derivative of order α of g is defined by

$${}^{C}D_{0^+}^{\alpha,\varphi}g(\vartheta)=\frac{1}{\Gamma(n-\alpha)}\int_0^\vartheta\varphi'(s)(\varphi(\vartheta)-\varphi(s))^{n-\alpha-1}g_\varphi^{[n]}(s)ds, \tag{2.2}$$

where $g_\varphi^{[n]}(\epsilon)=\big(\frac{1}{\varphi'(\epsilon)}\frac{d}{d\epsilon}\big)^n g(\epsilon)$, $n=[\alpha]+1$ and $\lfloor\alpha\rceil$ is the integer part of α.
In particular, if $0<\alpha<1$, then

$${}^{C}D_{0^+}^{\alpha,\varphi}g(\vartheta)=\frac{1}{\Gamma(\alpha)}\int_0^\vartheta(\varphi(\vartheta)-\varphi(s))^{\alpha-1}g'(s)ds,$$

and

$${}^{C}D_{0^+}^{\alpha,\varphi}g(\vartheta)=I_{0^+}^{1-\alpha,\varphi}\big(\frac{g'(\vartheta)}{\varphi'(\vartheta)}\big).$$

Remark 2.4. It's worth noting that when $\varphi(\vartheta)=\log(\vartheta)$ or $\varphi(\vartheta)=\vartheta$, Equation (2.2) simplifies to the Hadamard and Riemann-Liouville fractional derivatives, respectively.

Proposition 2.5. *[2] Let $\alpha>0$ and $g\in\mathcal{C}^{n-1}(\mathcal{U},\mathbb{R})$. Then, we can get the following*

1) $I_{0^+}^{\alpha,\varphi}\ {}^{C}D_{0^+}^{\alpha,\varphi}g(\vartheta)=g(\vartheta)-\sum\limits_{k=0}^{n-1}\frac{g_\varphi^{[k]}(0)}{k!}(\varphi(\vartheta)-\varphi(0))^k.$
2) ${}^{C}D_{0^+}^{\alpha,\varphi}I_{0^+}^{\alpha,\varphi}g(\vartheta)=g(\vartheta).$
3) $I_{0^+}^{\alpha,\varphi}$ *is bounded and linear from* $\mathcal{C}(\mathcal{U},\mathbb{R})$ *to* $\mathcal{C}(\mathcal{U},\mathbb{R})$.

Proposition 2.6. *[2] Let $\vartheta>0$ and $\alpha,\delta>0$. Then, we have the following*

1) $I_{0^+}^{\alpha,\varphi}(\varphi(\vartheta)-\varphi(0))^{\delta-1}=\dfrac{\Gamma(\delta)}{\Gamma(\delta+\alpha)}(\varphi(\vartheta)-\varphi(0))^{\alpha+\delta-1}.$
2) ${}^{C}D_{0^+}^{\alpha,\varphi}(\varphi(\vartheta)-\varphi(0))^{\delta-1}=\dfrac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}(\varphi(\vartheta)-\varphi(0))^{\alpha-\delta-1}.$

3) ${}^{C}D_{0+}^{\alpha,\varphi}(\varphi(\vartheta)-\varphi(0))^{k}=0, \quad \forall k<n\in\mathbb{N}.$

Definition 2.7. [7] Consider a function $v:\mathcal{U}\to\mathbb{X}$. The generalized Laplace transform of v is expressed as

$$\mathcal{L}_{\varphi}\{v(\vartheta)\}(s):=\widehat{v}(s)=\int_{0}^{\infty}\varphi'(\vartheta)e^{-s(\varphi(\vartheta)-\varphi(0))}v(\vartheta)d\vartheta.$$

Definition 2.8. [7] Consider two functions, f and g, that are of exponential order and piecewise continuous on $\mathcal{U}$. The definition of the generalized φ-convolution of the functions g and f is as follows:

$$(f\underset{\varphi}{*}g)(\vartheta)=\int_{0}^{\vartheta}g[\varphi^{-1}(\varphi(\vartheta)+\varphi(0)-\varphi(s))]\varphi'(s)f(s)ds.$$

Lemma 2.9. *(See [7]) Suppose $\alpha>0$ and v is a piecewise continuous function on $[0,\vartheta]$ with $\varphi(\vartheta)$-exponential order. In this case, we have the following*

(1) $\mathcal{L}_{\varphi}\{I_{0+}^{\alpha,\varphi}v(\vartheta)\}(s)=\dfrac{\widehat{v}(s)}{s^{\alpha}}$.

(2) $\mathcal{L}_{\varphi}\{{}^{C}D_{0+}^{\alpha,\varphi}v(\vartheta)\}(s)=s^{\alpha}\big[\mathcal{L}_{\varphi}\{v(\vartheta)\}-\displaystyle\sum_{k=0}^{n-1}s^{-k-1}v^{(k)}(0)\big]$, *such that* $n=[\alpha]+1$.

Definition 2.10. (See [10]) For any $\rho\in[0,\infty)$. The one-sided stable probability density is expressed by the following equation

$$\omega_{\alpha}(\rho)=\frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{n-1}(\varphi(\rho)-\varphi(0))^{-\alpha n-1}\frac{\Gamma(\alpha n+1)}{n!}\sin(n\pi\alpha).$$

Lemma 2.11. *[10] The Laplace transform of $\omega_{\alpha}(\vartheta)$ is represented by*

$$\int_{0}^{\infty}e^{-\lambda(\varphi(\vartheta)-\varphi(0))}\omega_{\alpha}(\vartheta)\varphi'(\vartheta)d\vartheta=e^{-\lambda^{\alpha}}.$$

Lemma 2.12. *[3] Let H and G be two non-empty and bounded subsets in a Banach space $\mathbb{X}$. The following properties are verified*

(1) $\mu(G)\leq\mu(H)$ *when* $G\subset H$,
(2) G *is pre-compact if and only if* $\mu(G)=0$,
(3) $\mu(G)=\mu(\overline{G})=\mu(convG)$, *where $convG$ and $\overline{G}$ denote the closure and convex hull of G, respectively,*
(4) $\mu(G+H)\leq\mu(G)+\mu(H)$, *where* $G+H=\{z+x,z\in G,x\in H\}$,
(5) $\mu(H\cup G)\leq\max\{\mu(H),\mu(G)\}$,
(6) $\mu(\ell G)=|\ell|\mu(G)$ *for every* $\ell\in\mathbb{R}$,
(7) *If the mapping $\mathcal{F}:D(\mathcal{F})\subset\mathbb{X}\to\mathcal{Y}$ is Lipschitz continuous with the constant ℓ. Then, for each bounded subset G in $D(\mathcal{F})$, we have $\mu(Q(G))\leq\ell\mu(G)$, where $\mathcal{Y}$ is a Banach space.*
(8) *We have*

$$\begin{aligned}\mu(G)&=\inf\big\{d(G,H),H\subset\mathbb{X}\text{ be pre-compact}\big\}\\&=\inf\{d(G,H),H\subset\mathbb{X}\text{ be finite valued}\}.\end{aligned}$$

Here, $d(G,H)$ denotes the symmetric (or nonsymmetric) Hausdorff distance between the two subsets G and H in $\mathbb{X}$.

