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Soliton solutions in the nonlinear conformable Wu-Zhang system

Hira Tariq[1], Hira Ashraf[1], Mohammad Ali Hosseinzadeh[2], Hadi Rezazadeh[2,*], and Soheila Biria[3]
[1]Department of Mathematics, Government College Women University, Sialkot, Pakistan.
[2]Faculty of Engineering Modern Technologies, Amol University of Special Modern Technologies, Amol, Iran.
[3]Higher Education Extension Office, Vice-Chancellor of Education, Ministry of Science, Research and Technology, Tehran, Iran.

Abstract

In this paper, new analytical solutions of nonlinear fractional Wu-Zhang system are determined with the aid of two analytical approaches, that is, generalized projective Riccati equation method and Sardar sub-equation method via conformable derivative. The system describes $(1+1)$-dimensional dispersive long wave in two horizontal directions on shallow waters. Some new solitary wave solutions are demonstrated by the means of computer softwares maple or mathematica. The obtained results reveals that the proposed method is very efficacious and straightforward in the determination of the solution for the nonlinear fractional Wu-Zhang system.

Keywords. Fractional Wu-Zhang system, Conformable fractional derivative, Generalized projective Riccati equation method, Sardar sub-equation method.

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1. Introduction

In past few years, the fractional integrals and the fractional derivatives (fractional calculus) [6, 15] have been the focus of many studies because of the frequent appearances in numerous fields such as chemistry, physics, fluid mechanics, geology, optical fibers, mechanics, ecology, and engineering. In recent decades, nonlinear fractional differential equations (NFDEs) are efficiently used to model various of phenomena in different fields, namely, physics, mechanical engineering, continuous-time random walk, dynamical systems, shallow water waves, wave propagation phenomenon, dynamical systems and so on [21, 30]. Among the investigations for fractional nonlinear partial differential equations (NPDEs), research for constructing travelling wave solutions of fractional NPDEs is a main topic, which can also be beneficial and helpful reference for other related research. A lot of considerable approaches are established to investigate fractional models, for instant, Caputo [53], Riemann-Liouville [20], conformable derivatives [2] etc. Many powerful techniques are applied to study and discuss the explicit solutions and their physical behavior of these models. Some of the techniques are the extended tanh-function method [12], the modified auxiliary equation method [55], the multipliers method [14], the first integral method [54], the simplest equation method [49], the jacobi elliptic function method [13], the sine-cosine method [58], the Hirota bilinear methd [18], the (G'/G)-expansion method [60], the extended trial equation method [16], the Sine-Gordon expansion method [50], the new transfer function method [7] and so on [17, 31, 46]. Water waves dispersion is one of the fundamental properties in above discussed models. The frequency dispersion is described by this property [9]. In addition, it relates to the traveling of waves at variant speeds with disparate wavelengths. Physical phenomena such as floods and tsunamis are often caused by shallow water waves as well as dispersive water waves due to earthquakes. To control and minimize harmful effects of these situations, there are different mathematical models presented by numerous researchers. The non-linear Wu-Zhang system is one of the famous models due to its importance in describing long dispersive water waves in the sea. In this regard, the main focus of this study is to retrieve the some novel wave analytical solutions of the nonlinear Wu-Zhang Model

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* Corresponding author. Email: h.rezazadeh@ausmt.ac.ir.

via conformable derivative with the aid of two analytical approaches, that is, generalized projective Riccati equation method and Sardar sub-equation method for the first time. This system gives a description of (1+1)-dimensional dispersive long wave in two horizontal directions on shallow waters [51]. The Wu-Zhang system has the following form:

$$\begin{aligned} &r_t + rr_x + s_x = 0, \\ &s_t + (rs)_x + \frac{1}{3}s_{xxx} = 0. \end{aligned}$$

where $r(x,t)$ gives elevation of water and $s(x,t)$ gives surface velocity of the water, respectively. Many researchers in few of past decades studied this system and constructed numerous solitary wave solutions. For example, the first integral method [11], the ansatz method [59] and the modified auxiliary equation method [48], the extended Fan sub-equation method [57], by three different numerical schemes [47] and many other research papers have studied this system [47, 57]. In this paper, we consider the time-fractional Wu-Zhang system, which has the following form:

$$\begin{aligned} &D_t^k r + rr_x + s_x = 0, \\ &D_t^k s + (rs)_x + \frac{1}{3}s_{xxx} = 0, \end{aligned} \tag{1.1}$$

where time-fractional derivative is taken in conformable sense $(0 < k \leq 1)$. Two analytical approaches, GPREM and Sardar sub-equation method, are supposed to construct some novel solitary wave solutions (a special kind of travelling wave solutions) with the aid of maple or mathematica computer softwares.

A lot of the NFPDEs have been solved by the aid of Sardar sub-equation method as it is most powerful and direct tool to construct analytical solutions [19, 56]. The GPREM is also a well accepted analytical approach for analysis of the solitary wave solutions since 2003. This paper is arranged as follows, some preliminaries of conformable fractional derivative are given in section 2. Description of the fundamental steps of proposed methods are illustrated in section 3. Some physical meanings and graphical representation of obtained solutions are discussed in section 4. Finally, some conclusions drawn in last section.

2. The Conformable Derivative

We start this section with the following definition which is needed during this paper.

Definition 2.1. The conformable fractional derivative [1, 2] for a function $f : (0, \infty) \to R$ of order $0 < k \leq 1$ at $t > 0$ is defined as

$$D^k(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{(1-k)})}{\epsilon}. \tag{2.1}$$

Note that if the function f is k-differentiable in $(0, a)$ then $D^k(f)(t) = (t)^{1-k} f'(t)$. The definition of conformable derivative satisfies the properties stated in the theorem given below [1, 2].

Theorem 2.2. *Let f, h be the k-differentiable at a point t and $0 < k \leq 1$, then*

$$\begin{aligned} &D^k(af + bh) = aD^k(f) + bD^k(h), \ for \ a, b \in R, \\ &D^k(t^\eta) = \eta t^{\eta - k}, \ \forall \ \eta \ \in \ R, \\ &D^k(fh) = g_t D^k(h) + h_t D^k(f), \\ &D^k\Big(\frac{f}{h}\Big) = \frac{f_t D^k(h) - h_t D^k(f)}{h^2}. \end{aligned}$$

3. Description of the proposed methods

By letting a non-linear conformable fractional differential equation having the form given below:

$$U(D_t^k p, p_{xt}, D_t^{2k} p, p_{xx}, ...) = 0, \tag{3.1}$$

where U is a polynomial in $p(x,t)$ and its higher-order partial derivatives and D_t^k is time fractional derivative $(0 < k \leq 1)$.

