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An overlapping adaptive step-size multi-derivative hybrid block method for higher order initial value problems

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Abstract

The need for accurate solutions to mathematical models, particularly for linear and nonlinear higher-order initial value problems, is essential across various scientific and engineering fields. Traditional methods often face challenges with stability and precision, especially in non-linear cases, prompting the development of advanced numerical techniques. This study introduces a two-step overlapping adaptive step-size multi-derivative hybrid block method to address these challenges in solving higher-order initial value problems. The method incorporates overlapping elements, using the second-to-last intra-step point from the previous step within each integration block to enhance accuracy. The method uses error estimation and selects an appropriate step-size, ensuring the desired accuracy without wasting computational resources or introducing unnecessary errors. The non-linear initial value problems are efficiently linearized using a modified-Picard iteration. Numerical examples are provided to demonstrate the efficiency and accuracy of the proposed method, and its performance is compared against a similar non-overlapping method as well as other methods reported in the literature.

Keywords. Hybrid block method, Multi-derivative, Modified-Picard iteration, Overlapping, Adaptive step-size.
1991 Mathematics Subject Classification. 65L05, 65L04, 65L20.

1. Introduction

Problems that arise in the fields of applied sciences, such as biological systems, quantum mechanics, physics and engineering, can be modeled using higher-order ODEs. The complexity or impossibility of solving some of these problems through analytical means has emphasized the continued importance of numerical techniques. Higher-order DEs can either be solved directly or by first transforming them into an equivalent system of first-order ODEs [7, 8, 13], although this process can be more demanding compared to direct methods [5, 9, 11, 23].

Over the years, numerous discussions have revolved around various multi-derivative methods, implicit methods, and the utilization of intra-step points [2, 6, 16]. For instance, a three-point block technique involving fourth- and fifth-derivatives for third-order ODEs was proposed in [3], a two-point block approach utilizing a fifth-derivative to directly solve fourth-order ODEs was introduced in [4] and a third-derivative hybrid block method (HBM) for second-order IVPs was presented in [19]. Research findings suggest that multi-derivative methods achieve higher accuracy while maintaining strong stability properties, while implicit methods offer advantages in terms of stability and robustness. Various authors have developed BHMs for the direct solution of higher-order ODEs. These methods are formulated in terms of LMMs for IVPs. Ken et al. [10] developed a block hybrid collocation technique for third-order ODEs. A direct approach for solving third-order ODEs using a two-point four-step block technique was proposed in [12]. Orakwelu et al. [17] presented a block hybrid method (BHM) for third-order IVPs, exploring the efficiency of the method by imposing different off-points between grid points during the formulation process.

Recently, Ahmedai et al. [1] proposed a BHM with equally spaced intra-step points for solving third-order IVPs. This technique incorporates a simple iterative approach for linearizing the equations, which significantly improves both the convergence rate and accuracy of the method. Another researcher (see [15]), introduced an overlapping

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HBM by incorporating the second-to-last intra-step point of the previous step into each integrating block of the HBM, demonstrating an improvement in the method for solving first-order IVPs. Existing numerical methods for solving IVPs often rely on small fixed step-sizes across the entire integration interval, which can introduce inefficiencies by requiring unnecessarily small steps even where larger steps could suffice. To address this limitation, adaptive step-size methods have been explored. For instance, adaptive step-size HBMs are discussed in [21, 22] for integrating IVPs directly. The adaptive step-size helps in balancing accuracy and efficiency by using smaller step-size where the solution changes rapidly and larger step-size where the changes are more gradual. A variable step-size fourth-derivative HBM was introduced in [18] to solve third-order IVPs, demonstrating its applicability on the Blasius equation and various nonlinear thin-film flow problems. They found that the method exhibits efficiency and utility in solving real-life problems.

This research presents a new overlapping adaptive step-size MDHBM for higher-order IVPs. The method uses the modified-Picard iteration technique to linearize the nonlinear IVPs. The effectiveness of the method is examined through numerical experiment to demonstrate its convergence and accurate solutions.

2. Derivation of the Method

Consider a higher-order IVP of the form

$$z^{w} = f(t, z, z', \dots, z^{w-1}), \qquad z(a) = z_0, \quad z'(a) = z'_0, \quad \dots, \quad z^{w-1}(a) = z_0^{w-1}, \tag{2.1}$$

where w $= 3, 4$ and prime denotes the derivative with respect to the independent variable t, f is a continuous linear or nonlinear function in the interval $[\hat{a}, \hat{b}]$ and w is the order of the IVP.

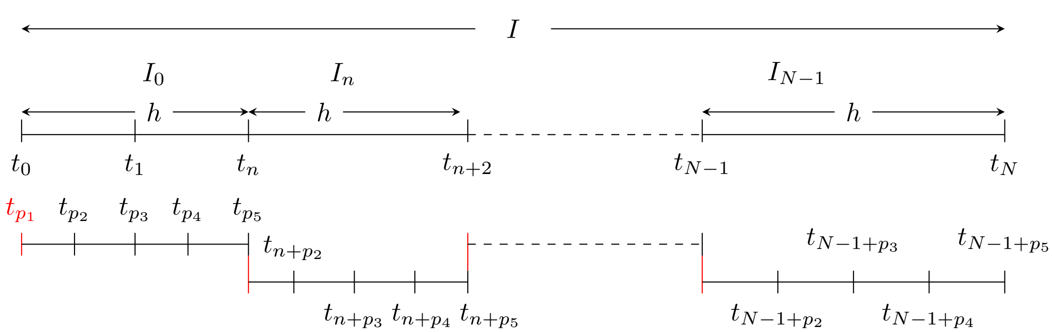

Figure 1. Non-overlapping grid [14].

2.1. Non-Overlapping Block Method.

Non-Overlapping Block Method. The non-overlapping method is a two-step method based on approximating the solution $z(t)$ to $z^{w-1}(t)$ at $2\Phi + w - 1$ collocation points

$$t_{n+p_i} = t_n + hp_i, \quad n = 0, 1, 2, \dots, N-1,$$

in the closed interval $[\hat{a}, \hat{b}]$, which is partitioned as $\hat{a} = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = \hat{b}$. h is the chosen step-length on a grid I, p_i denotes the collocation parameters: $p_1 = 0$, p_2, $p_3 = 1$, p_4, and p_5, where p_2, p_3, and p_4 are the intra-step points ($m = 3$). Φ is the number of collocation parameters. As shown in Figure 1, the representation of the non-overlapping grid I is illustrated for the case N.

The solution $z(t)$ of the non-overlapping block can be approximated by a power series polynomial of the form

$$Z(t) = \sum_{\mathbf{j=0}}^{\mathbf{2\Phi+w-1}} \tilde{\Lambda}_{n,j}(t - t_n)^j, \tag{2.2}$$

where $\tilde{\Lambda}_{n,j}$ represents the unknown coefficients, which are obtained by solving a system of $2\Phi+w-1$ equations. Here, $\Phi=5$ is the number of collocation parameters, resulting in $2\Phi+w-1$ unknowns generated from

$$
\begin{aligned}
Z'_{n+p_i} &= z'\left(t_{n+p_i}, z_{n+p_i}\right),\\
Z''_{n+p_i} &= z''\left(t_{n+p_i}, z_{n+p_i}, z'_{n+p_i}\right),\\
&\vdots\\
Z^{w}_{n+p_i} &= f\left(t_{n+p_i}, z_{n+p_i}, z'_{n+p_i}, \ldots, z^{w-1}_{n+p_i}\right),\\
Z^{w+1}_{n+p_i} &= g\left(t_{n+p_i}, z_{n+p_i}, z'_{n+p_i}, \ldots, f_{n+p_i}\right),
\end{aligned}
\tag{2.3}
$$

where $i=1,\,2,\,\ldots,\,\Phi$ and $w=3,\,4$, with the initial conditions

$$Z(t_n)=\tilde{\Lambda}_{n,0}=z_n, \tag{2.4}$$

$$Z'(t_n)=\tilde{\Lambda}_{n,1}=z'_n, \tag{2.5}$$

$$Z^{w-1}(t_n)=\tilde{\Lambda}_{n,w-1}=z^{w-1}_n, \qquad w=3,\,4. \tag{2.6}$$

Equations (2.3)-(2.6) are solved using Mathematica 13.0 to determine $\tilde{\Lambda}_{n,j}$, which are then substituted into (2.2) to obtain the continuous approximation of the non-overlapping fourth derivative method when $w=3$ is of the form

$$z_{n+p_i}=z_n+hp_{i\,1}z'_n+h^2\mu_{i\,1}z''_n+h^3\beta_{i\,1}f_n+h^3\left(\sum_{j=2}^{\Phi}\alpha_{i\,j}f_{n+p_j}\right)+h^4\tau_{i\,1}g_n+h^4\left(\sum_{j=2}^{\Phi}\gamma_{i\,j}g_{n+p_j}\right), \tag{2.7}$$

$$z'_{n+p_i}=z'_n+h\varepsilon_{i\,1}z''_n+h^2\eta_{i\,1}f_n+h^2\left(\sum_{j=2}^{\Phi}\zeta_{i\,j}f_{n+p_j}\right)+h^3\omega_{i\,1}g_n+h^3\left(\sum_{j=2}^{\Phi}\nu_{i\,j}g_{n+p_j}\right), \tag{2.8}$$

$$z''_{n+p_i}=z''_n+hd_{i\,1}f_n+h\left(\sum_{j=2}^{\Phi}\vartheta_{i\,j}f_{n+p_j}\right)+h^2\upsilon_{i\,1}g_n+h^2\left(\sum_{j=2}^{\Phi}\sigma_{i\,j}g_{n+p_j}\right), \tag{2.9}$$

where $i=2,3,....,\Phi$ and $p_{i,1},\ \mu_{i,1},\ \alpha_{i,j},\ \beta_{i,1},\ \tau_{i,1},\ \eta_{i,1},\ \gamma_{i,j},\ \nu_{i,j},\ \omega_{i,,1},\ \zeta_{i,j},\ d_{i,1},\ \varepsilon_{i,1},\ \upsilon_{i,1},\ \vartheta_{i,j}$ and $\sigma_{i\,j}$ are known constant coefficients. The non-overlapping fifth derivative method, when $w=4$ is obtained for solving fourth order IVPs is of the form

$$z_{n+p_i}=z_n+h\breve{p}_{i\,1}z'_n+h^2\breve{\mu}_{i\,1}z''_n+h^3\breve{\iota}_{i\,1}z'''_n+h^4\breve{\beta}_{i\,1}f_n+h^4\left(\sum_{j=2}^{\Phi}\breve{\alpha}_{i\,j}f_{n+p_j}\right)+h^5\breve{\tau}_{i\,1}g_n+h^5\left(\sum_{j=2}^{\Phi}\breve{\gamma}_{i\,j}g_{n+p_j}\right), \tag{2.10}$$

$$z'_{n+p_i}=z'_n+h\hat{p}_{i\,1}z''_n+h^2\hat{\mu}_{i\,1}z'''_n+h^3\hat{\beta}_{i\,1}f_n+h^3\left(\sum_{j=2}^{\Phi}\hat{\alpha}_{i\,j}f_{n+p_j}\right)+h^4\hat{\tau}_{i\,1}g_n+h^4\left(\sum_{j=2}^{\Phi}\hat{\gamma}_{i\,j}g_{n+p_j}\right), \tag{2.11}$$

$$z''_{n+p_i}=z''_n+h\hat{\varepsilon}_{i\,1}z'''_n+h^2\hat{\eta}_{i\,1}f_n+h^2\left(\sum_{j=2}^{\Phi}\hat{\zeta}_{i\,j}f_{n+p_j}\right)+h^3\hat{\omega}_{i\,1}g_n+h^3\left(\sum_{j=2}^{\Phi}\hat{\nu}_{i\,j}g_{n+p_j}\right), \tag{2.12}$$

$$z'''_{n+p_i}=z'''_n+h\hat{d}_{i\,1}f_n+h\left(\sum_{j=2}^{\Phi}\hat{\vartheta}_{i\,j}f_{n+p_j}\right)+h^2\hat{\upsilon}_{i\,1}g_n+h^2\left(\sum_{j=2}^{\Phi}\hat{\sigma}_{i\,j}g_{n+p_j}\right), \tag{2.13}$$

where $i=2,3,....,\Phi$ and $\hat{p}_{i,1},\ \breve{p}_{i,1},\ \hat{\mu}_{i,1},\ \breve{\mu}_{i,1},\ \breve{\iota}_{i,1},\ \hat{\alpha}_{i,1},\ \hat{\beta}_{i,1},\ \hat{\tau}_{i,1},\ \breve{\alpha}_{i,j},\ \breve{\beta}_{i,1},\ \breve{\tau}_{i,1},\ \hat{\eta}_{i,1},\ \hat{\gamma}_{i,j},\breve{\gamma}_{i,j},\ \hat{\nu}_{i,j},\ \hat{\omega}_{i,1},\ \hat{\zeta}_{i,j},\hat{d}_{i,1},\ \hat{\varepsilon}_{i,1},\ \hat{\upsilon}_{i,1},\ \hat{\vartheta}_{i,j}$, and $\hat{\sigma}_{i,j}$ are known constant coefficients. In addition, the collocation parameters used are $p_1=0,\ p_2=\frac{1}{2},\ p_3=1,\ p_4=\frac{3}{2}$ and $p_5=2$. In matrix form, Equations (2.7)–(2.9) can be

$$\boldsymbol{A}_1\,Z_{n+\Phi}=\boldsymbol{A}_0\,Z_n+h\,\boldsymbol{p}_0\,Z'_n+h^2\,\boldsymbol{\mu}_0\,Z''_n+h^3\,\boldsymbol{\alpha}_1\,F_{n+\Phi}+h^3\,\boldsymbol{\beta}_0\,F_n+h^4\,\boldsymbol{\gamma}_1\,G_{n+\Phi}+h^4\,\boldsymbol{\tau}_0\,G_n, \tag{2.14}$$

$$\boldsymbol{A}_1Z'_{n+\Phi}=\boldsymbol{A}_0Z'_n+h\,\boldsymbol{\varepsilon}_0\,Z''_n+h^2\,\boldsymbol{\zeta}_1\,F_{n+\Phi}+h^2\,\boldsymbol{\eta}_0\,F_n+h^3\,\boldsymbol{\nu}_1\,G_{n+\Phi}+h^3\,\boldsymbol{\omega}_0\,G_n, \tag{2.15}$$

$$\boldsymbol{A}_1Z''_{n+\Phi}=\boldsymbol{A}_0\,Z''_n+h\,\boldsymbol{d}_0\,F_n+h\,\boldsymbol{\vartheta}_1\,F_{n+\Phi}+h^2\,\boldsymbol{\sigma}_1\,G_{n+\Phi}+h^2\,\boldsymbol{\upsilon}_0\,G_n. \tag{2.16}$$