(9) *If $\{G_m\}_m^{\infty}=1$ is a decreasing sequence and $\lim_{m\to\infty}\mu(G_m)=0$, then $\cap_{m=1}^{\infty}G_m\neq\emptyset$ and compact in the Banach space $\mathbb{X}$, where G_m are bounded non-empty closed subsets of $\mathbb{X}$.*

Lemma 2.13. *[13] If $\{v_m\}_{m=1}^{\infty}\subset L^1(0,K,\mathbb{X})$ is uniformly integrable. Hence, $\mu(\{v_m\}_{m=1}^{\infty})$ is measurable. Moreover, we have*

$$\mu(\{\int_{0}^{\vartheta}v_m(s)ds\}_{m=1}^{\infty})\leq2\int_{0}^{\vartheta}\mu\{v_m(s)\}_{m=1}^{\infty}ds.$$

Lemma 2.14. *[4] If the set B_r is bounded. Then, for any given $\varepsilon > 0$, there exists a sequence $\{v_m\}_{m=1}^{\infty}$ contained within B_r with*

$$\mu(B_r) \leq 2\mu(\{v_m\}_{m=1}^{\infty}) + \varepsilon.$$

Theorem 2.15. *[13] Let B be a convex closed part of a Banach space $\mathbb{X}$ containing the origin 0 and $\mathcal{P} : B \to \mathbb{X}$ be a continuous map satisfying the condition mapping that satisfies Mönch condition, which asserts that for any ($G \subseteq B$ is countable, $G \subseteq \overline{co}(\{0\} \bigcup \mathcal{P}(G) \to \overline{G}$ is compact). Then, the map $\mathcal{P}$ possesses at least one fixed point in D.*

3. Results

In this section, we formulate the integral solution for the fractional problem (1.1) using the φ-Laplace transform. To achieve this, the following lemma must be established.

Lemma 3.1. *The following integral equation is equivalent to the fractional evolution problem (1.1):*

$$v(\epsilon) = v_0 + \Phi(v) + I_{0^+}^{\alpha;\varphi}\big[Av(\epsilon) + f\big(\epsilon, v(\epsilon), \int_0^{\epsilon} \varrho(\epsilon, s)h(\epsilon, s, v(s))ds\big)\big], \text{ for all } \epsilon \in \mathcal{U}. \tag{3.1}$$

Proof. Suppose v solves the problem (1.1). Applying the φ-fractional integral $I_{0^+}^{\alpha;\varphi}$ to both sides of the problem (1.1), we can obtain the following

$$I_{0^+}^{\alpha;\varphi}\, {}^C D_{0^+}^{\alpha;\varphi} v(\epsilon) = I_{0^+}^{\alpha;\varphi}\big[Av(\epsilon) + f\left(\epsilon, v(\epsilon), \int_0^{\epsilon} \varrho(\epsilon, s)h(\epsilon, s, v(s))ds\right)\big].$$

Utilizing Proposition 2.5, we derive

$$v(\epsilon) - v(0) = I_{0^+}^{\alpha;\varphi}[Av(\epsilon) + f(\epsilon, v(\epsilon), \int_0^{\epsilon} \varrho(\epsilon, s)h(\epsilon, s, v(s))ds)],$$

since $v(0) = v_0 + \Phi(v)$, it follows that

$$v(\epsilon) = v_0 + \Phi(v) + \frac{1}{\Gamma(\alpha)} \times \int_0^{\epsilon} \varphi'(s)(\varphi(\epsilon) - \varphi(s))^{\alpha-1}\big[Av(s) + f\big(s, v(s), \int_0^{s} \varrho(s, \vartheta)h(s, \vartheta, v(\vartheta))d\vartheta\big)\big]ds.$$

Hence, the integral Equation (3.1) holds.

Conversely, through straightforward calculations, it becomes obvious that if v satisfies the integral Equation (3.1), then the problem (1.1) is also satisfied. □

Lemma 3.2. *If the integral Equation (3.1) is satisfied, then we can obtain the following*

$$\begin{aligned} v(\epsilon) = S_{\alpha,\varphi}(\epsilon)(v_0 + \Phi(v)) + \int_0^{\epsilon} R_{\alpha,\varphi}(\varphi(\epsilon) - \varphi(s)) \\ \times f\left(s, v(s), \int_0^{s} \varrho(s, \vartheta)h(s, \vartheta, v(\vartheta))d\vartheta\right)\varphi'(s)ds. \end{aligned}$$

Proof. Let $\lambda > 0$. Utilizing the generalized Laplace transforms on the integral Equation (3.1) and employing Lemma 2.9, we get

$$\widehat{v}(\lambda) = \frac{1}{\lambda}(v_0 + \Phi(v)) + \frac{1}{\lambda^{\alpha}}\big(A\widehat{v}(\lambda) + \widehat{f}(\lambda)\big),$$

where $\widehat{v}(\lambda) = \int_0^{\infty} \varphi'(\epsilon)e^{-\lambda(\varphi(\epsilon)-\varphi(0))}v(\epsilon)d\epsilon$, and

$$\widehat{f}(\lambda) = \int_0^{\infty} \varphi'(\epsilon)e^{-\lambda(\varphi(\epsilon)-\varphi(0))} f\big(\epsilon, v(\epsilon), \int_0^{\epsilon} \varrho(\epsilon, s)h(\epsilon, s, v(s))ds\big)d\epsilon,$$

it follows that

$$\widehat{v}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}(v_0 + \Phi(v)) + (\lambda^{\alpha}I - A)^{-1}\widehat{f}(\lambda).$$