Using the wave transformation:

$$p(x,t) = P(\zeta), \text{ where } \zeta = x + c\frac{t^k}{k}, \tag{3.2}$$

on Eq. (3.1), the NFPDE is converted into an ODE of the form:

$$U(p, p', p'', ...) = 0. \tag{3.3}$$

3.1. **The Generalized Projective Riccati Equation Method (GPREM).** The fundamental steps of the GPREM are given below:

Step 1: It is assumed that the formal solution of the Eq. (3.3) has following form:

$$p(x,t) = A_0 + \sum_{i=0}^{M} \vartheta^{i-1}(\zeta)[A_i\vartheta(\zeta) + B_i\tau(\zeta)], \tag{3.4}$$

where the arbitrary constants A_i, A_0, and B_i will be determined later, while the functions $\vartheta(\zeta)$ and $\tau(\zeta)$ satisfies the following ODEs:

$$\vartheta'(\zeta) = \varepsilon\vartheta(\zeta)\tau(\zeta), \tag{3.5}$$

$$\tau'(\zeta) = R + \varepsilon\tau^2(\zeta) - m\vartheta(\zeta), \ \varepsilon \pm 1, \tag{3.6}$$

where

$$\tau^2(\zeta) = -\varepsilon\left(R - 2m\vartheta(\zeta) + \frac{m^2-1}{R}\vartheta^2(\zeta)\right), \tag{3.7}$$

where m and R are non-zero arbitrary constants.
If $R = m = 0$, then the formal solution of Eq. (3.3) have the form:

$$p(x,t) = \sum_{i=0}^{M}[A_i\tau^i(\zeta)], \tag{3.8}$$

and $\tau(\zeta)$ will satisfy the following ODE:

$$\tau'(\zeta) = \tau^2(\zeta). \tag{3.9}$$

Step 2: The positive integer M in Eq. (3.4) is determined by the aid of the homogeneous balance method.
Step 3: Using Eq. (3.5), (3.6), and (3.7) in Eq. (3.4) and then collecting the terms having same powers of $(\vartheta^j\tau^i)$, where $(j = 0, 1, 2, 3, ..., i = 0, 1)$ or (τ^j), where $(j = 0, 1, 2, 3, ...)$ and by equating these terms to zero, a system of algebraic equations is obtained that will be simplified by using wolfram Mathematica-9 or maple to get values of R, c, m, A_i, and B_i.
Step 4: It is acknowledged that [52], Eq. (3.4) accepts the following solutions.

(1) If $\varepsilon = -1$, $R \neq 0$,

$$\begin{aligned}\vartheta_1 &= \frac{R\,\text{sech}(\sqrt{R}\zeta)}{m\,\text{sech}(\sqrt{R}\zeta)+1}, \quad \tau_1 = \frac{R\tanh(\sqrt{R}\zeta)}{m\,\text{sech}(\sqrt{R}\zeta)+1},\\ \vartheta_2 &= \frac{R\,\text{csch}(\sqrt{R}\zeta)}{m\,\text{csch}(\sqrt{R}\zeta)+1}, \quad \tau_2 = \frac{R\coth(\sqrt{R}\zeta)}{m\,\text{csch}(\sqrt{R}\zeta)+1}.\end{aligned} \tag{3.10}$$

(2) If $\varepsilon = 1,\ R \neq 0$,

$$\vartheta_3 = \frac{R\sec(\sqrt{R}\zeta)}{m\sec(\sqrt{R}\zeta)+1},\ \tau_3 = \frac{R\tan(\sqrt{R}\zeta)}{m\sec(\sqrt{R}\zeta)+1},$$
$$\vartheta_4 = \frac{R\csc(\sqrt{R}\zeta)}{m\csc(\sqrt{R}\zeta)+1},\ \tau_4 = \frac{R\cot(\sqrt{R}\zeta)}{m\csc(\sqrt{R}\zeta)+1}. \tag{3.11}$$

(3) If $R = m = 0$,

$$\vartheta_5 = \frac{G}{\zeta},\ \tau_5 = \frac{1}{\varepsilon\zeta}, \tag{3.12}$$

where G is a non-zero arbitrary constant.

Step 5: The values of A_0, A_i, B_i, c, m, and R and the solutions Eq. (3.10), (3.11), and (3.12) are substituted in Eq. (3.4), the analytical solutions of Eq. (3.1) are gained. In this study, both cases $\varepsilon = 1$ and $\varepsilon = -1$ are done.

3.2. **The Sardar Sub-Equation Method.** It is supposed that the Equation (3.3) has a formal solution of the form given below:

$$p(x,t) = \sum_{i=0}^{M} \nu_i \psi^i(\zeta), \tag{3.13}$$

where ν_i, $(i = 0, 1, 2..., M.)$ are the coefficients to be determined later and $\psi^{'}(\zeta)$ satisfies the following ODE.

$$(\psi'(\zeta))^2 = \alpha + g\psi^2 + \psi^4, \tag{3.14}$$

where α and g are arbitrary constants. The solutions of the ODE are

Case 1:

If $g > 0$ and $\alpha = 0$, then

$$\psi_1^{\pm}(\zeta) = \pm\sqrt{-mng}\,\mathrm{sech}_{mn}(\sqrt{g}\zeta),$$

$$\psi_2^{\pm}(\zeta) = \pm\sqrt{-mng}\,\mathrm{csch}_{mn}(\sqrt{g}\zeta),$$

where

$$\mathrm{sech}_{mn}(\zeta) = \frac{2}{me^{\zeta} + ne^{-\zeta}},\ \mathrm{csch}_{mn}(\zeta) = \frac{2}{me^{\zeta} - ne^{-\zeta}}.$$

Case 2:

If $g < 0$ and $\alpha = 0$, then

$$\psi_3^{\pm}(\zeta) = \pm\sqrt{-mng}\,\sec_{mn}(\sqrt{-g}\zeta),$$
$$\psi_4^{\pm}(\zeta) = \pm\sqrt{-mng}\,\csc_{mn}(\sqrt{-g}\zeta),$$

where

$$\sec_{mn}(\zeta) = \frac{2}{me^{i\zeta} + ne^{-i\zeta}},\ \csc_{mn}(\zeta) = \frac{2}{me^{i\zeta} - ne^{-i\zeta}}.$$

Case 3:

If $g < 0$ and $\alpha = \frac{g^2}{4}$, then

$$\psi_5^{\pm}(\zeta) = \pm\sqrt{\frac{-g}{2}}\tanh_{mn}\left(\sqrt{\frac{-g}{2}}\zeta\right),$$

$$\psi_6^{\pm}(\zeta) = \pm\sqrt{\frac{-g}{2}}\coth_{mn}\left(\sqrt{\frac{-g}{2}}\zeta\right),$$

$$\psi_7^{\pm}(\zeta) = \pm\sqrt{\frac{-g}{2}}\left(\tanh_{mn}\left(\sqrt{-2g}\zeta\right) \pm i\sqrt{rs}\,\text{sech}_{mn}\left(\sqrt{-2g}\zeta\right)\right),$$

$$\psi_8^{\pm}(\zeta) = \pm\sqrt{\frac{-g}{2}}\left(\coth_{mn}\left(\sqrt{-2g}\zeta\right) \pm \sqrt{rs}\,\text{csch}_{mn}\left(\sqrt{-2g}\zeta\right)\right),$$

$$\psi_9^{\pm}(\zeta) = \pm\sqrt{\frac{-g}{8}}\left(\tanh_{mn}\left(\sqrt{\frac{-g}{8}}\zeta\right) \pm \sqrt{rs}\coth_{mn}\left(\sqrt{\frac{-g}{8}}\zeta\right)\right),$$

where

$$\tanh_{mn}(\zeta) = \frac{me^{\zeta} - ne^{-\zeta}}{me^{\zeta} + ne^{-\zeta}}, \ \coth_{mn}(\zeta) = \frac{me^{\zeta} + ne^{-\zeta}}{me^{\zeta} - ne^{-\zeta}}.$$