The constant coefficients are

$$
\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad
\boldsymbol{A}_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{p}_0 = \begin{pmatrix} p_{2,1} & 0 & \cdots & 0 \\ p_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ p_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix},
$$

$$
\boldsymbol{\mu}_0 = \begin{pmatrix} \mu_{2,1} & 0 & \cdots & 0 \\ \mu_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{\alpha}_1 = \begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,\Phi} \\ \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{\Phi,2} & \alpha_{\Phi,3} & \cdots & \alpha_{\Phi,\Phi} \end{pmatrix},
$$

$$
\boldsymbol{\beta}_0 = \begin{pmatrix} \beta_{2,1} & 0 & \cdots & 0 \\ \beta_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{\gamma}_1 = \begin{pmatrix} \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,\Phi} \\ \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{\Phi,2} & \gamma_{\Phi,3} & \cdots & \gamma_{\Phi,\Phi} \end{pmatrix},
$$

$$
\boldsymbol{\tau}_0 = \begin{pmatrix} \tau_{2,1} & 0 & \cdots & 0 \\ \tau_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \tau_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{\varepsilon}_0 = \begin{pmatrix} \varepsilon_{2,1} & 0 & \cdots & 0 \\ \varepsilon_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \varepsilon_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \tag{2.17}
$$

$$
\boldsymbol{\zeta}_1 = \begin{pmatrix} \zeta_{2,2} & \zeta_{2,3} & \cdots & \zeta_{2,\Phi} \\ \zeta_{3,2} & \zeta_{3,3} & \cdots & \zeta_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_{\Phi,2} & \zeta_{\Phi,3} & \cdots & \zeta_{\Phi,\Phi} \end{pmatrix}, \quad
\boldsymbol{\eta}_0 = \begin{pmatrix} \eta_{2,1} & 0 & \cdots & 0 \\ \eta_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix},
$$

$$
\boldsymbol{\omega}_0 = \begin{pmatrix} \omega_{2,1} & 0 & \cdots & 0 \\ \omega_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{\nu}_1 = \begin{pmatrix} \nu_{2,2} & \nu_{2,3} & \cdots & \nu_{2,\Phi} \\ \nu_{3,2} & \nu_{3,3} & \cdots & \nu_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{\Phi,2} & \nu_{\Phi,3} & \cdots & \nu_{\Phi,\Phi} \end{pmatrix},
$$

$$
\boldsymbol{d}_0 = \begin{pmatrix} d_{2,1} & 0 & \cdots & 0 \\ d_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ d_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad
\boldsymbol{\vartheta}_1 = \begin{pmatrix} \vartheta_{2,2} & \vartheta_{2,3} & \cdots & \vartheta_{2,\Phi} \\ \vartheta_{3,2} & \vartheta_{3,3} & \cdots & \vartheta_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \vartheta_{\Phi,2} & \vartheta_{\Phi,3} & \cdots & \vartheta_{\Phi,\Phi} \end{pmatrix},
$$

$$
\boldsymbol{\sigma}_1 = \begin{pmatrix} \sigma_{2,2} & \sigma_{2,3} & \cdots & \sigma_{2,\Phi} \\ \sigma_{3,2} & \sigma_{3,3} & \cdots & \sigma_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{\Phi,2} & \sigma_{\Phi,3} & \cdots & \sigma_{\Phi,\Phi} \end{pmatrix}, \quad
\boldsymbol{\upsilon}_0 = \begin{pmatrix} \upsilon_{2,1} & 0 & \cdots & 0 \\ \upsilon_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \upsilon_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}.
$$

Equations (2.10)–(2.13) can be expressed in matrix form as

$$
\boldsymbol{A}_1 Z_{n+\Phi} = \boldsymbol{A}_0 Z_n + h\,\breve{\boldsymbol{p}}_0\, Z'_n + h^2\,\breve{\boldsymbol{\mu}}_0\, Z''_n + h^3\,\breve{\boldsymbol{\iota}}_0\, Z'''_n + h^4\,\breve{\boldsymbol{\alpha}}_1\, F_{n+\Phi} + h^4\,\breve{\boldsymbol{\beta}}_0\, F_n + h^5\,\breve{\boldsymbol{\gamma}}_1\, G_{n+\Phi} + h^5\,\breve{\boldsymbol{\tau}}_0\, G_n, \tag{2.18}
$$

$$
\boldsymbol{A}_1 Z'_{n+\Phi} = \boldsymbol{A}_0 Z'_n + h\,\hat{\boldsymbol{p}}_0\, Z''_n + h^2\,\hat{\boldsymbol{\mu}}_0\, Z''_n + h^3\,\hat{\boldsymbol{\alpha}}_1\, F_{n+\Phi} + h^3\,\hat{\boldsymbol{\beta}}_0\, F_n + h^4\,\hat{\boldsymbol{\gamma}}_1\, G_{n+\Phi} + h^4\,\hat{\boldsymbol{\tau}}_0\, G_n, \tag{2.19}
$$

$$
\boldsymbol{A}_1 Z''_{n+\Phi} = \boldsymbol{A}_0 Z''_n + h\,\hat{\boldsymbol{\varepsilon}}_0\, Z'''_n + h^2\,\hat{\boldsymbol{\zeta}}_1\, F_{n+\Phi} + h^2\,\hat{\boldsymbol{\eta}}_0\, F_n + h^3\,\hat{\boldsymbol{\nu}}_1\, G_{n+\Phi} + h^3\,\hat{\boldsymbol{\omega}}_0\, G_n, \tag{2.20}
$$

$$
\boldsymbol{A}_1 Z'''_{n+\Phi} = \boldsymbol{A}_0\, Z'''_n + h\,\hat{\boldsymbol{d}}_0\, F_n + h\,\hat{\boldsymbol{\vartheta}}_1\, F_{n+\Phi} + h^2\,\hat{\boldsymbol{\sigma}}_1\, G_{n+\Phi} + h^2\,\hat{\boldsymbol{\upsilon}}_0\, G_n, \tag{2.21}
$$

where

$$
\begin{aligned}
&\breve{\mathbf{p}}_0 = \begin{pmatrix} \breve{p}_{2,1} & 0 & \cdots & 0 \\ \breve{p}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \breve{p}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad \breve{\boldsymbol{\mu}}_0 = \begin{pmatrix} \breve{\mu}_{2,1} & 0 & \cdots & 0 \\ \breve{\mu}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\mu}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\breve{\boldsymbol{\alpha}}_1 = \begin{pmatrix} \breve{\alpha}_{2,2} & \breve{\alpha}_{2,3} & \cdots & \breve{\alpha}_{2,\Phi} \\ \breve{\alpha}_{3,2} & \breve{\alpha}_{3,3} & \cdots & \breve{\alpha}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\alpha}_{\Phi,2} & \breve{\alpha}_{\Phi,3} & \cdots & \breve{\alpha}_{\Phi,\Phi} \end{pmatrix}, \quad \breve{\boldsymbol{\beta}}_0 = \begin{pmatrix} \breve{\beta}_{2,1} & 0 & \cdots & 0 \\ \breve{\beta}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\beta}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\breve{\boldsymbol{\gamma}}_1 = \begin{pmatrix} \breve{\gamma}_{2,2} & \breve{\gamma}_{2,3} & \cdots & \breve{\gamma}_{2,\Phi} \\ \breve{\gamma}_{3,2} & \breve{\gamma}_{3,3} & \cdots & \breve{\gamma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\gamma}_{\Phi,2} & \breve{\gamma}_{\Phi,3} & \cdots & \breve{\gamma}_{\Phi,\Phi} \end{pmatrix}, \quad \breve{\boldsymbol{\tau}}_0 = \begin{pmatrix} \breve{\tau}_{2,1} & 0 & \cdots & 0 \\ \breve{\tau}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\tau}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\hat{\boldsymbol{\varepsilon}}_0 = \begin{pmatrix} \hat{\varepsilon}_{2,1} & 0 & \cdots & 0 \\ \hat{\varepsilon}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\varepsilon}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{\boldsymbol{\zeta}}_1 = \begin{pmatrix} \hat{\zeta}_{2,2} & \hat{\zeta}_{2,3} & \cdots & \hat{\zeta}_{2,\Phi} \\ \hat{\zeta}_{3,2} & \hat{\zeta}_{3,3} & \cdots & \hat{\zeta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\zeta}_{\Phi,2} & \hat{\zeta}_{\Phi,3} & \cdots & \hat{\zeta}_{\Phi,\Phi} \end{pmatrix}, \\
&\hat{\boldsymbol{\eta}}_0 = \begin{pmatrix} \hat{\eta}_{2,1} & 0 & \cdots & 0 \\ \hat{\eta}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\eta}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{\boldsymbol{\omega}}_0 = \begin{pmatrix} \hat{\omega}_{2,1} & 0 & \cdots & 0 \\ \hat{\omega}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\omega}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\hat{\boldsymbol{\nu}}_1 = \begin{pmatrix} \hat{\nu}_{2,2} & \hat{\nu}_{2,3} & \cdots & \hat{\nu}_{2,\Phi} \\ \hat{\nu}_{3,2} & \hat{\nu}_{3,3} & \cdots & \hat{\nu}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\nu}_{\Phi,2} & \hat{\nu}_{\Phi,3} & \cdots & \hat{\nu}_{\Phi,\Phi} \end{pmatrix}, \quad \hat{\boldsymbol{d}}_0 = \begin{pmatrix} \hat{d}_{2,1} & 0 & \cdots & 0 \\ \hat{d}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{d}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\hat{\boldsymbol{\vartheta}}_1 = \begin{pmatrix} \hat{\vartheta}_{2,2} & \hat{\vartheta}_{2,3} & \cdots & \hat{\vartheta}_{2,\Phi} \\ \hat{\vartheta}_{3,2} & \hat{\vartheta}_{3,3} & \cdots & \hat{\vartheta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\vartheta}_{\Phi,2} & \hat{\vartheta}_{,3} & \cdots & \hat{\vartheta}_{\Phi,\Phi} \end{pmatrix}, \quad \hat{\boldsymbol{p}}_0 = \begin{pmatrix} \hat{p}_{2,1} & 0 & \cdots & 0 \\ \hat{p}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{p}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\hat{\boldsymbol{\sigma}}_1 = \begin{pmatrix} \hat{\sigma}_{2,2} & \hat{\sigma}_{2,3} & \cdots & \hat{\sigma}_{2,\Phi} \\ \hat{\sigma}_{3,2} & \hat{\sigma}_{3,3} & \cdots & \hat{\sigma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\sigma}_{\Phi,2} & \hat{\sigma}_{\Phi,3} & \cdots & \hat{\sigma}_{\Phi,\Phi} \end{pmatrix}, \quad \hat{\boldsymbol{\upsilon}}_0 = \begin{pmatrix} \hat{\upsilon}_{2,1} & 0 & \cdots & 0 \\ \hat{\upsilon}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\upsilon}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\breve{\boldsymbol{\iota}}_0 = \begin{pmatrix} \breve{\iota}_{2,1} & 0 & \cdots & 0 \\ \breve{\iota}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \breve{\iota}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{\boldsymbol{\mu}}_0 = \begin{pmatrix} \hat{\mu}_{2,1} & 0 & \cdots & 0 \\ \hat{\mu}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\mu}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix}, \\
&\hat{\boldsymbol{\alpha}}_1 = \begin{pmatrix} \hat{\alpha}_{2,2} & \hat{\alpha}_{2,3} & \cdots & \hat{\alpha}_{2,\Phi} \\ \hat{\alpha}_{3,2} & \hat{\alpha}_{3,3} & \cdots & \hat{\alpha}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\alpha}_{\Phi,2} & \hat{\alpha}_{\Phi,3} & \cdots & \hat{\alpha}_{\Phi,\Phi} \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_0 = \begin{pmatrix} \hat{\beta}_{2,1} & 0 & \cdots & 0 \\ \hat{\beta}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\beta}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix},
\end{aligned} \tag{2.22}
$$

$$\hat{\boldsymbol{\gamma}}_1 = \begin{pmatrix} \hat{\gamma}_{2,2} & \hat{\gamma}_{2,3} & \cdots & \hat{\gamma}_{2,\Phi} \\ \hat{\gamma}_{3,2} & \hat{\gamma}_{3,3} & \cdots & \hat{\gamma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\gamma}_{\Phi,2} & \hat{\gamma}_{\Phi,3} & \cdots & \hat{\gamma}_{\Phi,\Phi} \end{pmatrix}, \quad \hat{\boldsymbol{\tau}}_0 = \begin{pmatrix} \hat{\tau}_{2,1} & 0 & \cdots & 0 \\ \hat{\tau}_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\tau}_{\Phi,1} & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\begin{aligned} Z_{n+\Phi} &= \left(z_{n+p_2}, z_{n+p_3}, z_{n+p_4}, z_{n+2}\right)^T, & Z_n &= \left(z_n, z_{n-p_2}, z_{n-p_3}, z_{n-p_4}\right)^T, \\ Z'_{n+\Phi} &= \left(z'_{n+p_2}, z'_{n+p_3}, z'_{n+p_4}, z'_{n+2}\right)^T, & Z'_n &= \left(z'_n, z'_{n-p_2}, z'_{n-p_3}, z'_{n-p_4}\right)^T, \\ Z''_{n+\Phi} &= \left(z''_{n+p_2}, z''_{n+p_3}, z''_{n+p_4}, z''_{n+2}\right)^T, & Z''_n &= \left(z''_n, z''_{n-p_2}, z''_{n-p_3}, z''_{n-p_4}\right)^T, \\ Z'''_{n+\Phi} &= \left(z'''_{n+p_2}, z'''_{n+p_3}, z'''_{n+p_4}, z'''_{n+2}\right)^T, & Z'''_n &= \left(z'''_n, z'''_{n-p_2}, z'''_{n-p_3}, z'''_{n-p_4}\right)^T, \\ F_{n+\Phi} &= \left(f_{n+p_2}, f_{n+p_3}, f_{n+p_4}, f_{n+2}\right)^T, & F_n &= \left(f_n, f_{n-p_2}, f_{n-p_3}, f_{n-p_4}\right)^T, \\ G_{n+\Phi} &= \left(g_{n+p_2}, g_{n+p_3}, g_{n+p_4}, g_{n+2}\right)^T, & G_n &= \left(g_n, g_{n-p_2}, g_{n-p_3}, g_{n-p_4}\right)^T. \end{aligned} \tag{2.23}$$