Thus,

$$\widehat{v}(\lambda) = \lambda^{\alpha-1}\int_0^{+\infty} e^{-\lambda^\alpha s}T(s)(v_0+\Phi(v))ds + \int_0^{+\infty} e^{-\lambda^\alpha s}T(s)\widehat{f}(\lambda)ds.$$

Choosing $s=\vartheta^\alpha$ with $\vartheta = \varphi(\epsilon)-\varphi(0)$, we get

$$\widehat{v}(\lambda) = \alpha\lambda^{\alpha-1}\int_0^{+\infty} \vartheta^{\alpha-1}e^{-(\lambda\vartheta)^\alpha}T(\vartheta^\alpha)(v_0+\Phi(v))d\vartheta + \alpha\int_0^{+\infty}\vartheta^{\alpha-1}e^{-(\lambda\vartheta)^\alpha}T(\vartheta^\alpha)\widehat{f}(\lambda)d\vartheta := I_1+I_2,$$

where

$$\begin{aligned}
I_1 &= \alpha\lambda^{\alpha-1}\int_0^{+\infty}\Big[(\varphi(\epsilon)-\varphi(0))^{\alpha-1}e^{-\lambda^\alpha(\varphi(\epsilon)-\varphi(0))^\alpha}T((\varphi(\epsilon)-\varphi(0))^\alpha)(v_0+\Phi(v))\varphi'(\epsilon)\Big]d\epsilon\\
&= \alpha\lambda^{\alpha-1}\int_0^{+\infty}\int_0^{+\infty}\Big[e^{-\lambda(\varphi(\epsilon)-\varphi(0))(\varphi(\rho)-\varphi(0))}\omega_\alpha(\rho)\varphi'(\rho)\\
&\times T((\varphi(\epsilon)-\varphi(0))^\alpha)(v_0+\Phi(v))(\varphi(\epsilon)-\varphi(0))^{\alpha-1}\varphi'(\epsilon)\Big]d\epsilon d\rho\\
&= \alpha\lambda^{\alpha-1}\iint_0^{+\infty}\Big[e^{-\lambda(\varphi(\epsilon)-\varphi(0))}T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha\frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)(v_0+\Phi(v))\varphi'(\rho)\varphi'(\epsilon)\Big]d\epsilon d\rho\\
&= \lambda^{\alpha-1}\int_0^{+\infty}\Big[e^{-\lambda(\varphi(\epsilon)-\varphi(0))}\times\int_0^{+\infty}\Big(\alpha\omega_\alpha(\rho)T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha\frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}(v_0+\Phi(v))\varphi'(\rho)d\rho\Big)\varphi'(\epsilon)\Big]d\epsilon\\
&= \lambda^{\alpha-1}\mathcal{L}_\varphi(\mathcal{R}_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(0)))(v_0+\Phi(v)),
\end{aligned}$$

with

$$\mathcal{R}_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(0)) = \int_0^{+\infty}\Big[\alpha\omega_\alpha(\rho)T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha\frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\varphi'(\rho)\Big]d\rho.$$

We can write

$$\mathcal{R}_{\alpha,\varphi}(\epsilon) = \int_0^{+\infty}\sigma_\alpha(\rho)T\left(t^\alpha(\varphi(\rho)-\varphi(0))\right)t^{\alpha-1}\varphi'(\rho)d\rho,$$

such that $\sigma_\alpha(\rho) = (\varphi(\rho)-\varphi(0))^{-\frac{1}{\alpha}}\omega_\alpha\Big[\varphi^{-1}\Big(\big(\frac{1}{\varphi(\rho)-\varphi(0)}\big)^{\frac{1}{\alpha}}+\varphi(0)\Big)\Big]$. On the other hand, we have

$$\mathcal{L}_\varphi\Big(\frac{(\varphi(\epsilon)-\varphi(0))^{-\alpha}}{\Gamma(1-\alpha)}\Big)(\lambda) = \lambda^{\alpha-1}.$$

Then,

$$\begin{aligned}
I_1 &= \mathcal{L}_\varphi\Big(\frac{(\varphi(.)-\varphi(0))^{-\alpha}}{\Gamma(1-\alpha)} * R_{\alpha,\varphi}(.)\Big)(\epsilon)(v_0+\Phi(v))\\
&= \mathcal{L}_\varphi\big(I^{1-\alpha,\varphi}R_{\alpha,\varphi}(\epsilon)\big)(v_0+\Phi(v)).
\end{aligned}$$

Let $\varphi_\epsilon^\vartheta = \varphi(\vartheta)+\varphi(\epsilon)-\varphi(0)$ and let us calculate I_2, we have

$$\begin{aligned}
I_2 &= \int_0^{+\infty} e^{-\lambda^\alpha(\varphi(\epsilon)-\varphi(0))}T((\varphi(\epsilon)-\varphi(0)))\widehat{f}(\lambda)\varphi'(\epsilon)d\epsilon\\
&= \iint_0^{+\infty}\Big[e^{-\lambda^\alpha(\varphi(\epsilon)-\varphi(0))}e^{-\lambda(\varphi(s)-\varphi(0))}T((\varphi(\epsilon)-\varphi(0)))\\
&\times f\Big(s,v(s),\int_0^s\varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)\varphi'(\epsilon)\Big]dsd\epsilon\\
&= \iint_0^{+\infty}\Big[\alpha\left(\varphi(\epsilon)-\varphi(0)\right)^{\alpha-1}e^{-(\lambda(\varphi(\epsilon)-\varphi(0)))^\alpha}e^{-\lambda(\varphi(s)-\varphi(0))}\\
&\times T((\varphi(\epsilon)-\varphi(0))^\alpha)\ f\Big(s,v(s),\int_0^s\varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)\varphi'(\epsilon)\Big]d\epsilon ds
\end{aligned}$$