Case 4:
If $g > 0$ and $\alpha = \frac{g^2}{4}$, then

$$\psi_{10}^{\pm}(\zeta) = \pm\sqrt{\frac{g}{2}}\tan_{mn}\left(\sqrt{\frac{-g}{2}}\zeta\right),$$

$$\psi_{11}^{\pm}(\zeta) = \pm\sqrt{\frac{g}{2}}\cot_{mn}\left(\sqrt{\frac{-g}{2}}\zeta\right),$$

$$\psi_{12}^{\pm}(\zeta) = \pm\sqrt{\frac{g}{2}}\left(\tan_{mn}\left(\sqrt{2g}\zeta\right) \pm i\sqrt{rs}\sec_{mn}\left(\sqrt{2g}\zeta\right)\right),$$

$$\psi_{13}^{\pm}(\zeta) = \pm\sqrt{\frac{g}{2}}\left(\cot_{mn}\left(\sqrt{2g}\zeta\right) \pm \sqrt{rs}\csc_{mn}\left(\sqrt{2g}\zeta\right)\right),$$

$$\psi_{14}^{\pm}(\zeta) = \pm\sqrt{\frac{g}{8}}\left(\tan_{mn}\left(\sqrt{\frac{g}{8}}\zeta\right) \pm \sqrt{rs}\cot_{mn}\left(\sqrt{\frac{g}{8}}\zeta\right)\right),$$

where

$$\tan_{mn}(\zeta) = -i\frac{me^{i\zeta} - ne^{-i\zeta}}{me^{i\zeta} + ne^{-i\zeta}}, \ \cot_{mn}(\zeta) = i\frac{me^{i\zeta} + ne^{-i\zeta}}{me^{i\zeta} - ne^{-i\zeta}}.$$

By balancing method value of M will be evaluated. After finding value of M, the predicted solutions as well as necessary derivatives will be substituted to the Eq. (3.13). All the coefficients of power of $\psi(\zeta)$ are further equated to the zero and an algebraic system of equations is obtained. The resultant algebraic system is solved on maple or wolfram mathematica for values of ν_i's and g. At the last, put $\zeta = x + c\frac{t^k}{k}$ into the obtained solutions.

4. Implementation of the previous analytical approaches on nonlinear conformable Wu-Zhang system

Here, the above analytical computational methods GPREM and Sardar sub-equation method are applied on non-linear fractional Wu-Zhang system which describes (1 + 1)- dimensional dispersive long wave in two horizontal directions on shallow waters. The system has the following form:

$$\begin{aligned} D_t^k r &= -rr_x - s_x, \\ D_t^k s &= -sr_x - rs_x - \frac{1}{3}r_{xxx}. \end{aligned} \tag{4.1}$$

By the following wave transformation and conformable derivative definition,

$$\begin{aligned} &r(x,t) = r(\zeta), \text{ where } \zeta = x + \frac{ct^k}{k}, \\ &s(x,t) = s(\zeta), \text{ where } \zeta = x + \frac{ct^k}{k}, \end{aligned} \tag{4.2}$$

the non-linear fractional system is converted into the nonlinear ordinary differential system of the following form:

$$\begin{aligned} &cr' = -rr' - s', \\ &cs' = -sr' - rs' - \frac{1}{3}r'''. \end{aligned} \tag{4.3}$$

By integrating the first equation of above system and substituting the result in the second, and then integrating the final equation with respect to ζ, the following equation is obtained:

$$3r^3 + 9cr^2 + 6c^2r - 2r'' = 0. \tag{4.4}$$

By balancing principle, the value of M is obtained as $M = 1$.

4.1. **Analysis Of Wu-Zhang system through GPREM.** According to the GPREM, the formal solution of Eq. (4.4) for $M = 1$ has the following form:

$$r(\zeta) = A_0 + A_1\vartheta(\zeta) + B_1\tau(\zeta). \tag{4.5}$$

Equating all the coefficients to zero and considering the case $\varepsilon = 1$, the following system of algebraic equations is obtained:

$$\begin{aligned} \varepsilon^3\tau^0 &: \ 3A_1^3R - 9A_1B_1^2(m^2-1) + 4A_1(m^2-1) = 0, \\ \varepsilon^2\tau^0 &: \ 9A_0A_1^2R - 9A_0B_1^2(m^2-1) + 18mA_1B_1^2R + 9A_1^2R - 9cB_1^2(m^2-1) - 6mRA_1 = 0, \\ \varepsilon^1\tau^0 &: \ 9A_1A_0^2R + 18mRA_0A_1^2 - 9R^2A_1B_1^2 + 18cmRB_1^2 + 18cRA_0A_1 + 6A_1Rc^2 + 2A_1R^2 = 0, \\ \varepsilon^2\tau^1 &: \ -3B_1^3(m^2-1) + 9RA_1^2B_1 + 4B_1(m^2-1) = 0, \\ \varepsilon^1\tau^1 &: \ 6mRB_1^3 + 18A_0A_1B_1R + 18RA_1B_1c - 2B_1Rm = 0, \\ \varepsilon^0\tau^1 &: \ -3B_1^3R^2 + 9A_0^2B_1R + 18RcA_0B_1 + 6c^2B_1R = 0, \\ Const. &: \ 3A_0^3R - 9R^2A_0B_1^2 + 9cRA_0^2 - 9R^2B_1^2 + 6A_0Rc^2 = 0. \end{aligned} \tag{4.6}$$

The above gained algebraic equations are further solved by the aid of the Maple or Mathematica 9, results of the following form are gained:

Family 1:

$$A_0 = 0, \ A_1 = \frac{2}{3\sqrt{1-c^2}}, \ B_1 = 0, \ R \to -3c^2, \ m = \frac{1}{\sqrt{1-c^2}}. \tag{4.7}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = \frac{2}{3\sqrt{1-c^2}}\left(\frac{-3c^2\sec(\sqrt{-3c^2}\zeta)}{\frac{1}{\sqrt{1-c^2}}\sec(\sqrt{-3c^2}\zeta) + 1}\right), \tag{4.8}$$

or

$$r(\zeta) = \frac{2}{3\sqrt{1-c^2}}\left(\frac{-3c^2\csc(\sqrt{-3c^2}\zeta)}{\frac{1}{\sqrt{1-c^2}}\csc(\sqrt{-3c^2}\zeta) + 1}\right), \tag{4.9}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 2:

$$A_0 = -2c, \ A_1 = \frac{2}{3\sqrt{1-4c+3c^2}}, \ B_1 = 0, \ R = -3c^2, \ m = \frac{1-2c}{\sqrt{1-4c+3c^2}}. \tag{4.10}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = -2c + \frac{2}{3\sqrt{1-4c+3c^2}}\left(\frac{-3c^2\sec(\sqrt{-3c^2}\zeta)}{\frac{1-2c}{\sqrt{1-4c+3c^2}}\sec(\sqrt{-3c^2}\zeta)+1}\right), \tag{4.11}$$