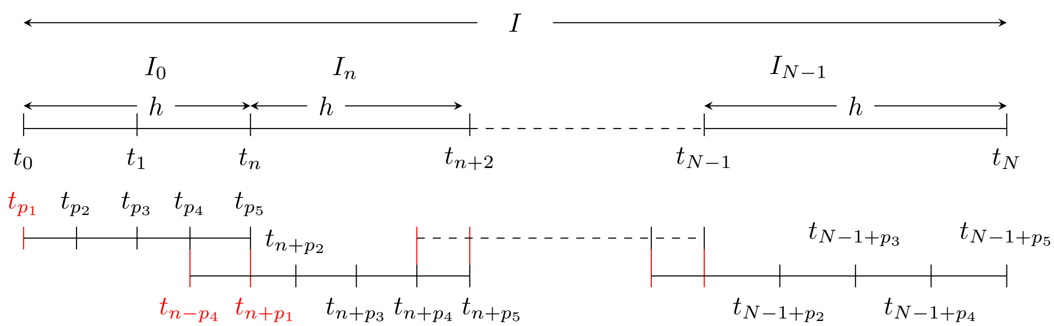

FIGURE 2. Overlapping grid [14, 15].

2.2. **Overlapping Block Method.** For the overlapping method, we choose $2\Phi + w + 1$ collocation points

$$t_{n+p_i} = t_n + hp_i, \qquad t_{n-p_4} = t_n - hp_4, \quad i = 1, 2, \ldots \Phi,$$

over a closed interval $[t_n, t_{n+2}]$, where $n > 1$. Unlike the non-overlapping method, which uses $2\Phi + w - 1$ collocation points, the number of collocation points for the overlapping method increases to $2\Phi + w + 1$ due to the inclusion of two overlap conditions. We consider the collocation parameters

$$-p_4 < p_1 = 0 < p_2 < \ldots < p_{\Phi-1} < p_\Phi.$$

As shown in Figure 2, the representation of the overlapping grid I is illustrated for the case N. Here, the grid I_n overlaps with a grid point $t_{n-2+p_{\Phi-1}} = t_{n-p_4}$ from the previous grid I_{n-2}. Using the power series polynomial of the form

$$Z(t) = \sum_{j=0}^{2\Phi+w+1} \tilde{\Lambda}_{n,j} \left(t - t_n\right)^j, \tag{2.24}$$

combining (2.3), the initial conditions (2.4)–(2.6) and the overlap conditions

$$\begin{aligned} z^w(t_{n-p_4}) &= f_{n-p_4}, \\ z^{w+1}(t_{n-p_4}) &= g_{n-p_4}, \qquad w = 3,\, 4. \end{aligned} \tag{2.25}$$

$\Lambda_{n,j}$ is determined and then substituted into (2.2) to obtain a continuous approximation of the overlapping fourth derivative method for third-order IVPs

$$\begin{aligned} z_{n+p_i} &= z_n + hp_{i\,1}z'_n + h^2\mu_{i\,1}z''_n + h^3\left(\sum_{j=1,\ 4}\tilde{\beta}_{i,j}f_{n-p_j}\right) + h^3\left(\sum_{j=2}^{\Phi}\tilde{\alpha}_{i\,j}f_{n+p_j}\right) \\ &\quad + h^4\left(\sum_{j=1,\ 4}\tilde{\tau}_{i,j}g_{n-p_j}\right) + h^4\left(\sum_{j=2}^{\Phi}\tilde{\gamma}_{i\,j}g_{n+p_j}\right), \end{aligned} \tag{2.26}$$

$$\begin{aligned} z'_{n+p_i} &= z'_n + h\varepsilon_{i\,1}z''_n + h^2\left(\sum_{j=1,\ 4}\tilde{\eta}_{i\,j}f_{n-p_j}\right) + h^2\left(\sum_{j=2}^{\Phi}\tilde{\zeta}_{i\,j}f_{n+p_j}\right) \\ &\quad + h^3\left(\sum_{j=1,\ 4}\tilde{\omega}_{i\,j}g_{n-p_j}\right) + h^3\left(\sum_{j=2}^{\Phi}\tilde{\nu}_{i\,j}g_{n+p_j}\right), \end{aligned} \tag{2.27}$$

$$z''_{n+p_i} = z''_n + h\left(\sum_{j=1,\ 4}\tilde{d}_{i\,j}f_{n-p_j}\right) + h\left(\sum_{j=2}^{\Phi}\tilde{\vartheta}_{i\,j}f_{n+p_j}\right) + h^2\left(\sum_{j=1,\ 4}\tilde{\upsilon}_{i\,j}g_{n-p_j}\right) + h^2\left(\sum_{j=2}^{\Phi}\tilde{\sigma}_{i\,j}g_{n+p_j}\right), \tag{2.28}$$

where $i = 2, 3, ..., \Phi$. The overlapping fifth derivative method obtained for fourth-order IVPs takes the form

$$\begin{aligned} z_{n+p_i} &= z_n + h\breve{p}_{i\,1}z'_n + h^2\breve{\mu}_{i\,1}z''_n + h^3\breve{\iota}_{i\,1}z'''_n + h^4\left(\sum_{j=1,\ 4}\bar{\beta}_{i\,j}f_{n-p_j}\right) + h^4\left(\sum_{j=2}^{\Phi}\bar{\alpha}_{i\,j}f_{n+p_j}\right) \\ &\quad + h^5\left(\sum_{j=2,\ 4}\bar{\tau}_{i\,j}g_{n-p_j}\right) + h^5\left(\sum_{j=1}^{\Phi}\bar{\gamma}_{i\,j}g_{n+p_j}\right), \end{aligned} \tag{2.29}$$

$$\begin{aligned} z'_{n+p_i} &= z'_n + h\hat{p}_{i\,1}z''_n + h^2\hat{\mu}_{i\,1}z'''_n + h^3\left(\sum_{j=1,4}\bar{\bar{\beta}}_{i\,j}f_{n-p_j}\right) + h^3\left(\sum_{j=2}^{\Phi}\bar{\bar{\alpha}}_{i\,j}f_{n+p_j}\right) \\ &\quad + h^4\left(\sum_{j=1,\ 4}\bar{\bar{\tau}}_{i\,j}g_{n-p_j}\right) + h^4\left(\sum_{j=2}^{\Phi}\bar{\bar{\gamma}}_{i\,j}g_{n+p_j}\right), \end{aligned} \tag{2.30}$$

$$z''_{n+p_i} = z''_n + h\hat{\varepsilon}_{i\,1}z'''_n + h^2\left(\sum_{j=1,\ 4}\bar{\eta}_{i\,j}f_{n-p_j}\right) + h^2\left(\sum_{j=2}^{\Phi}\bar{\zeta}_{i\,j}f_{n+p_j}\right) + h^3\left(\sum_{j=1,\ 4}\bar{\omega}_{i\,j}g_{n-p_j}\right) + h^3\left(\sum_{j=2}^{\Phi}\bar{\nu}_{i\,j}g_{n+p_j}\right), \tag{2.31}$$

$$z'''_{n+p_i} = z'''_n + h\left(\sum_{j=1,\ 4}\bar{d}_{i\,j}f_{n-p_j}\right) + h\left(\sum_{j=2}^{\Phi}\bar{\vartheta}_{i\,j}f_{n+p_j}\right) + h^2\left(\sum_{j=1,\ 4}\bar{\upsilon}_{i\,j}g_{n-p_j}\right) + h^2\left(\sum_{j=2}^{\Phi}\bar{\sigma}_{i\,j}g_{n+p_j}\right), \tag{2.32}$$

where $i = 2, 3, ..., \Phi$. The matrix representation of Equations (2.26)–(2.28) are

$$\boldsymbol{A}_1 Z_{n+\Phi} = \boldsymbol{A}_0 Z_n + h\,\boldsymbol{p}_0\,Z'_n + h^2\,\boldsymbol{\mu}_0\,Z''_n + h^3\,\tilde{\boldsymbol{\alpha}}_1\,F_{n+\Phi} + h^3\,\tilde{\boldsymbol{\beta}}_0\,F_n + h^4\,\tilde{\boldsymbol{\gamma}}_1\,G_{n+\Phi} + h^4\,\tilde{\boldsymbol{\tau}}_0\,G_n, \tag{2.33}$$

$$\boldsymbol{A}_1 Z'_{n+\Phi} = \boldsymbol{A}_0 Z'_n + h\,\boldsymbol{\varepsilon}_0\,Z''_n + h^2\,\tilde{\boldsymbol{\zeta}}_1\,F_{n+\Phi} + h^2\,\tilde{\boldsymbol{\eta}}_0\,F_n + h^3\,\tilde{\boldsymbol{\nu}}_1\,G_{n+\Phi} + h^3\,\tilde{\boldsymbol{\omega}}_0\,G_n, \tag{2.34}$$

$$\boldsymbol{A}_1 Z''_{n+\Phi} = \boldsymbol{A}_0\,Z''_n + h\,\tilde{\boldsymbol{d}}_0\,F_n + h\,\tilde{\boldsymbol{\vartheta}}_1\,F_{n+\Phi} + h^2\,\tilde{\boldsymbol{\sigma}}_1\,G_{n+\Phi} + h^2\,\tilde{\boldsymbol{\upsilon}}_0\,G_n. \tag{2.35}$$