$$= \iiint_0^\infty \Big[\alpha\left(\varphi(\epsilon)-\varphi(0)\right)^{\alpha-1} e^{-\lambda(\varphi(\epsilon)-\varphi(0))(\varphi(\rho)-\varphi(0))} e^{-\lambda(\varphi(s)-\varphi(0))}\omega_\alpha(\rho)$$
$$\times T((\varphi(\epsilon)-\varphi(0))^\alpha) f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(\rho)\varphi'(s)\varphi'(\epsilon)\Big] d\rho ds d\epsilon$$
$$= \iiint_0^\infty \Big[\alpha e^{-\lambda(\varphi(\epsilon)+\varphi(s)-2\varphi(0))} T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha \frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)$$
$$\times f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(\rho)\varphi'(s)\varphi'(\epsilon)\Big] d\rho ds d\epsilon$$
$$= \iiint_0^\infty \Bigg[\alpha e^{-\lambda(\varphi(\vartheta)-\varphi(0))} T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha \frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)$$
$$\times f\Big(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right), v(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right)), \int_0^{\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right)} \varrho(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right),\vartheta)h(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right),\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(\rho)\varphi'(\vartheta)\varphi'(\epsilon)\Bigg] d\rho d\vartheta d\epsilon$$
$$= \iiint_0^\infty \Bigg[\alpha e^{-\lambda(\varphi(\vartheta)-\varphi(0))} T\left(\frac{\varphi(\epsilon)-\varphi(0)}{\varphi(\rho)-\varphi(0)}\right)^\alpha \frac{(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)$$
$$\times f\Big(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right), v(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right)), \int_0^{\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right)} \varrho(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right),\xi)h(\varphi^{-1}\left(\varphi_\epsilon^\vartheta\right),\xi,v(\xi))d\xi\Big)\varphi'(\rho)\varphi'(\vartheta)\varphi'(\epsilon)\Bigg] d\rho d\epsilon d\vartheta.$$

Hence,

$$I_2 = \int_0^\infty e^{-\lambda(\varphi(\vartheta)-\varphi(0))}\Bigg(\int_0^\vartheta\int_0^{+\infty} \alpha T\left(\frac{\varphi(\vartheta)-\varphi(s)}{\varphi(\rho)-\varphi(0)}\right)^\alpha \frac{(\varphi(\vartheta)-\varphi(s))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)$$
$$\times f\left(s,v(s),\int_0^s \varrho(s,\xi)h(s,\xi,v(\xi))d\xi\right)\varphi'(\rho)\varphi'(s)d\rho\, ds\Bigg)\varphi'(\vartheta)d\vartheta.$$

It follows that,

$$\widehat{v}(\lambda) = \mathcal{L}_\varphi\big(I^{1-\alpha,\varphi}R_{\alpha,\varphi}(\epsilon)\big) + \int_0^\infty e^{-\lambda(\varphi(\epsilon)-\varphi(0))}\Bigg[\int_0^\epsilon\int_0^{+\infty} \alpha T\left(\frac{\varphi(\epsilon)-\varphi(s)}{\varphi(\rho)-\varphi(0)}\right)^\alpha \frac{(\varphi(\epsilon)-\varphi(s))^{\alpha-1}}{(\varphi(\rho)-\varphi(0))^\alpha}\omega_\alpha(\rho)$$
$$\times f\left(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\right)\varphi'(\rho)\varphi'(s)d\rho\, ds\Bigg]\varphi'(\epsilon)d\epsilon.$$

Applying the inverse Laplace transform, we acquire

$$v(\epsilon) = I^{1-\alpha,\varphi}R_{\alpha,\varphi}(\epsilon)(v_0+\Phi(v)) + \int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\times f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds$$
$$= S_{\alpha,\varphi}(\epsilon)(v_0+\Phi(v)) + \int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\times f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds,$$

which completes the proof. □

Throughout the paper, we impose the following assumption on A:

H(A): The operator A generates a strongly continuous semigroup $\{T(\vartheta):\ \vartheta\geq 0\}$ in $\mathbb{X}$, and there is a constant real number $M\geq 1$ with $\sup_{\vartheta\in[0,\infty)}\{|T(\vartheta)|_{L(\mathbb{X})}\}\leq M$.

Proposition 3.3. *We assume that $H(A)$ holds, then*

(1) *For a fixed $\epsilon>0, \{R_{\alpha,\varphi}(\epsilon)\}_{\epsilon>0}$ and $\{S_{\alpha,\varphi}(\epsilon)\}_{\epsilon>0}$ are linear operators.*
(2) *For any $x\in\mathbb{X}$, then $\parallel R_{\alpha,\varphi}(\epsilon)x\parallel\leq\frac{\epsilon^{\alpha-1}M}{\Gamma(1+\alpha)}\|x\|$ and $\parallel S_{\alpha,\varphi}(\epsilon)x\parallel\ \leq\frac{M}{\alpha}\parallel x\parallel$.*
(3) *$\{R_{\alpha,\varphi}(\epsilon)\}_{\epsilon>0}$ and $\{S_{\alpha,\varphi}(\epsilon)\}_{\epsilon>0}$ are strongly continuous.*

Proof. Since $\displaystyle\int_0^{+\infty} \alpha\rho\omega_\alpha(\rho)\varphi'(\rho)d\rho = \frac{1}{\Gamma(1+\alpha)}$. Then, $\| R_{\alpha,\varphi}(\epsilon)x \| \leq \frac{\epsilon^{\alpha-1}M}{\Gamma(1+\alpha)}\| x \|$. From the above inequality it follows that

$$\begin{aligned}
\left\| S_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(0))x \right\| &= \left\| I^{1-\alpha,\varphi} R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(0))x \right\| \\
&\leq \frac{M I^{1-\alpha,\varphi}(\varphi(\epsilon)-\varphi(0))^{\alpha-1}}{\Gamma(1+\alpha)}\| x \| \\
&\leq \frac{M\Gamma(\alpha)}{\Gamma(1+\alpha)}\| x \| \\
&\leq \frac{M}{\alpha}\| x \|.
\end{aligned}$$

This implies that $\| S_{\alpha,\varphi}(\epsilon)x \| \leq \frac{M}{\alpha}\| x \|$. Moreover, let $x \in \mathbb{X}$ and $0 < \epsilon_1 < \epsilon_2 \leq \epsilon$, from a simple calculation, it follows that $\lim\limits_{\epsilon_1 \to \epsilon_2} \left\| R_{\alpha,\varphi}(\epsilon_1)x - R_{\alpha,\varphi}(\epsilon_2)x \right\| = 0$ and $\lim\limits_{\epsilon_1 \to \epsilon_2} \left\| S_{\alpha,\varphi}(\epsilon_1)x - S_{\alpha,\varphi}(\epsilon_2)x \right\| = 0$. □

Now, we give the following hypotheses:

(H1) The function $\Phi : \mathbb{X} \to \mathbb{X}$ is continuous, and there exist two positive constants K and d with $\|\Phi(v) - \Phi(u)\| \leq K\|v-u\|_\infty$ and $\|\Phi(v)\| \leq K\|v\|_\infty + d$, for all $u,\ v \in \mathcal{C}(\mathcal{U},\mathbb{X})$.