or

$$r(\zeta) = -2c + \frac{2}{3\sqrt{1-4c+3c^2}}\left(\frac{-3c^2\csc(\sqrt{-3c^2}\zeta)}{\frac{1-2c}{\sqrt{1-4c+3c^2}}\csc(\sqrt{-3c^2}\zeta)+1}\right), \tag{4.12}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 3:

$$A_0 = -c,\ A_1 = \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{3(-1+c)\sqrt{2-4c+3c^2}},\ B_1 = 0,\ R = \frac{3c^2}{2},\ m = \frac{\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{2-4c+3c^2}}. \tag{4.13}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = -c + \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{3(-1+c)\sqrt{2-4c+3c^2}}\left(\frac{-3c^2\sec(\sqrt{-3c^2}\zeta)}{\frac{\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{2-4c+3c^2}}\sec(\sqrt{-3c^2}\zeta)+1}\right), \tag{4.14}$$

or

$$r(\zeta) = -c + \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{3(-1+c)\sqrt{2-4c+3c^2}}\left(\frac{-3c^2\csc(\sqrt{-3c^2}\zeta)}{\frac{\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{2-4c+3c^2}}\csc(\sqrt{-3c^2}\zeta)+1}\right), \tag{4.15}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 4:

$$A_0 = c,\ A_1 = 0,\ B_1 = \frac{2}{\sqrt{3}},\ R = 0,\ m = 0. \tag{4.16}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = c + \frac{2}{\sqrt{3}}\left(\frac{1}{\varepsilon\zeta}\right), \tag{4.17}$$

where $\zeta = x + \frac{ct^k}{k}$.

Again, equating all the coefficients to zero and considering the case $\varepsilon = -1$, the following system of algebraic equations is obtained:

$$\begin{aligned}
\varepsilon^3\tau^0 &: \ 3A_1^3R + 9A_1B_1^2(m^2-1) - 4A_1(m^2-1) = 0,\\
\varepsilon^2\tau^0 &: \ 9A_0A_1^2R + 9A_0B_1^2(m^2-1) - 18mA_1B_1^2R + 9A_1^2R + 9cB_1^2(m^2-1) - 2mRA_1 = 0,\\
\varepsilon^1\tau^0 &: \ 9A_1A_0^2R - 18mRA_0A_1^2 + 9R^2A_1B_1^2 - 18cmRB_1^2 + 18cRA_0A_1 + 6A_1Rc^2 - 2A_1R^2 = 0,\\
\varepsilon^2\tau^1 &: \ 3B_1^3(m^2-1) + 9RA_1^2B_1 - 4B_1(m^2-1) = 0,\\
\varepsilon^1\tau^1 &: \ -6mRB_1^3 + 18A_0A_1B_1R + 18RA_1B_1c + 2B_1Rm = 0,\\
\varepsilon^0\tau^1 &: \ 3B_1^3R^2 + 9A_0^2B_1R + 18RcA_0B_1 + 6c^2B_1R = 0,\\
Const. &: \ 3A_0^3R + 9R^2A_0B_1^2 + 9cRA_0^2 + 9R^2B_1^2 + 6A_0Rc^2 = 0.
\end{aligned} \tag{4.18}$$

The above gained system of algebraic equations are further solved by the aid of the Maple or Mathematica 9, results of the following form are gained:

Family 1:

$$A_0 = 0,\ A_1 = \frac{2}{3\sqrt{9-c^2}},\ B_1 = 0,\ R = 3c^2,\ m = \frac{3}{\sqrt{9-c^2}}. \tag{4.19}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = \frac{2}{3\sqrt{9-c^2}}\left(\frac{3c^2\operatorname{sech}(\sqrt{3c^2}\zeta)}{\frac{3}{\sqrt{9-c^2}}\operatorname{sech}(\sqrt{3c^2}\zeta)+1}\right), \tag{4.20}$$

or

$$r(\zeta) = \frac{2}{3\sqrt{9-c^2}}\left(\frac{3c^2\operatorname{csch}(\sqrt{3c^2}\zeta)}{\frac{3}{\sqrt{9-c^2}}\operatorname{csch}(\sqrt{3c^2}\zeta)+1}\right), \tag{4.21}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 2:

$$A_0 = -2c,\ A_1 = \frac{2\sqrt{(-1+2c)^2}}{(3-6c)\sqrt{9-36c+35c^2}},\ B_1 = 0,\ R = 3c^2,\ m = \frac{3\sqrt{(-1+2c)^2}}{\sqrt{9-36c+35c^2}}. \tag{4.22}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = -2c + \frac{2\sqrt{(-1+2c)^2}}{(3-6c)\sqrt{9-36c+35c^2}}\left(\frac{3c^2\operatorname{sech}(\sqrt{-3c^2}\zeta)}{\frac{3\sqrt{(-1+2c)^2}}{\sqrt{9-36c+35c^2}}\operatorname{sech}(\sqrt{3c^2}\zeta)+1}\right), \tag{4.23}$$

or

$$r(\zeta) = -2c + \frac{2\sqrt{(-1+2c)^2}}{(3-6c)\sqrt{9-36c+35c^2}}\left(\frac{3c^2\operatorname{csch}(\sqrt{-3c^2}\zeta)}{\frac{3\sqrt{(-1+2c)^2}}{\sqrt{9-36c+35c^2}}\operatorname{csch}(\sqrt{3c^2}\zeta)+1}\right), \tag{4.24}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 3:

$$A_0 = -c,\ A_1 = \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{(3-3c)\sqrt{18-36c+19c^2}},\ B_1 = 0,\ R = \frac{-3c^2}{2},\ m = \frac{3\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{18-36c+19c^2}}. \tag{4.25}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = -c + \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{(3-3c)\sqrt{18-36c+19c^2}}\left(\frac{\frac{-3c^2}{2}\operatorname{sech}(\sqrt{\frac{-3c^2}{2}}\zeta)}{\frac{3\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{18-36c+19c^2}}\operatorname{sech}(\frac{-3c^2}{2}\zeta)+1}\right), \tag{4.26}$$

or

$$r(\zeta) = -c + \frac{2\sqrt{2}\sqrt{(-1+c)^2}}{(3-3c)\sqrt{18-36c+19c^2}}\left(\frac{\frac{-3c^2}{2}\operatorname{csch}(\sqrt{\frac{-3c^2}{2}}\zeta)}{\frac{3\sqrt{2}\sqrt{(-1+c)^2}}{\sqrt{18-36c+19c^2}}\operatorname{csch}(\frac{-3c^2}{2}\zeta)+1}\right), \tag{4.27}$$

where $\zeta = x + \frac{ct^k}{k}$.

Family 4:

$$A_0 = -c,\ A_1 = 0,\ B_1 = \frac{-2}{\sqrt{3}},\ R = 0,\ m = 0. \tag{4.28}$$

From (3.5), (3.6), (3.7), and (4.4) the following analytical solutions are deduced:

$$r(\zeta) = -c - \frac{2}{\sqrt{3}}\left(\frac{1}{\varepsilon\zeta}\right), \tag{4.29}$$

where $\zeta = x + \frac{ct^k}{k}$.