The coefficients obtained above are

$$\begin{aligned}
\tilde{\boldsymbol{\alpha}}_1 &= \begin{pmatrix} \tilde{\alpha}_{2,2} & \tilde{\alpha}_{2,3} & \cdots & \tilde{\alpha}_{2,\Phi} \\ \tilde{\alpha}_{3,2} & \tilde{\alpha}_{3,3} & \cdots & \tilde{\alpha}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\alpha}_{\Phi,2} & \tilde{\alpha}_{\Phi,3} & \cdots & \tilde{\alpha}_{\Phi,\Phi} \end{pmatrix}, \quad \tilde{\boldsymbol{\beta}}_0 = \begin{pmatrix} \tilde{\beta}_{2,1} & 0 & \cdots & \tilde{\beta}_{2,-4} \\ \tilde{\beta}_{3,1} & 0 & \cdots & \tilde{\beta}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\beta}_{\Phi,1} & 0 & \cdots & \tilde{\beta}_{\Phi,-4} \end{pmatrix}, \\
\tilde{\boldsymbol{\gamma}}_1 &= \begin{pmatrix} \tilde{\gamma}_{2,2} & \tilde{\gamma}_{2,3} & \cdots & \tilde{\gamma}_{2,\Phi} \\ \tilde{\gamma}_{3,2} & \tilde{\gamma}_{3,3} & \cdots & \tilde{\gamma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\gamma}_{\Phi,2} & \tilde{\gamma}_{\Phi,3} & \cdots & \tilde{\gamma}_{\Phi,\Phi} \end{pmatrix}, \quad \tilde{\boldsymbol{\tau}}_0 = \begin{pmatrix} \tilde{\tau}_{2,1} & 0 & \cdots & \tilde{\tau}_{2,-4} \\ \tilde{\tau}_{3,1} & 0 & \cdots & \tilde{\tau}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\tau}_{\Phi,1} & 0 & \cdots & \tilde{\tau}_{\Phi,-4} \end{pmatrix}, \\
\tilde{\boldsymbol{\zeta}}_1 &= \begin{pmatrix} \tilde{\zeta}_{2,2} & \tilde{\zeta}_{2,3} & \cdots & \tilde{\zeta}_{2,\Phi} \\ \tilde{\zeta}_{3,2} & \tilde{\zeta}_{3,3} & \cdots & \tilde{\zeta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\zeta}_{\Phi,2} & \tilde{\zeta}_{\Phi,3} & \cdots & \tilde{\zeta}_{\Phi,\Phi} \end{pmatrix}, \quad \tilde{\boldsymbol{\eta}}_0 = \begin{pmatrix} \tilde{\eta}_{2,1} & 0 & \cdots & \tilde{\eta}_{2,-4} \\ \tilde{\eta}_{3,1} & 0 & \cdots & \tilde{\eta}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\eta}_{\Phi,1} & 0 & \cdots & \tilde{\eta}_{\Phi,-4} \end{pmatrix}, \\
\tilde{\boldsymbol{\omega}}_0 &= \begin{pmatrix} \tilde{\omega}_{2,1} & 0 & \cdots & \tilde{\omega}_{2,-4} \\ \tilde{\omega}_{3,1} & 0 & \cdots & \tilde{\omega}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\omega}_{\Phi,1} & 0 & \cdots & \tilde{\omega}_{\Phi,-4} \end{pmatrix}, \quad \tilde{\boldsymbol{\nu}}_1 = \begin{pmatrix} \tilde{\nu}_{2,2} & \tilde{\nu}_{2,3} & \cdots & \tilde{\nu}_{2,\Phi} \\ \tilde{\nu}_{3,2} & \tilde{\nu}_{3,3} & \cdots & \tilde{\nu}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\nu}_{\Phi,2} & \tilde{\nu}_{\Phi,3} & \cdots & \tilde{\nu}_{\Phi,\Phi} \end{pmatrix}, \\
\tilde{\boldsymbol{d}}_0 &= \begin{pmatrix} \tilde{d}_{2,1} & 0 & \cdots & \tilde{d}_{2,-4} \\ \tilde{d}_{3,1} & 0 & \cdots & \tilde{d}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{d}_{\Phi,1} & 0 & \cdots & \tilde{d}_{\Phi,-4} \end{pmatrix}, \quad \tilde{\boldsymbol{\vartheta}}_1 = \begin{pmatrix} \tilde{\vartheta}_{2,2} & \tilde{\vartheta}_{2,3} & \cdots & \tilde{\vartheta}_{2,\Phi} \\ \tilde{\vartheta}_{3,2} & \tilde{\vartheta}_{3,3} & \cdots & \tilde{\vartheta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\vartheta}_{\Phi,2} & \tilde{\vartheta}_{\Phi,3} & \cdots & \tilde{\vartheta}_{\Phi,\Phi} \end{pmatrix}, \\
\tilde{\boldsymbol{\sigma}}_1 &= \begin{pmatrix} \tilde{\sigma}_{2,2} & \tilde{\sigma}_{2,3} & \cdots & \tilde{\sigma}_{2,\Phi} \\ \tilde{\sigma}_{3,2} & \tilde{\sigma}_{3,3} & \cdots & \tilde{\sigma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\sigma}_{\Phi,2} & \tilde{\sigma}_{\Phi,3} & \cdots & \tilde{\sigma}_{\Phi,\Phi} \end{pmatrix}, \quad \tilde{\boldsymbol{\upsilon}}_0 = \begin{pmatrix} \tilde{\upsilon}_{2,1} & 0 & \cdots & \tilde{\upsilon}_{2,-4} \\ \tilde{\upsilon}_{3,1} & 0 & \cdots & \tilde{\upsilon}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\upsilon}_{\Phi,1} & 0 & \cdots & \tilde{\upsilon}_{\Phi,-4} \end{pmatrix}.
\end{aligned} \tag{2.36}$$

Equations (2.29) – (2.32) can be represented in matrix form as

$$\boldsymbol{A}_1 Z_{n+\Phi} = \boldsymbol{A}_0 Z_n + h\,\breve{\boldsymbol{p}}_0\, Z'_n + h^2\,\breve{\boldsymbol{\mu}}_0\, Z''_n + h^3\,\breve{\boldsymbol{\iota}}_0\, Z'''_n + h^4\,\bar{\boldsymbol{\alpha}}_1\, F_{n+\Phi} + h^4\,\bar{\boldsymbol{\beta}}_0\, F_n + h^5\,\bar{\boldsymbol{\gamma}}_1\, G_{n+\Phi} + h^5\,\bar{\boldsymbol{\tau}}_0\, G_n, \tag{2.37}$$

$$\boldsymbol{A}_1 Z'_{n+\Phi} = \boldsymbol{A}_0 Z'_n + h\,\hat{\boldsymbol{p}}_0\, Z''_n + h^2\,\hat{\boldsymbol{\mu}}_0\, Z''_n + h^3\,\bar{\bar{\boldsymbol{\alpha}}}_1\, F_{n+\Phi} + h^3\,\bar{\bar{\boldsymbol{\beta}}}_0\, F_n + h^4\,\bar{\bar{\boldsymbol{\gamma}}}_1\, G_{n+\Phi} + h^4\,\bar{\bar{\boldsymbol{\tau}}}_0\, G_n, \tag{2.38}$$

$$\boldsymbol{A}_1 Z''_{n+\Phi} = \boldsymbol{A}_0 Z''_n + h\,\hat{\boldsymbol{\varepsilon}}_0\, Z'''_n + h^2\,\bar{\boldsymbol{\zeta}}_1\, F_{n+\Phi} + h^2\,\bar{\boldsymbol{\eta}}_0\, F_n + h^3\,\bar{\boldsymbol{\nu}}_1\, G_{n+\Phi} + h^3\,\bar{\boldsymbol{\omega}}_0\, G_n, \tag{2.39}$$

$$\boldsymbol{A}_1 Z'''_{n+\Phi} = \boldsymbol{A}_0\, Z'''_n + h\,\bar{\boldsymbol{d}}_0\, F_n + h\,\bar{\boldsymbol{\vartheta}}_1\, F_{n+\Phi} + h^2\,\bar{\boldsymbol{\sigma}}_1\, G_{n+\Phi} + h^2\,\bar{\boldsymbol{\upsilon}}_0\, G_n, \tag{2.40}$$

where

$$\bar{\boldsymbol{\alpha}}_1 = \begin{pmatrix} \bar{\alpha}_{2,2} & \bar{\alpha}_{2,3} & \cdots & \bar{\alpha}_{2,\Phi} \\ \bar{\alpha}_{3,2} & \bar{\alpha}_{3,3} & \cdots & \bar{\alpha}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\alpha}_{\Phi,2} & \bar{\alpha}_{\Phi,3} & \cdots & \bar{\alpha}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\boldsymbol{\beta}}_0 = \begin{pmatrix} \bar{\beta}_{2,1} & 0 & \cdots & \bar{\beta}_{2,-4} \\ \bar{\beta}_{3,1} & 0 & \cdots & \bar{\beta}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\beta}_{\Phi,1} & 0 & \cdots & \bar{\beta}_{\Phi,-4} \end{pmatrix},$$

$$
\begin{aligned}
\bar{\boldsymbol{\gamma}}_1 &= \begin{pmatrix} \bar{\gamma}_{2,2} & \bar{\gamma}_{2,3} & \cdots & \bar{\gamma}_{2,\Phi} \\ \bar{\gamma}_{3,2} & \bar{\gamma}_{3,3} & \cdots & \bar{\gamma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\gamma}_{\Phi,2} & \bar{\gamma}_{\Phi,3} & \cdots & \bar{\gamma}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\boldsymbol{\tau}}_0 = \begin{pmatrix} \bar{\tau}_{2,1} & 0 & \cdots & \bar{\tau}_{2,-4} \\ \bar{\tau}_{3,1} & 0 & \cdots & \bar{\tau}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\tau}_{\Phi,1} & 0 & \cdots & \bar{\tau}_{\Phi,-4} \end{pmatrix}, \\
\bar{\boldsymbol{\zeta}}_1 &= \begin{pmatrix} \bar{\zeta}_{2,2} & \bar{\zeta}_{2,3} & \cdots & \bar{\zeta}_{2,\Phi} \\ \bar{\zeta}_{3,2} & \bar{\zeta}_{3,3} & \cdots & \bar{\zeta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\zeta}_{\Phi,2} & \bar{\zeta}_{\Phi,3} & \cdots & \bar{\zeta}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\boldsymbol{\eta}}_0 = \begin{pmatrix} \bar{\eta}_{2,1} & 0 & \cdots & \bar{\eta}_{2,-4} \\ \bar{\eta}_{3,1} & 0 & \cdots & \bar{\eta}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\eta}_{\Phi,1} & 0 & \cdots & \bar{\eta}_{\Phi,-4} \end{pmatrix}, \\
\bar{\boldsymbol{\omega}}_0 &= \begin{pmatrix} \bar{\omega}_{2,1} & 0 & \cdots & \bar{\omega}_{2,-4} \\ \bar{\omega}_{3,1} & 0 & \cdots & \bar{\omega}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\omega}_{\Phi,1} & 0 & \cdots & \bar{\omega}_{\Phi,-4} \end{pmatrix}, \quad \bar{\boldsymbol{\nu}}_1 = \begin{pmatrix} \bar{\nu}_{2,2} & \bar{\nu}_{2,3} & \cdots & \bar{\nu}_{2,\Phi} \\ \bar{\nu}_{3,2} & \bar{\nu}_{3,3} & \cdots & \bar{\nu}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\nu}_{\Phi,2} & \bar{\nu}_{\Phi,3} & \cdots & \bar{\nu}_{\Phi,\Phi} \end{pmatrix}, \\
\bar{\boldsymbol{d}}_0 &= \begin{pmatrix} \bar{d}_{2,1} & 0 & \cdots & \bar{d}_{2,-4} \\ \bar{d}_{3,1} & 0 & \cdots & \bar{d}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{d}_{\Phi,1} & 0 & \cdots & \bar{d}_{\Phi,-4} \end{pmatrix}, \quad \bar{\boldsymbol{\vartheta}}_1 = \begin{pmatrix} \bar{\vartheta}_{2,2} & \bar{\vartheta}_{2,3} & \cdots & \bar{\vartheta}_{2,\Phi} \\ \bar{\vartheta}_{3,2} & \bar{\vartheta}_{3,3} & \cdots & \bar{\vartheta}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\vartheta}_{\Phi,2} & \bar{\vartheta}_{,3} & \cdots & \bar{\vartheta}_{\Phi,\Phi} \end{pmatrix}, \\
\bar{\boldsymbol{\sigma}}_1 &= \begin{pmatrix} \bar{\sigma}_{2,2} & \bar{\sigma}_{2,3} & \cdots & \bar{\sigma}_{2,\Phi} \\ \bar{\sigma}_{3,2} & \bar{\sigma}_{3,3} & \cdots & \bar{\sigma}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\sigma}_{\Phi,2} & \bar{\sigma}_{\Phi,3} & \cdots & \bar{\sigma}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\boldsymbol{\upsilon}}_0 = \begin{pmatrix} \bar{\upsilon}_{2,1} & 0 & \cdots & \bar{\upsilon}_{2,-4} \\ \bar{\upsilon}_{3,1} & 0 & \cdots & \bar{\upsilon}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\upsilon}_{\Phi,1} & 0 & \cdots & \bar{\upsilon}_{\Phi,-4} \end{pmatrix}, \\
\bar{\bar{\boldsymbol{\alpha}}}_1 &= \begin{pmatrix} \bar{\bar{\alpha}}_{2,2} & \bar{\bar{\alpha}}_{2,3} & \cdots & \bar{\bar{\alpha}}_{2,\Phi} \\ \bar{\bar{\alpha}}_{3,2} & \bar{\bar{\alpha}}_{3,3} & \cdots & \bar{\bar{\alpha}}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\bar{\alpha}}_{\Phi,2} & \bar{\bar{\alpha}}_{\Phi,3} & \cdots & \bar{\bar{\alpha}}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\bar{\boldsymbol{\beta}}}_0 = \begin{pmatrix} \bar{\bar{\beta}}_{2,1} & 0 & \cdots & \bar{\bar{\beta}}_{2,-4} \\ \bar{\bar{\beta}}_{3,1} & 0 & \cdots & \bar{\bar{\beta}}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\bar{\beta}}_{\Phi,1} & 0 & \cdots & \bar{\bar{\beta}}_{\Phi,-4} \end{pmatrix}, \\
\bar{\bar{\boldsymbol{\gamma}}}_1 &= \begin{pmatrix} \bar{\bar{\gamma}}_{2,2} & \bar{\bar{\gamma}}_{2,3} & \cdots & \bar{\bar{\gamma}}_{2,\Phi} \\ \bar{\bar{\gamma}}_{3,2} & \bar{\bar{\gamma}}_{3,3} & \cdots & \bar{\bar{\gamma}}_{3,\Phi} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\bar{\gamma}}_{\Phi,2} & \bar{\bar{\gamma}}_{\Phi,3} & \cdots & \bar{\bar{\gamma}}_{\Phi,\Phi} \end{pmatrix}, \quad \bar{\bar{\boldsymbol{\tau}}}_0 = \begin{pmatrix} \bar{\bar{\tau}}_{2,1} & 0 & \cdots & \bar{\bar{\tau}}_{2,-4} \\ \bar{\bar{\tau}}_{3,1} & 0 & \cdots & \bar{\bar{\tau}}_{3,-4} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\bar{\tau}}_{\Phi,1} & 0 & \cdots & \bar{\bar{\tau}}_{\Phi,-4} \end{pmatrix}.
\end{aligned} \tag{2.41}
$$

3. Analysis of the Method

In this section, we discuss the local truncation error (LTE), order, zero stability, consistency, convergence and absolute stability of the method.