(H2) $f : \mathcal{U} \times \mathbb{X}^2 \to \mathbb{X}$ is of Carathéodory type, i.e. $f(\epsilon,.,.)$ is continuous for a.e $\epsilon \in \mathcal{U}$ and $f(.,v,u)$ is measurable for every $v,\ u \in \mathbb{X}$.

(H3) (i) There exist a non-decreasing continuous function $\sigma_f : \mathbb{X} \to \mathbb{X}$ and a function $m_f \in \mathcal{C}(\mathcal{U},\mathbb{X})$ with $\|f(\epsilon,u,v)\| \leq m_f\sigma_f(\|v\| + \|u\|)$, for every $u,v \in \mathbb{X}$ a.e $\epsilon \in \mathcal{U}$.
(ii) There is an integrable function $\eta : \mathcal{U} \to [0,+\infty)$ such that for almost every $\epsilon \in \mathcal{U}$, $\mu(f(\epsilon,D_1,D_2)) \leq \eta(\epsilon)\left[\sup\limits_{-\infty<\vartheta\leq 0}\mu(D_1(\vartheta)) + \mu(D_2)\right]$, where μ represents the Hausdorff measure of noncompactness, and $D_1, D_2 \subset \mathbb{X}$ are bounded subsets. Additionally, we have

$$\int_0^\epsilon \eta(s)ds \leq \kappa^*.$$

(H4) (i) The function $h(.,.,v) : \Delta \to \mathbb{X}$ is measurable for every $v \in \mathbb{X}$, and $h(\epsilon,s,.) : \mathbb{X} \to \mathbb{X}$ is a function continuous for all $(\epsilon,s) \in \Delta$. Additionally, there is a function $u : \Delta \to \mathbb{R}^+$ with $\|h(\epsilon,s,v)\| \leq u(\epsilon,s)\|v\|$ such that $\sup\limits_{\epsilon\in\mathcal{U}}\displaystyle\int_0^\epsilon u(\epsilon,s)ds := u^* < \infty$, for every $v \in \mathbb{X}$.
(ii) For any $0 \leq s \leq \epsilon \leq a$ and bounded set $D_1 \subset \mathbb{X}$, there is a functions $\sigma : \Delta \to \mathbb{R}^+$ with $\mu(h(\epsilon,s,D_1)) \leq \sigma(s,\epsilon)\mu(D_1)$ such that $\sup\limits_{\epsilon\in\mathcal{U}}\displaystyle\int_0^\epsilon \sigma(s,\epsilon)ds := \sigma^*$.

(H5) For any $\epsilon \in \mathcal{U}$, the function $\varrho(\epsilon,.)$ is measurable on $[0,\epsilon]$ and $\varrho_s(\epsilon) = \sup\{|\varrho(\epsilon,s)|,\ 0 \leq s \leq \epsilon\}$ is bounded on $\mathcal{U}$ and $l = \sup\limits_{\epsilon\in\mathcal{U}}\varrho_s(\epsilon)$.

Theorem 3.4. *Under the assumptions (H_A) and $(H_1)-(H_5)$, if the inequality*

$$M\left(\frac{K}{\alpha} + \frac{2\kappa^*}{\Gamma(1+\alpha)}(\varphi(a)-\varphi(0))^\alpha\left[1+2l\sigma^*\right]\right) < 1, \tag{3.2}$$

holds, then the fractional problem (1.1) has at least one solution.

Proof. We define a mapping $\mathit{\Gamma} : \mathcal{C}(\mathcal{U},\mathbb{X}) \longrightarrow \mathcal{C}(\mathcal{U},\mathbb{X})$ by

$$\mathit{\Gamma} v(\epsilon) = S_{\alpha,\varphi}(\epsilon)(\Phi(v)+v_0) + \int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds.$$

The proof is carried out in several steps.

Step 1: We will show that $\mathit{\Gamma}$ is continuous. Let $\{v_m\}$ be a sequence with v_m converge to v in $\mathcal{C}(\mathcal{U},\mathbb{X})$. Then by (H_2), when $n\to\infty$ we have the following

$$f\Big(s,v_m(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v_m(\vartheta))d\vartheta\Big)\to f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big),$$

for every $s\in\mathcal{U}$. To simplify the expressions, we pose that

$$K_{f_n}=f\Big(s,v_m(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v_m(\vartheta))d\vartheta\Big),$$
$$K_f=f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big).$$

For any $\epsilon\in\mathcal{U}$, we have

$$\begin{aligned}\|\mathit{\Gamma}v_m(\epsilon)-\mathit{\Gamma}v(\epsilon)\| &= \|S_{\alpha,\varphi}(\epsilon)(v_0+\Phi(v_m))-S_{\alpha,\varphi}(\epsilon)(v_0+\Phi(v))+\int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\times\Big[K_{f_n}-K_f\Big]\varphi'(s)ds\|\\ &\leq \|S_{\alpha,\varphi}(\epsilon)(\Phi(v_m))-\Phi(v)))\|+\int_0^\epsilon \|R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\|\varphi'(s)\|K_{f_n}-K_f\|ds\\ &\leq \frac{MK}{\alpha}\|v_m-v\|_\infty+\frac{M}{\Gamma(1+\alpha)}\int_0^\epsilon(\varphi(\epsilon)-\varphi(s))^{\alpha-1}\varphi'(s)\|K_{f_n}-K_f\|ds.\end{aligned}$$

Thus, we infer that $\|\mathit{\Gamma}v_m-\mathit{\Gamma}v\|_\infty\to 0$ as $n\to\infty$, it's follows that the map $\mathit{\Gamma}$ is continuous on $\mathcal{C}(\mathcal{U},\mathbb{X})$.
Step 2: we will prove that $\mathit{\Gamma}$ is bounded. We claim that $\mathit{\Gamma}(B_r)\subset B_r$ for any $v\in B_r$ and $\epsilon\in\mathcal{U}$, we have

$$\begin{aligned}\|\mathit{\Gamma}v(\epsilon)\| \leq{}& \frac{M}{\alpha}(\|v_0\|+K\|v\|_\infty+d)+\frac{M}{\Gamma(1+\alpha)}\int_0^\epsilon(\varphi(\epsilon)-\varphi(s))^{\alpha-1}\\ &\times\|f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\|\varphi'(s)ds\\ &\leq \frac{M}{\alpha}(\|v_0\|+K\|v\|_\infty+d)+\frac{M(\varphi(b)-\varphi(0))^\alpha}{\Gamma(1+\alpha)}\Big[m_f\sigma_f(\|v\|_\infty+l\int_0^s u(s,\vartheta)d\vartheta\|v\|_\infty)\Big]\\ &\leq \frac{M}{\alpha}(\|v_0\|+Kr+d)+\frac{M(\varphi(a)-\varphi(0))^\alpha}{\Gamma(1+\alpha)}m_f\sigma_f r(1+lu^*)\\ &\leq r.\end{aligned}$$