4.2. **Analysis Of Wu-Zhang system through the Sardar sub-equation method.** In this subsection, the Sardar sub-equation method is applied to the nonlinear fractional Wu-zhang system to construct some of the new and distinct solitary wave solutions. By homogenous balancing principle the value of M is obtained as $M = 1$. According to the mentioned method, Eq. (4.4) becomes

$$P(\xi) = \nu_0 + \nu_1\psi^1(\xi), \tag{4.30}$$

by substituting Eq. (4.24) into Eq. (3.10) and then equate all coefficients of $\psi(\xi)$ to zero, an algebraic system of equations is gained as given below,

$$\begin{aligned}
&\psi^3 : \; 3\nu_1^3 - 4\nu_1 = 0,\\
&\psi^2 : \; 9\nu_0\nu_1^2 + 9c\nu_1^2 = 0,\\
&\psi^1 : \; 9\nu_1\nu_0^2 g + 18c\nu_0\nu_1 + 6c^2\nu_1 - 2g\nu_1 = 0,\\
&\psi^0 : \; 3\nu_0^3 + 9c\nu_0^2 + 6c^2\nu_0.
\end{aligned} \tag{4.31}$$

By solving the above system with aid of computer softwares maple or mathematica to get the following results

$$\nu_0 = -c, \; \nu_1 = \frac{2}{\sqrt{3}}, \; g = \frac{3c^2}{2}. \tag{4.32}$$

The formal solutions of Eq. (3.10) corresponding to the Eq. (4.24), along with solution Eq. (3.8) are

Case 1:

If $g = \frac{3c^2}{2} > 0$ and $\alpha = 0$, then

$$P_1^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{2}mn}(\operatorname{sech}_{mn}(\sqrt{\frac{3c^2}{2}}(x + c\frac{t^k}{k}))), \tag{4.33}$$

$$P_2^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{-3c^2}{2}mn}(\operatorname{csch}_{mn}(\sqrt{\frac{3c^2}{2}}(x + c\frac{t^k}{k}))), \tag{4.34}$$

Case 2:

If $g = \frac{3c^2}{2} < 0$ and $\alpha = 0$, then

$$P_3^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{-3c^2}{2}mn}(\sec_{mn}(\sqrt{\frac{-3c^2}{2}}(x + c\frac{t^k}{k}))), \tag{4.35}$$

$$P_4^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{-3c^2}{2}mn}(\csc_{mn}(\sqrt{\frac{-3c^2}{2}}(x + c\frac{t^k}{k}))), \tag{4.36}$$

Case 3:

If $g = \frac{3c^2}{2} < 0$ and $\alpha = (\frac{\frac{3c^2}{2}}{2})^2$, then

$$P_5^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{-\frac{3c^2}{4}}\tanh_{mn}\left(\sqrt{-\frac{3c^2}{4}}(x + c\frac{t^k}{k})\right), \tag{4.37}$$

$$P_6^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{-\frac{3c^2}{4}}\coth_{mn}\left(\sqrt{-\frac{3c^2}{4}}(x + c\frac{t^k}{k})\right), \tag{4.38}$$

$$P_7^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{-\frac{3c^2}{4}}\left(\tanh_{mn}\left(\sqrt{-3c^2}(x + c\frac{t^k}{k})\right) \pm i\sqrt{mn}\operatorname{sech}_{mn}\left(\sqrt{-3c^2}(x + c\frac{t^k}{k})\right)\right), \tag{4.39}$$

$$P_8^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{-\frac{3c^2}{4}}\left(\coth_{mn}\left(\sqrt{-3c^2}(x + c\frac{t^k}{k})\right) \pm \sqrt{mn}\operatorname{csch}_{mn}\left(\sqrt{-3c^2}(x + c\frac{t^k}{k})\right)\right), \tag{4.40}$$

$$P_9^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{-\frac{3c^2}{16}}\left(\tanh_{mn}\left(\sqrt{-\frac{3c^2}{16}}(x + c\frac{t^k}{k})\right) \pm \sqrt{mn}\coth_{mn}\left(\sqrt{-\frac{3c^2}{16}}(x + c\frac{t^k}{k})\right)\right). \tag{4.41}$$

Case 4:
If $g = \frac{3c^2}{2} > 0$ and $\alpha = (\frac{\frac{3c^2}{2}}{2})^2$, then

$$P_{10}^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{4}}\tan_{mn}\left(\sqrt{-\frac{3c^2}{4}}(x + c\frac{t^k}{k})\right), \tag{4.42}$$

$$P_{11}^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{4}}\cot_{mn}\left(\sqrt{-\frac{3c^2}{4}}(x + c\frac{t^k}{k})\right), \tag{4.43}$$

$$P_{12}^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{4}}\left(\tan_{mn}\left(\sqrt{3c^2}(x + c\frac{t^k}{k})\right) \pm i\sqrt{mn}\sec_{mn}\left(\sqrt{3c^2}(x + c\frac{t^k}{k})\right)\right), \tag{4.44}$$

$$P_{13}^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{4}}\left(\cot_{mn}\left(\sqrt{3c^2}(x + c\frac{t^k}{k})\right) \pm \sqrt{mn}\csc_{mn}\left(\sqrt{3c^2}(x + c\frac{t^k}{k})\right)\right), \tag{4.45}$$

$$P_{14}^{\pm} = -c \pm \frac{2}{\sqrt{3}}\sqrt{\frac{3c^2}{16}}\left(\tan_{mn}\left(\sqrt{\frac{3c^2}{16}}(x + c\frac{t^k}{k})\right) \pm \sqrt{mn}\cot_{mn}\left(\sqrt{\frac{3c^2}{16}}(x + c\frac{t^k}{k})\right)\right). \tag{4.46}$$

5. Physical and geometrical interpretation

The graphical illustration is an important tool used for understanding the properties and mechanism of solutions physically. The graphical phenomena are shown by setting the values of arbitrary parameters, and the graphical representations reveal the mechanism of wave behavior. The obtained results for the nonlinear conformable Wu-Zhang system retrieved via generalized projective Ricatti equation method and the Sardar sub-equation method will be supportive to conduct new laboratory experiments for a deeper insight into the possible physical changes. In water wave theory, a (1+1)-dimensional two horizontal directional dispersive long wave is described by this system. A few new kinds of solutions, which have not been added to the literature previously, are successfully developed in this article.

In Figure 1, 3D-plot, 2D-line graph and contour plot of an anti-bell shaped dark soliton of Eq. (4.9) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-30, 30]$ and $t \in [-15, 15]$. A 2D representation for suitable values of k and for $x = 0.1$ is also given. In Figure 2, 3D-plot, 2D-line graph and contour plot of a bright-dark soliton of Eq. (4.15) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. Also, a 2D representation for suitable values of k and for $x = 0.1$ is given. In Figure 3, 3D-plot, 2D-line graph and contour plot of a singular soliton of Eq. (4.13) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-50, 50]$ and $t \in [-15, 15]$. Also, 2D representation for suitable values of k and for $x = 0.1$ is given.