3.1. Order of Accuracy.

Theorem 3.1. *The LTE for the non-overlapping methods in the closed interval* $[t_n, t_{n+2}]$ *are expressed as follows: For the first case (2.7)*

$$
\begin{aligned}
\mathfrak{L}_i[z(t_n);h] = \frac{h^{m+10}}{(m+10)!}&\left[p_i^{m+10} - (m+10)(m+9)(m+8)\sum_{j=1}^{\Phi}\alpha_{i\,j}p_i^{m+7}\right]z^{(m+10)}(t_n) \\
-\frac{h^{m+11}}{(m+11)!}&\left[(m+11)(m+10)(m+9)(m+8)\sum_{j=1}^{\Phi}\gamma_{i\,j}p_j^{m+7}\right]z^{(m+11)}(t_n) + O(h^{m+12}).
\end{aligned} \tag{3.1}
$$

For the second case (2.10)

$$\mathfrak{L}_i[z(t_n);h] = \frac{h^{m+11}}{(m+11)!}\left[p_i^{m+11} - (m+11)(m+10)(m+9)(m+8)\sum_{j=1}^{\Phi}\breve{\alpha}_{i\,j}p_i^{m+7}\right]z^{(m+11)}(t_n) \tag{3.2}$$
$$-\frac{h^{m+12}}{(m+12)!}\left[(m+12)(m+11)(m+10)(m+9)(m+8)\sum_{j=1}^{\Phi}\breve{\gamma}_{i\,j}p_j^{m+7}\right]z^{(m+12)}(t_n) + O(h^{m+13}).$$

Proof. Considering $z(t)$ as a sufficiently differentiable function, the local truncation error for the non-overlapping methods (2.7) and (2.10) can be examined using the linear operator $\hat{\mathfrak{L}}_i$ as follows

$$\hat{\mathfrak{L}}_{\text{i}}[z(t_n);h] = z(t_n+hp_i) - z(t_n) - hp_i z'(t_n) - h^2\mu_i z''(t_n) - h^3\sum_{j=1}^{\Phi}[\alpha_{i\,j}z'''(t_n+hp_j)]$$
$$-h^4\sum_{j=1}^{\Phi}\left[\gamma_{i\,j}z^{iv}(t_n+hp_j)\right], \tag{3.3}$$

and

$$\hat{\mathfrak{L}}_{\text{i}}[z(t_n);h] = z(t_n+hp_i) - z(t_n) - h\breve{p}_i z'(t_n) - h^2\breve{\mu}_i z''(t_n) - h^3\breve{\iota}_i z'''(t_n)$$
$$-h^4\sum_{j=1}^{\Phi}\left[\breve{\alpha}_{i\,j}z^{iv}(t_n+hp_j)\right] - h^5\sum_{j=1}^{\Phi}[\breve{\gamma}_{i\,j}z^{v}(t_n+hp_j)], \quad i = 2,3,\ldots\Phi. \tag{3.4}$$

Using Taylor's series to expand (3.3) and (3.4) about t_n, to obtain

$$\hat{\mathfrak{L}}[z(t_n);h] = U_0\,z(t_n) + U_1\,h\,z'(t_n) + U_2\,h^2\,z''(t_n) + U_3\,h^3\,z'''(t_n) + \ldots + U_{\hat{m}}\,h^{\hat{m}}\,z^{\hat{m}}(t_n), \tag{3.5}$$

where U_0, U_1, U_2, ..., $U_{\hat{m}}$ are constants. The block hybrid methods have an order $\hat{m}$ if

$$U_0 = U_1 = \ldots = U_{\hat{m}+(w-1)} = 0, \qquad U_{\hat{m}+w} \neq 0, \qquad w = 3,\,4.$$

For the first case (2.7):
According to [1, 15], expanding (3.3) and using the Taylor series, we obtain

$$\hat{\mathfrak{L}}_i[z(t_n);h] = \sum_{k=1}^{M}\frac{h^{k+2}p_i^{k+2}}{(k+2)!}z^{(k+2)}(t_n) - \sum_{k=1}^{M}\frac{kh^{k+2}}{k!}\sum_{j=1}^{\Phi}\alpha_{i\,j}p_j^{k-1}z^{(k+2)}(t_n)$$
$$-\sum_{k=1}^{M}\frac{kh^{k+3}}{k!}\sum_{j=1}^{\Phi}\gamma_{i\,j}p_j^{k-1}z^{(k+3)}(t_n) + O(h^{M+1}). \tag{3.6}$$

Expanding (3.6), we obtain

$$\hat{\mathfrak{L}}_i[z(t_n);h] = \sum_{k=1}^{m+2}\frac{h^{k+2}p_i^{k+2}}{(k+2)!}z^{(k+2)}(t_n) - \sum_{k=1}^{m+2}\frac{kh^{k+2}}{k!}\sum_{j=1}^{\Phi}\alpha_{i\,j}p_j^{k-1}z^{(k+2)}(t_n)$$
$$-\sum_{k=1}^{m+2}\frac{kh^{k+3}}{k!}\sum_{j=1}^{\Phi}\gamma_{i\,j}p_j^{k-1}z^{(k+3)}(t_n) + \sum_{k=m+8}^{M}\frac{h^{k+2}p_i^{k+2}}{(k+2)!}z^{(k+2)}(t_n) \tag{3.7}$$
$$-\sum_{k=m+8}^{M}\frac{kh^{k+2}}{k!}\sum_{j=1}^{\Phi}\alpha_{i\,j}p_j^{k-1}z^{(k+2)}(t_n)$$
$$-\sum_{k=m+8}^{M}\frac{kh^{k+3}}{k!}\sum_{j=1}^{\Phi}\gamma_{i\,j}p_j^{k-1}z^{(k+3)}(t_n) + O(h^{M+1}),$$

where $m = 3$ and $M \geq m + 8$ is a positive integer. This can be expressed as

$$\begin{aligned}
\hat{\mathfrak{L}}_i[z(t_n); h] = &\sum_{k=3}^{m+2} \frac{h^k}{k!} \left[p_i^k - k(k-1)(k-2) \sum_{j=1}^{\Phi} \alpha_{i\,j} p_j^{k-3} \right] z^{(k)}(t_n) \\
&- \sum_{k=4}^{m+2} \frac{h^k}{k!} \left[k(k-1)(k-2)(k-3) \sum_{j=1}^{\Phi} \gamma_{i\,j} p_j^{k-4} \right] z^{(k)}(t_n) \\
&+ \sum_{k=m+10}^{M} \frac{h^k}{k!} \left[p_i^k - k(k-1)(k-2) \sum_{j=1}^{\Phi} \alpha_{i\,j} p_j^{k-3} \right] z^{(k)}(t_n) \\
&- \sum_{k=m+11}^{M} \frac{h^k}{k!} \left[k(k-1)(k-2)(k-3) \sum_{j=1}^{\Phi} \gamma_{i\,j} p_j^{k-4} \right] z^{(k)}(t_n) + O(h^{M+1}).
\end{aligned} \tag{3.8}$$

According to [1, 15], it is noted that

$$\sum_{j=1}^{\Phi} \alpha_{i\,j} p_j^{k-3} + (k-3) \sum_{j=1}^{\Phi} \gamma_{i\,j} p_j^{k-4} = \frac{p_i^k}{k(k-1)(k-2)}, \quad \text{for } k = 3, \ldots, m+2, \tag{3.9}$$

substituting (3.9) into (3.8), we obtain

$$\begin{aligned}
\hat{\mathfrak{L}}_i[z(t_n); h] = &\frac{h^{m+10}}{(m+10)!} \left[p_i^{m+10} - (m+10)(m+9)(m+8) \sum_{j=1}^{\Phi} \alpha_{i\,j} p_i^{m+7} \right] z^{(m+10)}(t_n) \\
&- \frac{h^{m+11}}{(m+11)!} \left[(m+11)(m+10)(m+9)(m+8) \sum_{j=1}^{\Phi} \gamma_{i\,j} p_j^{m+7} \right] z^{(m+11)}(t_n) + O(h^{m+12}).
\end{aligned} \tag{3.10}$$

Using (3.10), the error constant for (2.7) is

$$U_{i,m+10} = \frac{p_i^{m+10}}{(m+10)!} - \frac{1}{(m+7)!} \left(\sum_{j=1}^{\Phi} \alpha_{i\,j} p_i^{m+7} - \sum_{j=1}^{\Phi} \gamma_{i\,j} p_i^{m+7} \right). \tag{3.11}$$

For the second case (2.10):
Expanding (3.4) and using the Taylor series we obtain

$$\begin{aligned}
\hat{\mathfrak{L}}_i[z(t_n); h] = &\sum_{k=1}^{M} \frac{h^{k+3} p_i^{k+3}}{(k+3)!} z^{(k+3)}(t_n) - \sum_{k=1}^{M} \frac{k h^{k+3}}{k!} \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-1} z^{(k+3)}(t_n) \\
&- \sum_{k=1}^{M} \frac{k h^{k+4}}{k!} \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-1} z^{(k+4)}(t_n) + O(h^{M+1}).
\end{aligned} \tag{3.12}$$

Expanding (3.12), we obtain

$$\begin{aligned}
\hat{\mathfrak{L}}_i[z(t_n); h] = &\sum_{k=1}^{m+2} \frac{h^{k+3} p_i^{k+3}}{(k+3)!} z^{(k+3)}(t_n) - \sum_{k=1}^{m+2} \frac{k h^{k+3}}{k!} \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-1} z^{(k+3)}(t_n) \\
&- \sum_{k=1}^{m+2} \frac{k h^{k+4}}{k!} \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-1} z^{(k+4)}(t_n) + \sum_{k=m+8}^{M} \frac{h^{k+3} p_i^{k+3}}{(k+3)!} z^{(k+3)}(t_n)
\end{aligned} \tag{3.13}$$

$$- \sum_{k=m+8}^{M} \frac{kh^{k+3}}{k!} \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-1} z^{(k+3)}(t_n)$$
$$- \sum_{k=m+8}^{M} \frac{kh^{k+4}}{k!} \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-1} z^{(k+4)}(t_n) + O(h^{M+1}),$$

where $m = 3$ and $M \geq m+8$ is a positive integer. This can be written as

$$\begin{aligned}\hat{\mathfrak{L}}_i[z(t_n);h] = &\sum_{k=4}^{m+2} \frac{h^k}{k!} \left[p_i^k - k(k-1)(k-2)(k-3) \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-4} \right] z^{(k)}(t_n) \\ &- \sum_{k=5}^{m+2} \frac{h^k}{k!} \left[k(k-1)(k-2)(k-3)(k-4) \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-5} \right] z^{(k)}(t_n) \\ &+ \sum_{k=m+11}^{M} \frac{h^k}{k!} \left[p_i^k - k(k-1)(k-2)(k-3) \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-4} \right] z^{(k)}(t_n) \\ &- \sum_{k=m+12}^{M} \frac{h^k}{k!} \left[k(k-1)(k-2)(k-3)(k-4) \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-5} \right] z^{(k)}(t_n) + O(h^{M+1}).\end{aligned} \tag{3.14}$$

When

$$\sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_j^{k-4} + (k-4) \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{k-5} = \frac{p_i^k}{k(k-1)(k-2)(k-3)}, \quad \text{for } k = 4, \ldots, m+2, \tag{3.15}$$

substituting (3.15) into (3.14), we obtain

$$\begin{aligned}\hat{\mathfrak{L}}_i[z(t_n);h] = &\frac{h^{m+11}}{(m+11)!} \left[p_i^{m+11} - (m+11)(m+10)(m+9)(m+8) \sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_i^{m+7} \right] z^{(m+11)}(t_n) \\ &- \frac{h^{m+12}}{(m+12)!} \left[(m+12)(m+11)(m+10)(m+9)(m+8) \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_j^{m+7} \right] z^{(m+12)}(t_n) + O(h^{m+13}).\end{aligned} \tag{3.16}$$

From (3.16), the error constant for (2.10) is

$$U_{i,m+11} = \frac{p_i^{m+11}}{(m+11)!} - \frac{1}{(m+7)!} \left(\sum_{j=1}^{\Phi} \breve{\alpha}_{i\,j} p_i^{m+7} - \sum_{j=1}^{\Phi} \breve{\gamma}_{i\,j} p_i^{m+7} \right). \tag{3.17}$$

□

Corollary 3.2. *LTE for overlapping methods in the closed interval $[t_n, t_{n+2}]$ can be expressed as follows.*

- *For the first case (2.26)*

$$\begin{aligned}\mathfrak{L}_i[z(t_n);h] = &\frac{h^{m+12}}{(m+12)!} \left[p_i^{m+12} - (m+12)(m+11)(m+10) \left(\sum_{j=-4,1} \tilde{\beta}_{i\,j} p_j^{m+9} + \sum_{j=2}^{\Phi} \tilde{\alpha}_{i\,j} p_i^{m+9} \right) \right] z^{(m+12)}(t_n) \\ &- \frac{h^{m+13}}{(m+13)!} \left[(m+13)(m+12)(m+11)(m+10) \left(\sum_{j=-4,1} \tilde{\tau}_{i\,j} p_j^{m+9} + \sum_{j=2}^{\Phi} \tilde{\gamma}_{i\,j} p_j^{m+9} \right) \right] z^{(m+13)}(t_n) + O(h^{m+14}).\end{aligned} \tag{3.18}$$

- *For the second case (2.29)*

$$\mathfrak{L}_i[z(t_n);h] = \frac{h^{m+13}}{(m+13)!}\left[p_i^{m+13} - (m+13)(m+12)(m+11)(m+10)\left(\sum_{j=-4,1}\bar{\beta}_{i\,j}p_j^{m+9} + \sum_{j=2}^{\Phi}\bar{\alpha}_{i\,j}p_i^{m+9}\right)\right]z^{(m+13)}(t_n)$$
$$- \frac{h^{m+14}}{(m+14)!}\left[(m+14)(m+13)(m+12)(m+11)(m+10)\left(\sum_{j=-4,1}\bar{\tau}_{i\,j}p_j^{m+9} + \sum_{j=2}^{\Phi}\bar{\gamma}_{i\,j}p_j^{m+9}\right)\right]z^{(m+14)}(t_n) + O(h^{m+15}).$$