Step 3: $\mathit{\Gamma}(B_r)$ is equicontinuous. For every $\vartheta_1,\vartheta_2\in\mathcal{U}$ with $\vartheta_1\leq\vartheta_2$, we have the following

$$\begin{aligned}\|\mathit{\Gamma}v(\vartheta_2)-\mathit{\Gamma}v(\vartheta_1)\| \leq{}& \|S_{\alpha,\varphi}(\vartheta_2)-S_{\alpha,\varphi}(\vartheta_1)\|(\|v_0\|+K\|v\|_\infty+d)\\ &+\|\int_0^{\vartheta_1}(R_{\alpha,\varphi}(\varphi(\vartheta_2)-\varphi(s))-R_{\alpha,\varphi}(\varphi(\vartheta_1)-\varphi(s)))\\ &\times f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds\\ &+\int_{\vartheta_1}^{\vartheta_2} R_{\alpha,\varphi}(\varphi(\vartheta_2)-\varphi(s)\times f\Big(s,v(s),\int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds\|\\ &\leq \|S_{\alpha,\varphi}(\vartheta_2)-S_{\alpha,\varphi}(\vartheta_1)\|(\|v_0\|+K\|v\|_\infty+d)\\ &+\int_0^{\vartheta_1}\|R_{\alpha,\varphi}(\varphi(\vartheta_2)-\varphi(s))-R_{\alpha,\varphi}(\varphi(\vartheta_1)-\varphi(s))\|\times m_f\sigma_f r(1+lu^*)\varphi'(s)ds\\ &+\int_{\vartheta_1}^{\vartheta_2}\|R_{\alpha,\varphi}(\varphi(\vartheta_2)-\varphi(s))\|m_f\sigma_f r(1+lu^*)\varphi'(s)ds.\end{aligned}$$

From above, we get $\|\mathit{\Gamma}v(\vartheta_2)-\mathit{\Gamma}v(\vartheta_1)\|\to 0$ as $\vartheta_2\to\vartheta_1$. So $\mathit{\Gamma}$ is equicontinuous.

Step 4: We will verifies the Mönch condition is holds. Let $\mathcal{V} \subset B_r$ be countable such that $\mathcal{V} \subset \overline{\text{conv}}(0 \cup \mathit{\Gamma}(\mathcal{V}))$. We demonstrate that $\mu(\mathcal{V}) = 0$, where μ denotes the Hausdorff noncompactness measure. Without loss of generality, we may suppose $\mathcal{V} = \{u\}_{k=1}^{\infty}$, and it can be readily verified that $\mathcal{V}$ is equicontinuous and bounded. For any $\epsilon \in \mathcal{U}$, we have

$$\begin{aligned}\mathit{\Gamma} v(\epsilon) &= S_{\alpha,\varphi}(\epsilon)(\Phi(v) + v_0) + \int_0^{\epsilon} R_{\alpha,\varphi}(\varphi(\epsilon) - \varphi(s)) f\Big(s, v(s), \int_0^s \varrho(s,\vartheta) h(s,\vartheta,v(\vartheta)) d\vartheta\Big)\varphi'(s)ds \\ &= \mathit{\Gamma}_1 v(\epsilon) + \mathit{\Gamma}_2 v(\epsilon),\end{aligned}$$

where $\mathit{\Gamma}_1 v(\epsilon) = S_{\alpha,\varphi}(\epsilon)(\Phi(v) + v_0)$ and

$$\mathit{\Gamma}_2 v(\epsilon) = \int_0^{\epsilon} R_{\alpha,\varphi}(\varphi(\epsilon) - \varphi(s)) f\Big(s, v(s), \int_0^s \varrho(s,\vartheta) h(s,\vartheta,v(\vartheta))\Big) d\vartheta)\varphi'(s)ds.$$

Furthermore, the mapping $\mathit{\Gamma}_1 : \mathcal{V} \to \mathcal{C}(\mathcal{U}, \mathbb{X})$ is Lipschitz continuous with the real constant KM_α, as implied by conditions $(H1)$. Indeed, for any $v, u \in \mathcal{V}$, we have the following

$$\begin{aligned}\|\mathit{\Gamma}_1 v(\epsilon) - \mathit{\Gamma}_1 u(\epsilon)\| &\le \sup_{t\in[0,b]} \|S_{\alpha,\varphi}(\epsilon)\| \|\Phi(v) - \Phi(u)\| \\ &\le \frac{MK}{\alpha}\|v - u\|_\infty.\end{aligned}$$

So, from Lemmas 2.15, 2.13, and 2.14 and conditions $(H3)(ii)$, $(H4)(ii)$, we have