In Figure 4, 3D-plot, 2D-line graph, and contour plot of a periodic soliton of Eq. (4.13) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. A 2D representation for suitable values of k and for $x = 0.1$ is given. In Figure 5, 3D-Plot, 2D-line graph and contour plot of a lump-type bell shaped soliton of Eq. (4.9) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. Also, 2D representation for suitable values of k and for $x = 0.1$ and for $x = 0.1$ is given. In Figure 6, 3D-plot, 2D-line graph, and contour plot of a singular periodic soliton of Eq. (4.24) are shown for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. A 2D representation for suitable values of k and for $x = 0.1$ is given. In Figure 7, 3D-plot, 2D-line graph, and contour plot of a bell shaped bright soliton of Eq. (4.24) are shown for values of arbitrary constants taken as $c = 0.02, k = 0.8$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. A 2D representation for suitable values of k and for $x = 0.1$ is given. In Figure 8, 3D-plot, 2D-line graph, and contour plot of a kink shaped dark-bright soliton of Eq. (4.24) are shown for values of arbitrary constants taken as $c = 0.02, k = 0.8$ with $x \in [-40, 40]$ and $t \in [-15, 15]$. A 2D representation for suitable values of k and for $x = 0.1$ is given.

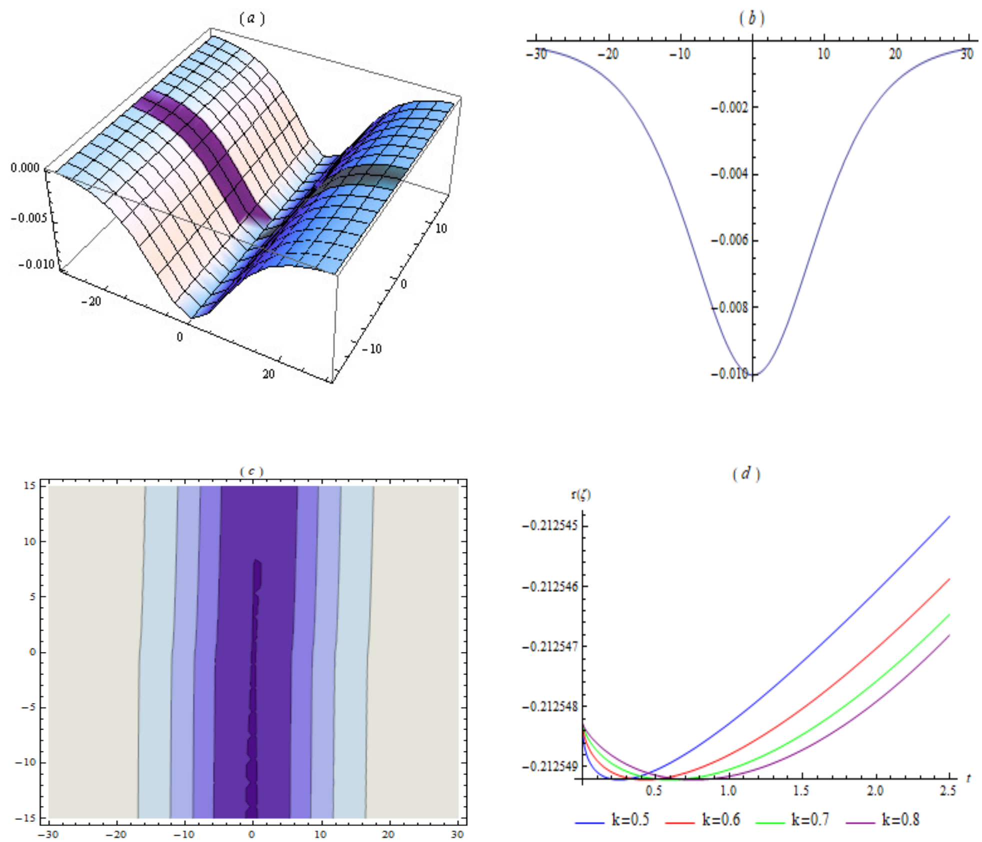

Figure 1. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) is depicted of Eq. (4.9) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows an anti-bell shaped dark soliton.

6. Discussion

In this article, two analytical approaches, the generalized projective Riccati equation method and the Sardar sub-equation method, have been applied to the nonlinear conformable time-fractional Wu-Zhang system for the first time to introduce a few novel solitary wave solutions. After successful implementation, 2D graph lines, 3D graphs, contour graphs as well as parametric graphs for some specific values of the constants are plotted to indicate novelty of obtained solutions. These solutions were also compared to rest of the literature over taken system to present novelty of the work. In [11], Eslami and Rezazadeh applied the first integral method on the given system and obtained some periodic solitary wave solutions. In [59], Yel and Baskonus applied the modified $\exp(-\Omega(\zeta))$-expansion function method on the given system to get soliton type solutions. In [48], Khater and Dianchen Lu applied the modified auxiliary equation method on the given system and obtained five solitary wave solutions. By comparing to the authors obtained in [11, 48, 59], we deduced that our solutions are novel and different. The comparison shows that the performance of the methods used in this article is an effective tool in constructing novel solitary wave solutions for nonlinear fractional PDEs. It also indicates that this discussion of the nonlinear time-fractional WuZhang system gives some novel soliton type solutions.

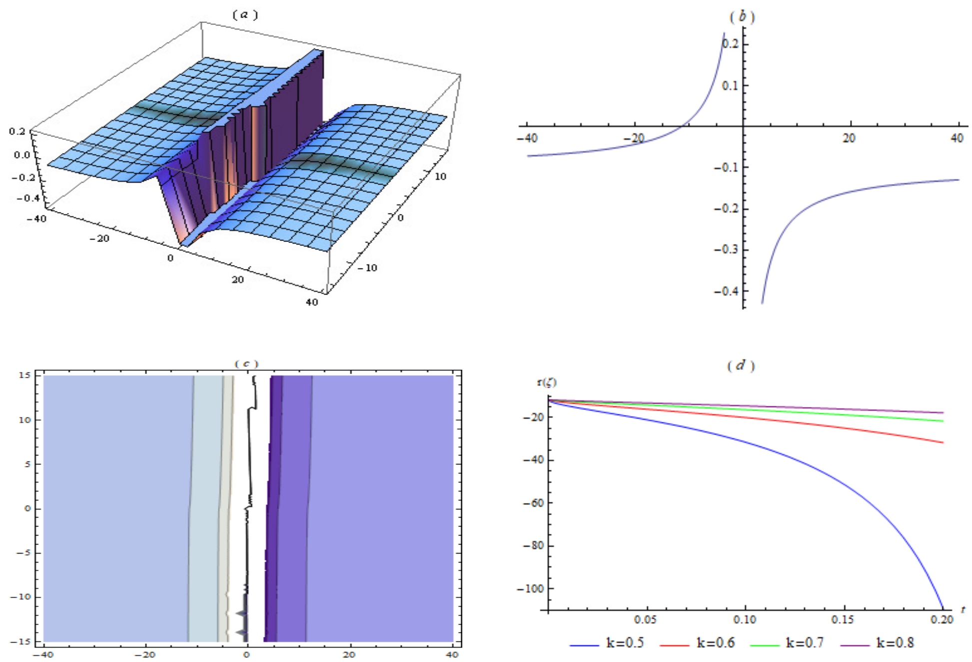

FIGURE 2. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.15) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a bright-dark soliton.