Proof. To analyze the local truncation error for the overlapping methods (2.26) and (2.29), we utilize the linear operator $\hat{\mathfrak{L}}_i$ as follows

$$\begin{aligned}\hat{\mathfrak{L}}_{\mathrm{i}}[z(t_n);h] = {} & z(t_n+hp_i) - z(t_n) - hp_iz'(t_n) - h^2\mu_iz''(t_n) - h^3\sum_{j=1,4}\left[\tilde{\beta}_{i\,j}z'''(t_n-hp_j)\right] \\ & - h^3\sum_{j=1}^{\Phi}[\tilde{\alpha}_{i\,j}z'''(t_n+hp_j)] - h^4\sum_{j=1,4}\left[\tilde{\tau}_{i\,j}z^{iv}(t_n-hp_j)\right] \\ & - h^4\sum_{j=1}^{\Phi}\left[\tilde{\gamma}_{i\,j}z^{iv}(t_n+hp_j)\right], \quad i = 2,3,\dots\Phi,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\hat{\mathfrak{L}}_{\mathrm{i}}[z(t_n);h] = {} & z(t_n+hp_i) - z(t_n) - h\breve{p}_iz'(t_n) - h^2\breve{\mu}_iz''(t_n) - h^3\breve{\iota}_iz'''(t_n) \\ & - h^4\sum_{j=1,4}\left[\bar{\beta}_{i\,j}z^{iv}(t_n-hp_j)\right] - h^4\sum_{j=1}^{\Phi}\left[\bar{\alpha}_{i\,j}z^{iv}(t_n+hp_j)\right] \\ & - h^5\sum_{j=1,4}[\bar{\tau}_{i\,j}z^{v}(t_n-hp_j)] - h^5\sum_{j=1}^{\Phi}[\bar{\gamma}_{i\,j}z^{v}(t_n+hp_j)], \quad i = 2,3,\dots\Phi.\end{aligned} \tag{3.20}$$

The proof is similar to that of Theorem 3.1, given above, and uses:
For the first case (2.26)

$$\sum_{j=-4,1}\tilde{\beta}_{i\,j}p_j^{k-3} + \sum_{j=2}^{\Phi}\tilde{\alpha}_{i\,j}p_j^{k-3} + (k-3)\left(\sum_{j=-4,1}\tilde{\tau}_{i\,j}p_j^{k-4} + \sum_{j=2}^{\Phi}\tilde{\gamma}_{i\,j}p_j^{k-4}\right) = \frac{p_i^k}{k(k-1)(k-2)}. \tag{3.21}$$

The error constant is

$$U_{i,m+10} = \frac{p_i^{m+12}}{(m+12)!} - \frac{1}{(m+9)!}\left(\sum_{j=-4,1}\tilde{\beta}_{i\,j}p_j^{m+9} + \sum_{j=2}^{\Phi}\tilde{\alpha}_{i\,j}p_j^{m+9} - \sum_{j=-4,1}\tilde{\tau}_{i\,j}p_j^{m+9} - \sum_{j=2}^{\Phi}\tilde{\gamma}_{i\,j}p_i^{m+9}\right). \tag{3.22}$$

For the second case (2.29)

$$\sum_{j=-4,1}\bar{\beta}_{i\,j}p_j^{k-4} + \sum_{j=1}^{\Phi}\bar{\alpha}_{i\,j}p_j^{k-4} + (k-4)\left(\sum_{j=-4,1}\bar{\tau}_{i\,j}p_j^{k-5} + \sum_{j=1}^{\Phi}\bar{\gamma}_{i\,j}p_j^{k-5}\right) = \frac{p_i^k}{k(k-1)(k-2)(k-3)}. \tag{3.23}$$

The error constant is

$$U_{i,m+13} = \frac{p_i^{m+13}}{(m+13)!} - \frac{1}{(m+9)!}\left(\sum_{j=-4,1}\breve{\beta}_{i\,j}p_j^{m+9} + \sum_{j=2}^{\Phi}\breve{\alpha}_{i\,j}p_j^{m+9} - \sum_{j=-4,1}\breve{\tau}_{i\,j}p_j^{m+9} - \sum_{j=2}^{\Phi}\breve{\gamma}_{i\,j}p_i^{m+9}\right). \tag{3.24}$$

□

TABLE 1. Truncation errors and order $\hat{m}$ for the overlapping and non-overlapping fourth and fifth derivative HBMs.

		Non-Overlapping		**Overlapping**	
w	$\hat{\mathfrak{L}}$	**LTE**	**Order** $\hat{m}$	**LTE**	**Order** $\hat{m}$
3	p_2	$4.0888 \times 10^{-11} z^{13}(t_n)h^{13}$	10	$8.5211 \times 10^{-13} z^{15}(t_n)h^{15}$	12
	p_3	$2.7989 \times 10^{-10} z^{13}(t_n)h^{13}$	10	$6.1109 \times 10^{-12} z^{15}(t_n)h^{15}$	12
	p_4	$7.6742 \times 10^{-10} z^{13}(t_n)h^{13}$	10	$1.7469 \times 10^{-11} z^{15}(t_n)h^{15}$	12
	p_5	$1.5539 \times 10^{-9} z^{13}(t_n)h^{13}$	10	$3.8213 \times 10^{-11} z^{15}(t_n)h^{15}$	12
4	p_2	$4.6756 \times 10^{-12} z^{14}(t_n)h^{14}$	10	$9.5568 \times 10^{-14} z^{16}(t_n)h^{16}$	12
	p_3	$7.5181 \times 10^{-11} z^{14}(t_n)h^{14}$	10	$1.6114 \times 10^{-12} z^{16}(t_n)h^{16}$	12
	p_4	$3.2598 \times 10^{-10} z^{14}(t_n)h^{14}$	10	$7.2178 \times 10^{-12} z^{16}(t_n)h^{16}$	12
	p_5	$8.9115 \times 10^{-10} z^{14}(t_n)h^{14}$	10	$2.0535 \times 10^{-11} z^{16}(t_n)h^{16}$	12

The LTE and order of the fourth and fifth derivative HBMs, both overlapping and non-overlapping, are presented in Table 1. It is observed that the overlapping method exhibits a higher order than the non-overlapping method, indicating an improvement in performance.

3.2. **Stability Analysis.** From matrices (2.14), (2.18), (2.33), and (2.37), as $h \to 0$ the matrix system reduces to

$$\boldsymbol{A}_1 Z_{n+1} = \boldsymbol{A}_0 Z_n. \tag{3.25}$$

Whose first characteristics polynomial $\aleph(R)$ given as

$$\aleph(R) = \det\left(R(\boldsymbol{A}_1) - \boldsymbol{A}_0\right). \tag{3.26}$$

Thus, the fourth and fifth derivative HBMs are zero stable for $\aleph(R) = 0$ and satisfies $|R_j| \leq 1$. For the root with $|R_j| = 1$, the multiplicity is not more than the order of the differential equation. Hence, the proposed methods converge, are zero stable, and consistent.

3.3. **Absolute Stability.** The RAS for the methods can be defined as

$$\Re(y) = \{y \in \mathbb{C} : \mathcal{H}(y) < 1\}.$$

These test equations

$$z^w = \hat{\lambda}^w z, \qquad z^{w+1} = \hat{\lambda}^{w+1} z, \qquad \hat{\lambda} < 0, \qquad w = 3,\, 4,$$

are applied on the matrix Equations (2.33) and (2.37) to obtain

$$Z_{n+1} = \mathcal{H}(y) Z_n, \qquad y = \hat{\lambda}^w h^w, \qquad w = 3,\, 4, \tag{3.27}$$

when $w = 3$, the $\mathcal{H}(y)$ is given by

$$\mathcal{H}(y) = \frac{\boldsymbol{A}_0 + y\,\boldsymbol{p}_0 + y^2 \tilde{\boldsymbol{\mu}}_0 + y^3 \tilde{\boldsymbol{\beta}}_0 + y^4 \tilde{\boldsymbol{\tau}}_0}{\boldsymbol{A}_1 - y^3\,\tilde{\boldsymbol{\alpha}}_1 - y^4\,\tilde{\boldsymbol{\gamma}}_1}, \tag{3.28}$$

when $w = 4$, the $\mathcal{H}(y)$ is given by

$$\mathcal{H}(y) = \frac{\boldsymbol{A}_0 + y\,\breve{\boldsymbol{p}}_0 + y^2 \breve{\boldsymbol{\mu}}_0 + y^3 \breve{\boldsymbol{\iota}}_0 + y^4 \bar{\boldsymbol{\beta}}_0 + y^5 \bar{\boldsymbol{\tau}}_0}{\boldsymbol{A}_1 - y^4\,\bar{\boldsymbol{\alpha}}_1 - y^5\,\bar{\boldsymbol{\gamma}}_1}. \tag{3.29}$$

The stability function $\Re(y)$ can be obtained by finding the dominant eigenvalues of the matrix $\mathcal{H}(y)$. As shown in Figures 3 and 4, the fourth derivative and fifth derivative of the non-overlapping and overlapping HBMs are absolute stable since their stability regions are contained entirely on the negative side of the plane.

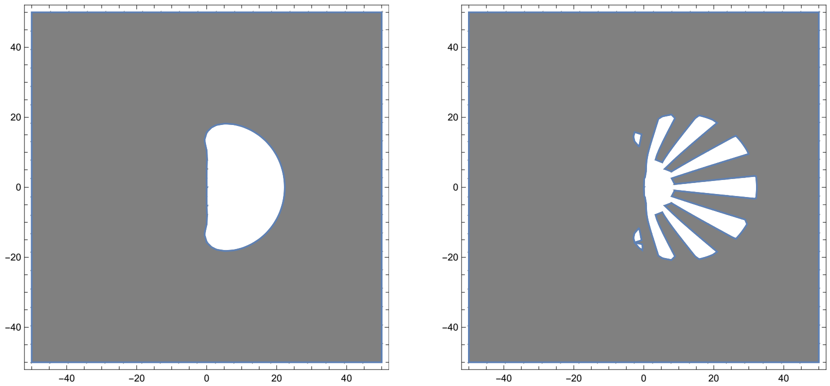

FIGURE 3. Stability region of non-overlapping and overlapping ($w = 3$).

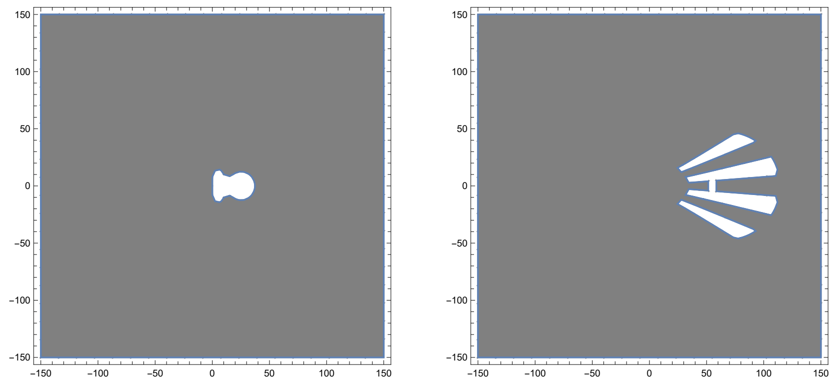

FIGURE 4. Stability region of non-overlapping and overlapping ($w = 4$).

4. Formulation of the method as an adaptive method

Adopting the approach considered in [20], the formulation of the method as an adaptive step-size form can be achieved by simultaneously executing the overlapping method and the non-overlapping (lower order) method. The adaptive step-size will improve the accuracy of the approximation, with smaller step-sizes generally leading to more accurate results. The use of a non-overlapping method alongside the overlapping method can help determine the appropriate step-size for the new step. The lower-order method is used to estimate the error at the final point for z_{n+2}, while the overlapping method is used to advance the integration process. The non-overlapping method (2.7) for third-order IVPs is

$$z^*_{n+2} = z_n + hp_{i\,1}z'_n + h^2\mu_{i\,1}z''_n + h^3\beta_{i\,1}f_n + h^3\left(\sum_{j=2}^{\Phi}\alpha_{i\,j}f_{n+p_j}\right) + h^4\tau_{i\,1}g_n + h^4\left(\sum_{j=2}^{\Phi}\gamma_{i\,j}g_{n+p_j}\right), \tag{4.1}$$

and the non-overlapping method (2.10) for fourth-order IVPs is

$$z^*_{n+2} = z_n + h\breve{p}_{i\,1}z'_n + h^2\breve{\mu}_{i\,1}z''_n + h^3\breve{\iota}_{i\,1}z'''_n + h^4\breve{\beta}_{i\,1}f_n + h^4\left(\sum_{j=2}^{\Phi}\breve{\alpha}_{i\,j}f_{n+p_j}\right) + h^5\breve{\tau}_{i\,1}g_n + h^5\left(\sum_{j=2}^{\Phi}\breve{\gamma}_{i\,j}g_{n+p_j}\right). \tag{4.2}$$

An estimate error is determined by the difference between the overlapping and non-overlapping methods, i.e

$$\boldsymbol{EstErr} = \|z_{n+2} - z^*_{n+2}\|.$$

Accept the estimate error obtained above, if $\|\boldsymbol{EstErr}\| < \text{Tol}$, where Tol is the user-defined tolerance. But if $\|\boldsymbol{EstErr}\| \geq \text{Tol}$, reject the results and adjust the step-size using the canonical formula proposed in [21] and recalculate.

$$h_{new} = \Upsilon h_{old}\left(\frac{Tol}{\|\boldsymbol{EstErr}\|}\right)^{\frac{1}{\hat{m}+w}} \qquad w = 3, 4,$$

where $\hat{m}$ represents the order of the non-overlapping method and $0 < \Upsilon < 1$ is the safety factor designed to prevent failure of steps. We take $\Upsilon = 0.9$. The condition **if** $h_{minimum} \leq h_{new} \leq h_{maximum}$ **then** $h_{old} = h_{new}$ is imposed during the implementation of the method.