$$\begin{aligned}\mu(\{\mathit{\Gamma} v_m\}_{m=1}^{\infty}) &\le \mu(\{\mathit{\Gamma}_1 v_m\}_{m=1}^{\infty}) + \mu(\{\mathit{\Gamma}_2 v_m\}_{m=1}^{\infty}) \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \mu\Big(\int_0^{\epsilon} R_{\alpha,\varphi}(\varphi(\epsilon) - \varphi(s)) \\ &\times f\big(s, v(s), \int_0^s \varrho(s,\vartheta) h(s,\vartheta,v(\vartheta)) d\vartheta)\varphi'(s)ds\Big) \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \frac{2M}{\Gamma(1+\alpha)}\int_0^{\epsilon} \mu\big((\varphi(\epsilon) - \varphi(s))^{\alpha-1} \\ &\times f(s, v(s), \int_0^s \varrho(s,\vartheta) h(s,\vartheta,v(\vartheta)) d\vartheta)\varphi'(s)ds \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \frac{2M}{\Gamma(1+\alpha)}\int_0^{\epsilon} (\varphi(\epsilon) - \varphi(s))^{\alpha-1}\eta(s) \\ &\times \Big[\sup_{0<s\le a} \mu(\{v_m\}_{m=1}^{\infty}(s)) + \mu\Big(\int_0^s \varrho(s,\vartheta) h(s,\vartheta,\{v_m\}_{m=1}^{\infty}(\vartheta)) d\vartheta\Big)\Big] ds \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \frac{2M}{\Gamma(1+\alpha)}\int_0^{\epsilon} (\varphi(\epsilon) - \varphi(s))^{\alpha-1}\eta(s) \\ &\times \Big[\sup_{0<s\le a} \mu(\{v_m\}_{m=1}^{\infty}(s)) + 2l\int_0^s \sigma(s,\vartheta) d\vartheta \mu\{v_m\}_{m=1}^{\infty}(\vartheta)\Big] ds \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \frac{2M}{\Gamma(1+\alpha)}\int_0^{\epsilon} (\varphi(\epsilon) - \varphi(s))^{\alpha-1}\eta(s) \\ &\times \Big[\sup_{0<s\le a} \mu(\{v_m\}_{m=1}^{\infty}(s)) + 2l\sigma^*\mu\{v_m\}_{m=1}^{\infty}(\vartheta)\Big] ds \\ &\le \frac{MK}{\alpha}\mu(\{v_m\}_{m=1}^{\infty}) + \frac{2M\kappa^*}{\Gamma(1+\alpha)}(\varphi(a) - \varphi(0))^{\alpha}\big[1 + 2l\sigma^*\big] \times \sup_{0<s\le a} \mu_C(\mathcal{V}(s)) \\ &\le M\Big(\frac{K}{\alpha} + \frac{2\kappa^*}{\Gamma(1+\alpha)}(\varphi(a) - \varphi(0))^{\alpha}\big[1 + 2l\sigma^*\big]\Big) \sup_{0<s\le a} \mu_C(\mathcal{V}(s)).\end{aligned}$$

From above, we have

$$\mu_C(\Gamma(\mathcal{V})) \leq M\Big[\frac{K}{\alpha} + \frac{2\kappa^*}{\Gamma(1+\alpha)}(\varphi(a)-\varphi(0))^\alpha[1+2l\sigma^*]\Big]\mu_C(\mathcal{V})$$
$$\leq \varpi^*\mu_C(\mathcal{V}),$$

where

$$\varpi^* = M\Big[\frac{K}{\alpha} + \frac{2\kappa^*}{\Gamma(1+\alpha)}(\varphi(a)-\varphi(0))^\alpha\big[1+2l\sigma^*\big]\Big].$$

Thus,

$$\mu_C(\mathcal{V}) \leq \mu_C((\overline{conv\{0\}}) \cup \Gamma(\mathcal{V})) = \mu_C(\Gamma(\mathcal{V})) \leq \varpi^*\mu_C(\mathcal{V}).$$

This implies $\mu_C(\mathcal{V}) = 0$. Therefore, by applying Theorem 2.15, we find a fixed point v of Γ in B_r. This fixed point serves as a solution to fractional initial value problem (1.1).

□

To establish the uniqueness of the solution, we apply Banach's fixed-point theorem. We assume the following hypotheses.

(H6) There exist constants $L_1, L_2 > 0$ such that:

$$\|f(\vartheta, v_1, u_1) - f(\vartheta, v_2, u_2)\| \leq L_1\|v_1 - v_2\| + L_2\|u_1 - u_2\|,$$

for all $\vartheta \in \mathcal{U}$ and $v_1, v_2, u_1, u_2 \in \mathbb{X}$.

(H7) There exists a constant $L_3 > 0$ such that:

$$|h(\vartheta, s, v_1) - h(\vartheta, s, v_2)| \leq L_3\|v_1 - v_2\|,$$

for all $(\vartheta, s) \in \Delta$ and $v_1, v_2 \in X$.

(H8) The function $\varrho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ is bounded, i.e., there exists a constant $L_4 > 0$ such that:

$$|\varrho(\vartheta, s)| \leq L_4, \quad \forall(\vartheta, s) \in \mathcal{U} \times \mathcal{U}.$$

Theorem 3.5. *Assume* $(H1)$ *and* $(H6) - (H8)$ *are verified. If*

$$\frac{M}{\alpha}\left(K + \frac{(\varphi(\epsilon)-\varphi(0))^\alpha}{\Gamma(1+\alpha)}(L_1 + L_2L_3L_4\epsilon)\right) < 1. \tag{3.3}$$

Then, the problem (1.1) *has a unique mild solution.*

Proof. For this, we define the operator $\Gamma : \mathcal{C}(\mathcal{U}, \mathbb{X}) \longrightarrow \mathcal{C}(\mathcal{U}, \mathbb{X})$ as follows:

$$\Gamma v(\epsilon) = S_{\alpha,\varphi}(\epsilon)(\Phi(v) + v_0) + \int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))f\Big(s, v(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big)\varphi'(s)ds.$$

Let $\epsilon \in \mathcal{U}$ and $v, u \in \mathcal{C}(\mathcal{U}, \mathbb{X})$. We have

$$\Gamma v(\epsilon) - \Gamma u(\epsilon) = S_{\alpha,\varphi}(\epsilon)(\Phi(v) - \Phi(u)) + \int_0^\epsilon R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\varphi'(s)$$
$$\left(f\Big(s, v(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big) - f\Big(s, u(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,u(\vartheta))d\vartheta\Big)\right)ds.$$

Then,

$$\|\Gamma v(\epsilon) - \Gamma u(\epsilon)\| \leq \|S_{\alpha,\varphi}(\epsilon)(\Phi(v) - \Phi(u))\| + \int_0^\epsilon \|R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\|\varphi'(s)$$
$$\|f\Big(s, v(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big) - f\Big(s, u(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,u(\vartheta))d\vartheta\Big)\|ds$$