7. Conclusions

In this paper, the GPREM and Sardar sub-equation method with the conformable fractional derivative are being successfully applied for developing some novel solitary wave solutions of the non-linear fractional Wu-Zhang system. These methods are more effective and convenient in the construction of solitary wave solutions of remarkable importance in the field of analytical solutions. With the help of these methods some novel analytical solitary wave explicit solutions of different kinds such as anti-bell shaped and bell shaped dark soliton, singular soliton, periodic singular solution, kink shaped soliton solutions etc. are obtained. Some of the obtained results are also plotted above in 3D solitary wave form, 2D line graphs form, contour plots and 2D representation for some suitable values for k with $x = 0.1$ for better understanding to these explicit solutions. The proposed methods have more advantages as these are more direct, reliable and powerful to apply. Therefore, it can be deduced that these methods can be extended to solve numerous non-linear fractional partial differential equations as well as many non-linear fractional partial differential equations systems.

Declarations

Ethical Approval. The authors declare that there are no animal studies in this work.
Conflict of Interest. The authors declare that they have no conflict of interest.
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Data Availability. There is no data set used.
Author Contributions. H. T. gave the first idea of investigation, concept and modeling. The methodology and computations has been done by M. A. H. Also, H. A. after modifying the concept, wrote the paper. H. R. and S. B. performed the reviewing and editing.

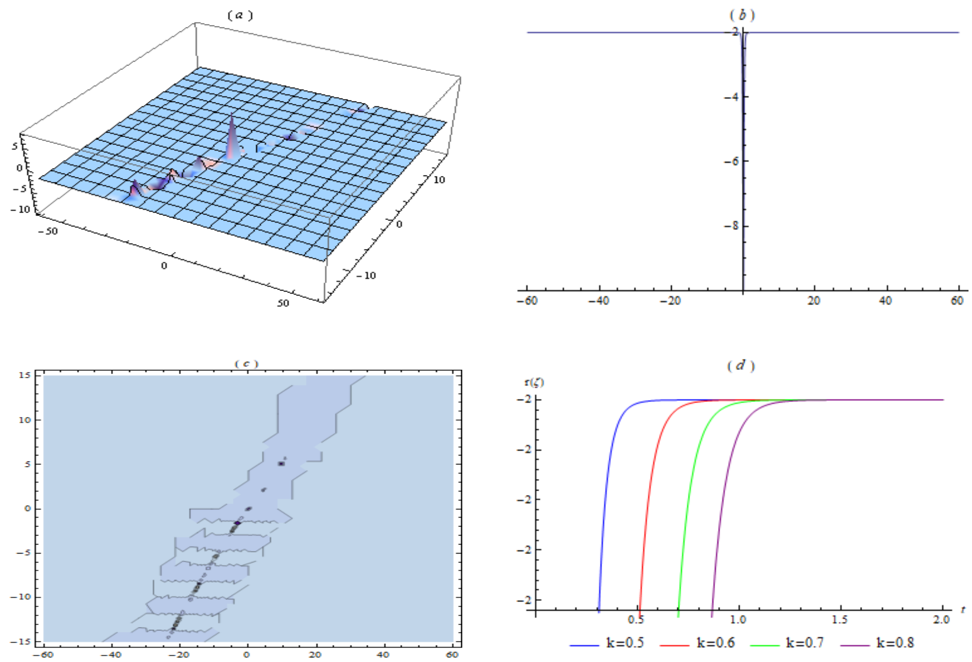

FIGURE 3. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.13) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a singular soliton.

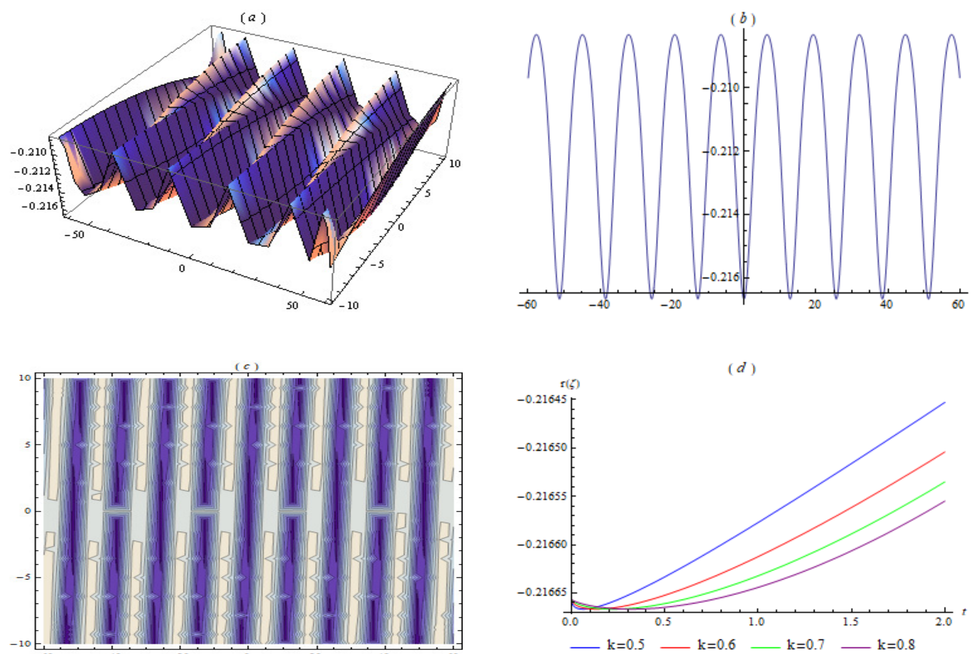

FIGURE 4. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.13) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a periodic soliton.

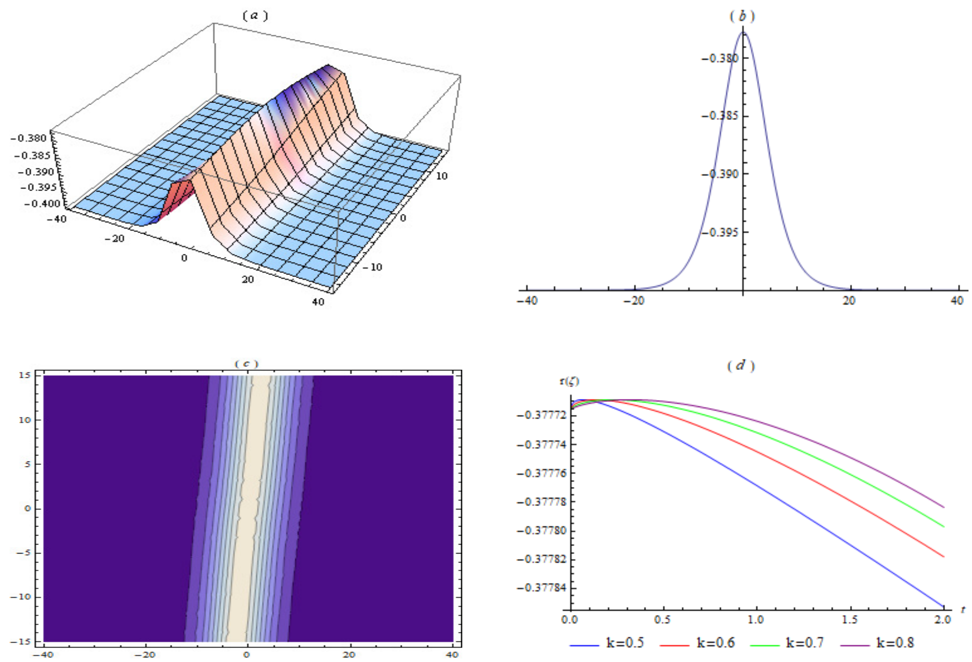

FIGURE 5. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.24) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a lump-type bell shaped soliton.