5. Implementation

5.1. **Linear higher-order differential equations.** Consider the general linear higher-order DE

$$z^{w} = f(t, z, z', \ldots, z^{w-1}), \qquad w = 3, 4, \tag{5.1}$$

when $w = 3$, we have

$$z''' = f(t, z, z', z'') = \psi(t) + \phi(t)z + \chi(t)z' + \kappa(t)z'', \tag{5.2}$$

and when $w = 4$, we have

$$z^{iv} = f(t, z, z', z'', z''') = l(t) + q(t)z + s(t)z' + v(t)z'' + w(t)z'''. \tag{5.3}$$

Differentiating (5.3), we have

$$z^{v} = f(t, z, z', z'', z''', z^{iv}) = \hat{l}(t) + \hat{q}(t)z + \hat{s}(t)z' + \hat{v}(t)z'' + \hat{w}(t)z''' + \hat{\lambda}(t)z^{iv}. \tag{5.4}$$

Substituting (5.2) and (5.3) into (2.14), (2.15), and (2.16) respectively, we obtain

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \mathbf{A_{13}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} \\ \mathbf{A_{31}} & \mathbf{A_{32}} & \mathbf{A_{33}} \end{bmatrix} \begin{bmatrix} Z_{n+\Phi} \\ Z'_{n+\Phi} \\ Z''_{n+\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{B_{1,n}} \\ \mathbf{B_{2,n}} \\ \mathbf{B_{3,n}} \end{bmatrix}. \tag{5.5}$$

The non-overlapping block for third-order DE on the closed interval $[t_n, t_{n+2}]$, we have

$$\begin{aligned}
\mathbf{A_{11}} &= \boldsymbol{A_1} - \left(h^3\boldsymbol{\alpha_1} + h^4\boldsymbol{\gamma_1}w_{n+2}\right)\phi_{n+2} - h^4\boldsymbol{\gamma_1}q_{n+2},\\
\mathbf{A_{12}} &= -\left(h^3\boldsymbol{\alpha_1} + h^4\boldsymbol{\gamma_1}w_{n+2}\right)\chi_{n+2} - h^4\boldsymbol{\gamma_1}s_{n+2},\\
\mathbf{A_{13}} &= -\left(h^3\boldsymbol{\alpha_1} + h^4\boldsymbol{\gamma_1}w_{n+2}\right)\kappa_{n+2} - h^4\boldsymbol{\gamma_1}v_{n+2},\\
\mathbf{A_{21}} &= -\left(h^2\boldsymbol{\zeta_1} + h^3\boldsymbol{\nu_1}w_{n+2}\right)\phi_{n+2} - h^3\boldsymbol{\nu_1}q_{n+2},\\
\mathbf{A_{22}} &= \boldsymbol{A_1} - \left(h^2\boldsymbol{\zeta_1} + h^3\boldsymbol{\nu_1}w_{n+2}\right)\chi_{n+2} - h^3\boldsymbol{\nu_1}s_{n+2},\\
\mathbf{A_{23}} &= -\left(h^2\boldsymbol{\zeta_1} + h^3\boldsymbol{\nu_1}w_{n+2}\right)\kappa_{n+2} - h^3\boldsymbol{\nu_1}v_{n+2},\\
\mathbf{A_{31}} &= -\left(h\boldsymbol{\vartheta_1} + h^2\boldsymbol{\sigma_1}w_{n+2}\right)\phi_{n+2} - h^2\boldsymbol{\sigma_1}q_{n+2},\\
\mathbf{A_{32}} &= -\left(h\boldsymbol{\vartheta_1} + h^2\boldsymbol{\sigma_1}w_{n+2}\right)\chi_{n+2} - h^2\boldsymbol{\sigma_1}s_{n+2},\\
\mathbf{A_{33}} &= \boldsymbol{A_1} - \left(h\boldsymbol{\vartheta_1} + h^2\boldsymbol{\sigma_1}w_{n+2}\right)\kappa_{n+2} - h^2\boldsymbol{\sigma_1}v_{n+2},\\
\mathbf{B_{1,n}} &= \boldsymbol{A_0}Z_n + h\boldsymbol{p_0}Z'_n + h^2\boldsymbol{\mu_0}Z''_n + h^3\boldsymbol{\beta_0}F_n + h^3\boldsymbol{\alpha_1}\psi_{n+2} + h^4\boldsymbol{\tau_0}G_n + h^4\boldsymbol{\gamma_1}l_{n+2} + h^4\boldsymbol{\gamma_1}w_{n+2}\psi_{n+2},\\
\mathbf{B_{2,n}} &= \boldsymbol{A_0}Z'_n + \boldsymbol{\varepsilon_0}Z''_n + h^2\boldsymbol{\zeta_1}\psi_{n+2} + h^2\boldsymbol{\eta_0}F_n + h^3\boldsymbol{\omega_0}G_n + h^3\boldsymbol{\nu_1}l_{n+2} + h^3\boldsymbol{\nu_1}w_{n+2}\psi_{n+2},\\
\mathbf{B_{3,n}} &= \boldsymbol{A_0}Z''_n + h\boldsymbol{\vartheta_1}\psi_{n+2} + h\boldsymbol{d_0}F_n + h^2\boldsymbol{\upsilon_0}G_n + h^2\boldsymbol{\sigma_1}l_{n+2} + h^2\boldsymbol{\sigma_1}w_{n+2}\psi_{n+2}.
\end{aligned} \tag{5.6}$$

For the overlapping block for third-order DE on the closed interval $[t_n, t_{n+2}]$, substitute (5.2) and (5.3) into (2.33), (2.34) and (2.35) respectively. We have

$$
\begin{aligned}
\mathbf{A_{11}} &= \boldsymbol{A_1} - \left(h^3\tilde{\boldsymbol{\alpha}}_\mathbf{1} + h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} w_{n+2}\right)\phi_{n+2} - h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} q_{n+2},\\
\mathbf{A_{12}} &= -\left(h^3\tilde{\boldsymbol{\alpha}}_\mathbf{1} + h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} w_{n+2}\right)\chi_{n+2} - h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} s_{n+2},\\
\mathbf{A_{13}} &= -\left(h^3\tilde{\boldsymbol{\alpha}}_\mathbf{1} + h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} w_{n+2}\right)\kappa_{n+2} - h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} v_{n+2},\\
\mathbf{A_{21}} &= -\left(h^2\tilde{\boldsymbol{\zeta}}_\mathbf{1} + h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} w_{n+2}\right)\phi_{n+2} - h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} q_{n+2},\\
\mathbf{A_{22}} &= \boldsymbol{A_1} - \left(h^2\tilde{\boldsymbol{\zeta}}_\mathbf{1} + h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} w_{n+2}\right)\chi_{n+2} - h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} s_{n+2},\\
\mathbf{A_{23}} &= -\left(h^2\tilde{\boldsymbol{\zeta}}_\mathbf{1} + h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} w_{n+2}\right)\kappa_{n+2} - h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} v_{n+2},\\
\mathbf{A_{31}} &= -\left(h\tilde{\boldsymbol{\vartheta}}_\mathbf{1} + h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} w_{n+2}\right)\phi_{n+2} - h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} q_{n+2},\\
\mathbf{A_{32}} &= -\left(h\tilde{\boldsymbol{\vartheta}}_\mathbf{1} + h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} w_{n+2}\right)\chi_{n+2} - h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} s_{n+2},\\
\mathbf{A_{33}} &= \mathbf{A_1} - \left(h\tilde{\boldsymbol{\vartheta}}_\mathbf{1} + h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} w_{n+2}\right)\kappa_{n+2} - h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} v_{n+2},\\
\mathbf{B_{1,n}} &= \left(\mathbf{A_0} + h^3\tilde{\boldsymbol{\beta}}_\mathbf{0}\phi_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} q_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} w_n\phi_n\right)Z_n + \left(h\boldsymbol{p_0} + h^3\tilde{\boldsymbol{\beta}}_\mathbf{0}\chi_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} s_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} w_n\chi_n\right)Z_n'\\
&\quad + \left(h^2\boldsymbol{\mu_0} + h^3\tilde{\boldsymbol{\beta}}_\mathbf{0}\kappa_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} v_n + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} w_n\kappa_n\right)Z_n'' + \left(h^3\tilde{\boldsymbol{\beta}}_\mathbf{0} + h^4\tilde{\boldsymbol{\tau}}_\mathbf{0} w_n\right)\psi_n\\
&\quad + \left(h^3\tilde{\boldsymbol{\alpha}}_\mathbf{1} + h^4\tilde{\boldsymbol{\gamma}}_\mathbf{1} w_{n+2}\right)\psi_{n+2} + h^4\left(\tilde{\boldsymbol{\gamma}}_\mathbf{1} l_{n+2} + \tilde{\boldsymbol{\tau}}_\mathbf{0} l_n\right),\\
\mathbf{B_{2,n}} &= \left(h^2\tilde{\boldsymbol{\eta}}_\mathbf{0}\phi_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} q_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} w_n\phi_n\right)Z_n + \left(\boldsymbol{A_0} + h^2\tilde{\boldsymbol{\eta}}_\mathbf{0}\chi_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} s_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} w_n\chi_n\right)Z_n'\\
&\quad + \left(h\boldsymbol{\varepsilon_0} + h^2\tilde{\boldsymbol{\eta}}_\mathbf{0}\kappa_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} v_n + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} w_n\kappa_n\right)Z_n'' + \left(h^2\tilde{\boldsymbol{\eta}}_\mathbf{0} + h^3\tilde{\boldsymbol{\omega}}_\mathbf{0} w_n\right)\psi_n\\
&\quad + \left(h^2\tilde{\boldsymbol{\zeta}}_\mathbf{1} + h^3\tilde{\boldsymbol{\nu}}_\mathbf{1} w_{n+2}\right)\psi_{n+2} + h^3\left(\tilde{\boldsymbol{\nu}}_\mathbf{1} l_{n+2} + \tilde{\boldsymbol{\omega}}_\mathbf{0} l_n\right),\\
\mathbf{B_{3,n}} &= \left(h\tilde{\boldsymbol{d}}_\mathbf{0}\phi_n + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} q_n + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} w_n\phi_n\right)Z_n + \left(h\tilde{\boldsymbol{d}}_\mathbf{0}\chi_n + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} s_n + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} w_n\chi_n\right)Z_n'\\
&\quad + \left(\boldsymbol{A_0} + h\tilde{\boldsymbol{d}}_\mathbf{0}\kappa_n + h^2\boldsymbol{\upsilon_0} v_n + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} w_n\kappa_n\right)Z_n'' + \left(h\tilde{\boldsymbol{d}}_\mathbf{0} + h^2\tilde{\boldsymbol{\upsilon}}_\mathbf{0} w_n\right)\psi_n\\
&\quad + \left(h\tilde{\boldsymbol{\vartheta}}_\mathbf{1} + h^2\tilde{\boldsymbol{\sigma}}_\mathbf{1} w_{n+2}\right)\psi_{n+2} + h^2\left(\tilde{\boldsymbol{\sigma}}_\mathbf{1} l_{n+2} + \tilde{\boldsymbol{\upsilon}}_\mathbf{0} l_n\right),
\end{aligned}
\tag{5.7}
$$

where $\phi_{n+2}, \chi_{n+2}, \kappa_{n+2}, q_{n+2}, s_{n+2}, v_{n+2}$ and w_{n+2} are $(\Phi-1)\times(\Phi-1)$ diagonal matrices,

$$
\psi_{n+2} = \left(\psi_{n+p_2}, \psi_{n+p_3}, \psi_{n+p_4}, \psi_{n+2}\right)^T, \qquad \psi_n = \left(\psi_n, \psi_{n-p_2}, \psi_{n-p_3}, \psi_{n-p_4}\right)^T,
$$
$$
l_{n+2} = \left(l_{n+p_2}, l_{n+p_3}, l_{n+p_4}, l_{n+2}\right)^T, \qquad l_n = \left(l_n, l_{n-p_2}, l_{n-p_3}, l_{n-p_4}\right)^T.
$$

For fourth-order DE, we substitute (5.3) and (5.4) into (2.18), (2.19), (2.20), and (2.21), respectively, we obtain

$$
\begin{bmatrix}
\mathbf{A_{11}} & \mathbf{A_{12}} & \mathbf{A_{13}} & \mathbf{A_{14}}\\
\mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} & \mathbf{A_{24}}\\
\mathbf{A_{31}} & \mathbf{A_{32}} & \mathbf{A_{33}} & \mathbf{A_{34}}\\
\mathbf{A_{41}} & \mathbf{A_{42}} & \mathbf{A_{43}} & \mathbf{A_{44}}
\end{bmatrix}
\begin{bmatrix}
Z_{n+\Phi}\\ Z'_{n+\Phi}\\ Z''_{n+\Phi}\\ Z'''_{n+\Phi}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{B_{1,n}}\\ \mathbf{B_{2,n}}\\ \mathbf{B_{3,n}}\\ \mathbf{B_{4,n}}
\end{bmatrix}.
\tag{5.8}
$$