$$
\begin{aligned}
&\leq \|S_{\alpha,\varphi}(\epsilon)\| \, \|(\Phi(v)-\Phi(u))\| + \int_0^{\epsilon} \|R_{\alpha,\varphi}(\varphi(\epsilon)-\varphi(s))\|\varphi'(s) \\
&\|f\Big(s, v(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,v(\vartheta))d\vartheta\Big) - f\Big(s, u(s), \int_0^s \varrho(s,\vartheta)h(s,\vartheta,u(\vartheta))d\vartheta\Big)\|ds \\
&\leq \frac{MK}{\alpha}\|v-u\|_\infty + \int_0^{\epsilon} \frac{(\varphi(\epsilon)-\varphi(s))^{\alpha-1}M}{\Gamma(1+\alpha)}\varphi'(s) \\
&\quad \left(L_1\|v(s)-u(s)\| + L_2\|\int_0^s \varrho(s,\vartheta)\big(h(s,\vartheta,v(\vartheta)) - h(s,\vartheta,u(\vartheta))\big)\,d\vartheta\|\right) ds \\
&\leq \frac{MK}{\alpha}\|v-u\|_\infty + \int_0^{\epsilon} \frac{(\varphi(\epsilon)-\varphi(s))^{\alpha-1}M}{\Gamma(1+\alpha)}\varphi'(s) \\
&\quad \left(L_1\|v(s)-u(s)\| + L_2L_3L_4\int_0^s \|v(\vartheta)-u(\vartheta)\|d\vartheta\right) ds \\
&\leq \frac{MK}{\alpha}\|v-u\|_\infty + \int_0^{\epsilon} \frac{(\varphi(\epsilon)-\varphi(s))^{\alpha-1}M}{\Gamma(1+\alpha)}\varphi'(s)\left(L_1\|v-u\|_\infty + L_2L_3L_4\epsilon\|v-u\|_\infty\right) ds \\
&\leq \frac{M}{\alpha}\left(K + \frac{(\varphi(\epsilon)-\varphi(0))^{\alpha}}{\Gamma(1+\alpha)}(L_1+L_2L_3L_4\epsilon)\right)\|v-u\|_\infty.
\end{aligned}
$$

Then,

$$
\|\Gamma v - \Gamma u\|_\infty \leq \frac{M}{\alpha}\left(K + \frac{(\varphi(\epsilon)-\varphi(0))^{\alpha}}{\Gamma(1+\alpha)}(L_1+L_2L_3L_4\epsilon)\right)\|v-u\|_\infty.
$$

Hence, by using the condition (3.3) and the Banach contraction mapping principle, we deduce that Γ has a unique fixed point in $\mathcal{C}(\mathcal{U},\mathbb{X})$. □

4. Application

In this section, we consider $\mathbb{X} = L^2([0,1],\mathbb{R})$ and define the operator A by

$$
Av = -\frac{\partial^2}{\partial \epsilon^2}v,
$$

with the domain

$$
D(A) = \{v \in \mathbb{X} \mid \frac{\partial^2}{\partial\epsilon^2}v \in \mathbb{X}, \quad v(\vartheta,0) = v(\vartheta,1) = 0\}.
$$

It is well known that A is a linear bounded operator generating a compact semigroup $\{T(\vartheta), \vartheta \geq 0\}$ on $\mathbb{X}$, satisfying $\|T(\vartheta)\| \leq e^{-\vartheta}$ for all $\vartheta \geq 0$. Then, $M = 1$. Now, we consider a fractional differential equation problem with nonlocal conditions.

$$
\begin{cases}
{}^{C}D_{0^+}^{\frac{4}{5},\vartheta} v(\vartheta,\epsilon) = -\frac{\partial^2}{\partial\epsilon^2}v(\vartheta,\epsilon) + 0.1(sin(v(\vartheta,\epsilon)) + \displaystyle\int_0^{\vartheta} 0.1\vartheta s \times 0.1\exp^{-(\vartheta-s)} v(s,\epsilon)ds, \\
v(0,\epsilon) + 0.1\int_0^1 e^{-s}v(s,\epsilon)\,ds = v_0(\epsilon), \quad \epsilon \in [0,1],
\end{cases}
\tag{4.1}
$$

Let $0 = \vartheta_0 < \vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_n \leq 1$ be fixed real numbers, with $v_0 \in \mathbb{X}$.

To express problem (4.1) in the abstract form of problem (1.1), we assume that:

- (H1) $\Phi : \mathbb{X} \to \mathbb{X}$ by: $\Phi(v) = 0.1\int_0^1 e^{-s}v(s)\,ds$. For all $u, v \in \mathbb{X}$, then we have $\|\Phi(v) - \Phi(u)\| \leq 0.1\|v-u\|$. Thus, $K = 0.1$.
- ((H2) and (H3)) Define $f : [0,1]\times\mathbb{X}^2 \to \mathbb{X}$ by: $f(\epsilon,v,u) = 0.1(\sin(v)+u)$. We have $\|f(\epsilon,v,u)\| \leq 0.1(\|v\|+\|u\|)$. Thus, $m_f(\epsilon) = 0.1, \quad \sigma_f(x) = x$. And $\mu(f(\epsilon,D_1,D_2)) \leq 0.1(\mu(D_1)+\mu(D_2))$, with $\int_0^1 0.1\,d\epsilon = 0.1 \leq \kappa^*$. Then, $\kappa^* = 0.1$

(H4) Define $h : \Delta \times X \to \mathbb{R}^+$ by: $h(\epsilon, s, v) = 0.1e^{-(\epsilon-s)}v(s)$.
we have, $\|h(\epsilon, s, v)\| \leq 0.1e^{-(\epsilon-s)}\|v\|$, with $u(\epsilon, s) = 0.1e^{-(\epsilon-s)}$ and $u^* = 0.1$. And $\mu(h(\epsilon, s, D_1)) \leq 0.1e^{-(\epsilon-s)}\mu(D_1)$, with $\sigma(s, \epsilon) = 0.1e^{-(\epsilon-s)}$ and $\sigma^* = 0.1$.

(H5) Define $\varrho : [0,1] \times [0,1] \to \mathbb{R}^+$ by: $\varrho(\epsilon, s) = 0.1\epsilon s$. and $\varrho_s(\epsilon) = \sup_{s\in[0,\epsilon]} |\varrho(\epsilon, s)| = 0.1\epsilon^2 \leq 0.1$, so $l = 0.1$.
Verification of the condition (3.2)

$$M\left(\frac{K}{\alpha} + \frac{2\kappa^*}{\Gamma(1+\alpha)}(\varphi(a) - \varphi(0))^\alpha\left[1 + 2l\sigma^*\right]\right) = 0.344 < 1. \tag{4.2}$$

The condition (3.2) is satisfied.

All the hypotheses of the Theorem 3.4 are verified; therefore, the fractional problem (4.1) has at least one solution.

5. Conclusion

We have established the existence and uniqueness of solutions for a class of fractional integro-differential equations involving the φ-Caputo fractional derivative under nonlocal conditions. By employing the measure of noncompactness, probability density functions, Mönch's fixed point theorem, and semigroup theory, we have derived sufficient conditions ensuring the well-posedness of the considered problem. The theoretical findings have been substantiated through an illustrative example, demonstrating the applicability and effectiveness of the proposed approach.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.
Data availability. The data used to support the findings of this study are included in the references within the article.
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