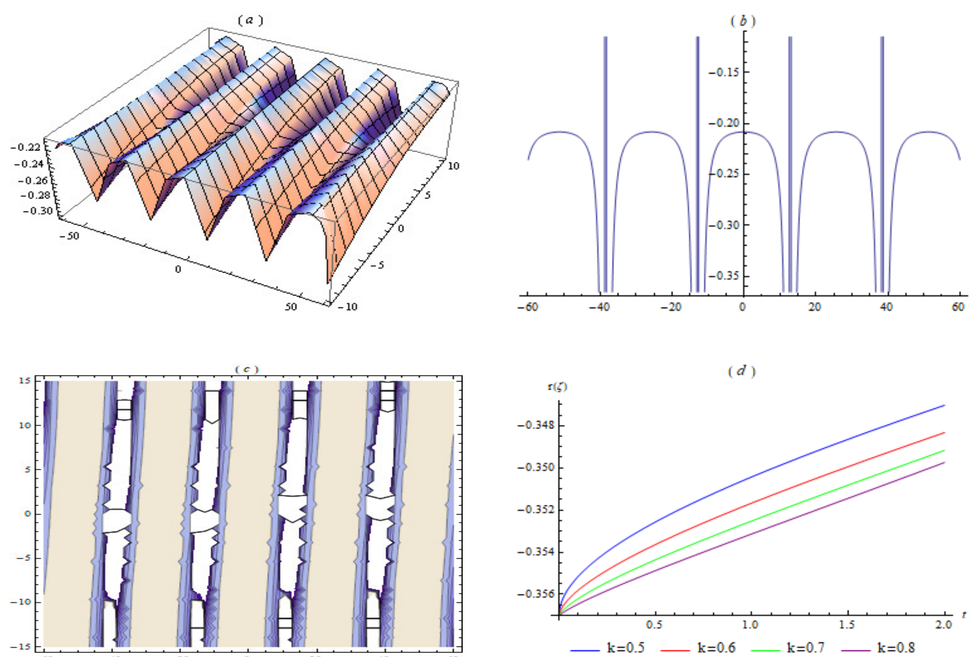

FIGURE 6. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.33) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a singular periodic soliton.

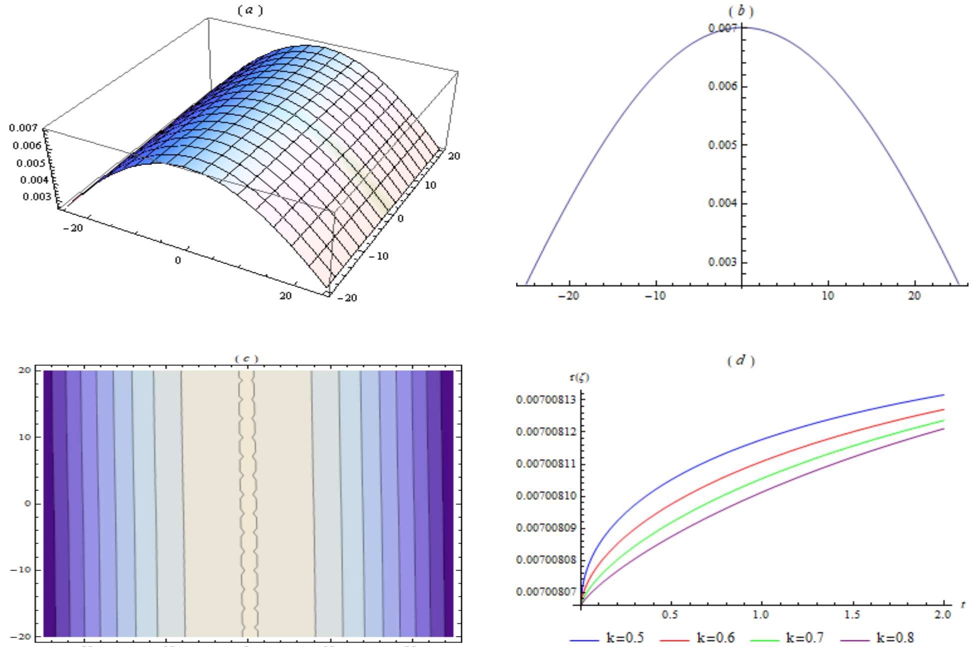

FIGURE 7. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.40) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows a bell-shaped bright soliton.

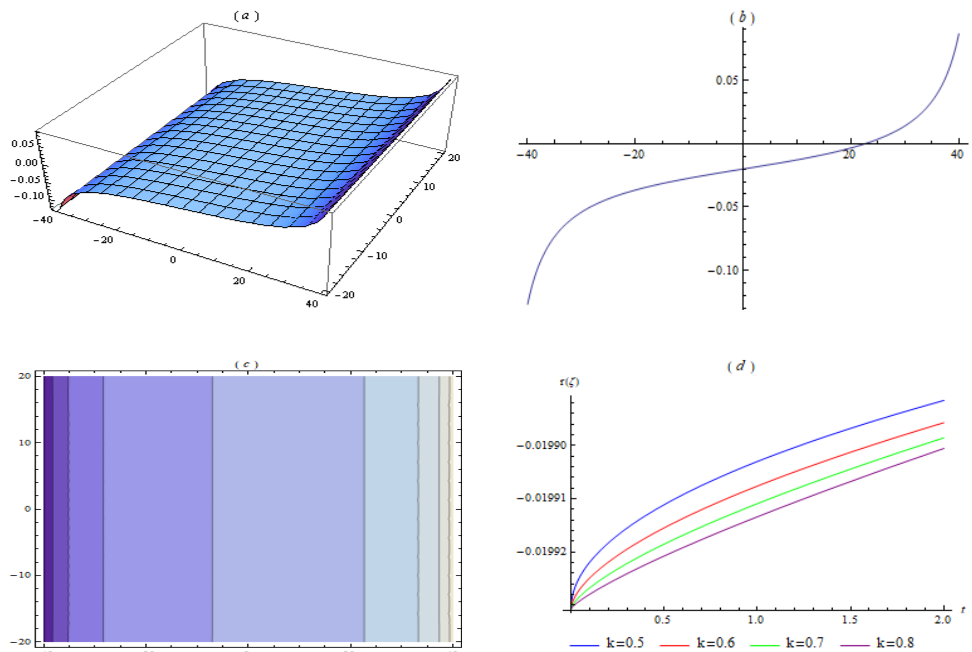

FIGURE 8. The 3D-Plot, 2D-line graph, contour plot and 2D parametric representation (for different values of k) of Eq. (4.45) for values of arbitrary constants taken as $c = 0.1, k = 0.9$ with $x \in [-40, 40]$ and $t \in [-15, 15]$ is depicted here. This profile shows kink shaped dark-bright soliton.

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