The non-overlapping block for fourth-order DE on the closed interval $[t_n, t_{n+2}]$, we have

$$
\begin{aligned}
\mathbf{A_{11}} &= \boldsymbol{A_1} - \left(h^4\breve{\boldsymbol{\alpha}}_\mathbf{1} + h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{\lambda}_{n+2}\right)q_{n+2} - h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{q}_{n+2},\\
\mathbf{A_{12}} &= -\left(h^4\breve{\boldsymbol{\alpha}}_\mathbf{1} + h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{\lambda}_{n+2}\right)s_{n+2} - h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{s}_{n+2},\\
\mathbf{A_{13}} &= -\left(h^4\breve{\boldsymbol{\alpha}}_\mathbf{1} + h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{\lambda}_{n+2}\right)v_{n+2} - h^5\breve{\boldsymbol{\gamma}}_\mathbf{1}\hat{v}_{n+2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A_{14}} &= -\left(h^4\breve{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\breve{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^5\breve{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{A_{21}} &= -\left(h^3\hat{\boldsymbol{\alpha}}_{\mathbf{1}} + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{22}} &= \boldsymbol{A_1} - \left(h^3\hat{\boldsymbol{\alpha}}_{\mathbf{1}} + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{23}} &= -\left(h^3\hat{\boldsymbol{\alpha}}_{\mathbf{1}} + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{24}} &= -\left(h^3\hat{\boldsymbol{\alpha}}_{\mathbf{1}} + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{A_{31}} &= -\left(h^2\hat{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{32}} &= -\left(h^2\hat{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{33}} &= \boldsymbol{A_1} - \left(h^2\hat{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{34}} &= -\left(h^2\hat{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{A_{41}} &= -\left(h\hat{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{42}} &= -\left(h\hat{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{43}} &= -\left(h\hat{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{44}} &= \boldsymbol{A_1} - \left(h\hat{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{B_{1,n}} &= \boldsymbol{A_0}Z_n + h\breve{\boldsymbol{p}}_{\mathbf{0}}Z_n' + h^2\breve{\boldsymbol{\mu}}_{\mathbf{0}}Z_n'' + h^3\breve{\boldsymbol{\iota}}_{\mathbf{0}}Z_n''' + h^4\breve{\boldsymbol{\beta}}_{\mathbf{0}}F_n + h^4\breve{\boldsymbol{\alpha}}_{\mathbf{1}}l_{n+2} + h^5\breve{\boldsymbol{\tau}}_{\mathbf{0}}G_n + h^5\breve{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{l}_{n+2} + h^5\breve{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}l_{n+2},\\
\mathbf{B_{2,n}} &= \boldsymbol{A_0}Z_n' + h\hat{\boldsymbol{p}}_{\mathbf{0}}Z_n'' + h^2\hat{\boldsymbol{\mu}}_{\mathbf{0}}Z_n''' + h^3\hat{\boldsymbol{\beta}}_{\mathbf{0}}F_n + h^3\hat{\boldsymbol{\alpha}}_{\mathbf{1}}l_{n+2} + h^4\hat{\boldsymbol{\tau}}_{\mathbf{0}}G_n + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{l}_{n+2} + h^4\hat{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}l_{n+2},\\
\mathbf{B_{3,n}} &= \boldsymbol{A_0}Z_n'' + h\hat{\boldsymbol{\varepsilon}}_{\mathbf{0}}Z_n''' + h^2\hat{\boldsymbol{\zeta}}_{\mathbf{1}}l_{n+2} + h^2\hat{\boldsymbol{\eta}}_{\mathbf{0}}F_n + h^3\hat{\boldsymbol{\omega}}_{\mathbf{0}}G_n + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{l}_{n+2} + h^3\hat{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}l_{n+2},\\
\mathbf{B_{4,n}} &= \boldsymbol{A_0}Z_n''' + h\hat{\boldsymbol{\vartheta}}_{\mathbf{1}}l_{n+2} + h\hat{\boldsymbol{d}}_{\mathbf{0}}F_n + h^2\hat{\boldsymbol{\upsilon}}_{\mathbf{0}}G_n + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{l}_{n+2} + h^2\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}l_{n+2},
\end{aligned}
\tag{5.9}
$$

for the overlapping block for fourth-order DE on the closed interval $[t_n,\, t_{n+2}]$, we have

$$
\begin{aligned}
\mathbf{A_{11}} &= \boldsymbol{A_1} - \left(h^4\bar{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{12}} &= -\left(h^4\bar{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{13}} &= -\left(h^4\bar{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{14}} &= -\left(h^4\bar{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{A_{21}} &= -\left(h^3\bar{\bar{\boldsymbol{\alpha}}}_{\mathbf{1}} + h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{22}} &= \boldsymbol{A_1} - \left(h^3\bar{\bar{\boldsymbol{\alpha}}}_{\mathbf{1}} + h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{23}} &= -\left(h^3\bar{\bar{\boldsymbol{\alpha}}}_{\mathbf{1}} + h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{24}} &= -\left(h^3\bar{\bar{\boldsymbol{\alpha}}}_{\mathbf{1}} + h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{w}_{n+2},\\
\mathbf{A_{31}} &= -\left(h^2\bar{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{q}_{n+2},\\
\mathbf{A_{32}} &= -\left(h^2\bar{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{s}_{n+2},\\
\mathbf{A_{33}} &= \boldsymbol{A_1} - \left(h^2\bar{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{v}_{n+2},\\
\mathbf{A_{34}} &= -\left(h^2\bar{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{w}_{n+2},
\end{aligned}
\tag{5.10}
$$

$$\mathbf{A_{41}} = -\left(h\bar{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) q_{n+2} - h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{q}_{n+2},$$
$$\mathbf{A_{42}} = -\left(h\bar{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) s_{n+2} - h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{s}_{n+2},$$
$$\mathbf{A_{43}} = -\left(h\bar{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) v_{n+2} - h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{v}_{n+2},$$
$$\mathbf{A_{44}} = \boldsymbol{A}_{\mathbf{1}} - \left(h\bar{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) w_{n+2} - h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{w}_{n+2},$$
$$\begin{aligned}\mathbf{B_{1,n}} = &\left(\boldsymbol{A}_{\mathbf{0}} + h^4\bar{\boldsymbol{\beta}}_{\mathbf{0}}q_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{q}_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{\lambda}_n q_n\right) Z_n + \left(h\breve{\boldsymbol{p}}_{\mathbf{0}} + h^4\bar{\boldsymbol{\beta}}_{\mathbf{0}}s_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{s}_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{\lambda}_n s_n\right) Z_n' \\ &+ \left(h^2\breve{\boldsymbol{\mu}}_{\mathbf{0}} + h^4\bar{\boldsymbol{\beta}}_{\mathbf{0}}v_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{v}_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{\lambda}_n v_n\right) Z_n'' + \left(h^3\breve{\boldsymbol{\iota}}_{\mathbf{0}} + h^4\bar{\boldsymbol{\beta}}_{\mathbf{0}}w_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{w}_n + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{\lambda}_n w_n\right) Z_n''' \\ &+ \left(h^4\bar{\boldsymbol{\beta}}_{\mathbf{0}} + h^5\bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{\lambda}_n\right) l_n + \left(h^4\bar{\boldsymbol{\alpha}}_{\mathbf{1}} + h^5\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) l_{n+2} + h^5\left(\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\hat{l}_{n+2} + \bar{\boldsymbol{\tau}}_{\mathbf{0}}\hat{l}_n\right),\end{aligned}$$
$$\begin{aligned}\mathbf{B_{2,n}} = &\left(h^3\bar{\bar{\boldsymbol{\beta}}}_{\mathbf{0}}q_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{q}_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{\lambda}_n q_n\right) Z_n + \left(\boldsymbol{A}_{\mathbf{0}} + h^3\bar{\bar{\boldsymbol{\beta}}}_{\mathbf{0}}s_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{s}_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{\lambda}_n s_n\right) Z_n' \\ &+ \left(h\hat{\boldsymbol{p}}_{\mathbf{0}} + h^3\bar{\bar{\boldsymbol{\beta}}}_{\mathbf{0}}v_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{v}_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{\lambda}_n v_n\right) Z_n'' + \left(h^2\hat{\boldsymbol{\mu}}_{\mathbf{0}} + h^3\bar{\bar{\boldsymbol{\beta}}}_{\mathbf{0}}w_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{w}_n + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{\lambda}_n w_n\right) Z_n''' \\ &+ \left(h^3\bar{\bar{\boldsymbol{\beta}}}_{\mathbf{0}} + h^4\bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{\lambda}_n\right) l_n + \left(h^3\bar{\bar{\boldsymbol{\alpha}}}_{\mathbf{1}} + h^4\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) l_{n+2} + h^4\left(\bar{\bar{\boldsymbol{\gamma}}}_{\mathbf{1}}\hat{l}_{n+2} + \bar{\bar{\boldsymbol{\tau}}}_{\mathbf{0}}\hat{l}_n\right),\end{aligned}$$
$$\begin{aligned}\mathbf{B_{3,n}} = &\left(h^2\bar{\boldsymbol{\eta}}_{\mathbf{0}}q_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{q}_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{\lambda}_n q_n\right) Z_n + \left(h^2\bar{\boldsymbol{\eta}}_{\mathbf{0}}s_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{s}_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{\lambda}_n s_n\right) Z_n' \\ &+ \left(\boldsymbol{A}_{\mathbf{0}} + h^2\bar{\boldsymbol{\eta}}_{\mathbf{0}}v_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{v}_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{\lambda}_n v_n\right) Z_n'' + \left(h\hat{\boldsymbol{\varepsilon}}_{\mathbf{0}} + h^2\bar{\boldsymbol{\eta}}_{\mathbf{0}}w_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{w}_n + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{\lambda}_n w_n\right) Z_n''' \\ &+ \left(h^2\bar{\boldsymbol{\eta}}_{\mathbf{0}} + h^3\bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{\lambda}_n\right) l_n + \left(h^2\bar{\boldsymbol{\zeta}}_{\mathbf{1}} + h^3\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) l_{n+2} + h^3\left(\bar{\boldsymbol{\nu}}_{\mathbf{1}}\hat{l}_{n+2} + \bar{\boldsymbol{\omega}}_{\mathbf{0}}\hat{l}_n\right),\end{aligned}$$
$$\begin{aligned}\mathbf{B_{4,n}} = &\left(h\bar{\boldsymbol{d}}_{\mathbf{0}}q_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{q}_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{\lambda}_n q_n\right) Z_n + \left(h\bar{\boldsymbol{d}}_{\mathbf{0}}s_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{s}_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{\lambda}_n s_n\right) Z_n' \\ &+ \left(h\bar{\boldsymbol{d}}_{\mathbf{0}}v_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{v}_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{\lambda}_n v_n\right) Z_n'' + \left(\boldsymbol{A}_{\mathbf{0}} + h\bar{\boldsymbol{d}}_{\mathbf{0}}w_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{w}_n + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{\lambda}_n w_n\right) Z_n''' \\ &+ \left(h\bar{\boldsymbol{d}}_{\mathbf{0}} + h^2\bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{\lambda}_n\right) l_n + \left(h\bar{\boldsymbol{\vartheta}}_{\mathbf{1}} + h^2\bar{\boldsymbol{\sigma}}_{\mathbf{1}}\hat{\lambda}_{n+2}\right) l_{n+2} + h^2\left(\bar{\boldsymbol{\sigma}}_{\mathbf{1}}l_{n+2} + \bar{\boldsymbol{\upsilon}}_{\mathbf{0}}\hat{l}_n\right),\end{aligned}$$

where q_{n+2}, s_{n+2}, v_{n+2}, w_{n+2}, $\hat{q}_{n+2}$, $\hat{s}_{n+2}$, $\hat{v}_{n+2}$, $\hat{w}_{n+2}$ and $\hat{\lambda}_{n+2}$ are $(\Phi-1)\times(\Phi-1)$ diagonal matrices,

$$l_{n+2} = (l_{n+p_2},\, l_{n+p_3},\, l_{n+p_4},\, l_{n+2})^T, \qquad l_n = (l_n,\, l_{n-p_2},\, l_{n-p_3},\, l_{n-p_4})^T,$$
$$\hat{l}_{n+2} = \left(\hat{l}_{n+p_2},\, \hat{l}_{n+p_3},\, \hat{l}_{n+p_4},\, \hat{l}_{n+2}\right)^T, \qquad \hat{l}_n = \left(\hat{l}_n,\, \hat{l}_{n-p_2},\, \hat{l}_{n-p_3},\, \hat{l}_{n-p_4}\right)^T.$$

5.2. **Nonlinear higher-order differential equations.** Consider a nonlinear higher-order DE in the form

$$z^w = \tilde{\Phi}_1(t)z + \tilde{\Phi}_2(t,\, z)z' + \ldots + \tilde{\Phi}_w(t,\, z,\, z^{w-2})z^{w-1} + \tilde{\Psi}(t, z, z', \ldots, z^{w-1}), \qquad w = 3,\, 4, \tag{5.11}$$

where $\tilde{\Phi}_1,\, \tilde{\Phi}_2,\, \ldots \tilde{\Phi}_w$ are coefficients of $z,\, z',\, \ldots,\, z^{w-1}$ respectively and $\tilde{\Psi}$ is the nonlinear term. We apply the modified-Picard iteration

$$z^w_{r+1} = \tilde{\Phi}_1(t)z_{r+1} + \tilde{\Phi}_2(t,\, z_r)z'_{r+1} + \ldots + \tilde{\Phi}_w(t,\, z_r,\, z_r^{w-2})z^{w-1}_{r+1} + \tilde{\Psi}(t,\, z_r,\, z_r',\, \ldots,\, z_r^{w-1}), \tag{5.12}$$

to solve (5.11). The current iteration is denoted by $r+1$ and the previous iteration is denoted by r. This equation is in the linear form (5.2), (5.3), and (5.4) with

$$\begin{aligned}&\psi(t) = \tilde{\Psi}(t,\, z_r,\, z_r',\, z_r''), \quad \phi(t) = \tilde{\Phi}_1(t), \quad \chi(t) = \tilde{\Phi}_2(t,\, z_r), \quad \kappa(t) = \tilde{\Phi}_3(t,\, z_r,\, z_r'),\\ &l(t) = \tilde{\Psi}(t,\, z_r,\, z_r',\, z_r'',\, z_r'''), \quad q(t) = \tilde{\Phi}_1(t), \quad s(t) = \tilde{\Phi}_2(t,\, z_r), \quad v(t) = \tilde{\Phi}_3(t,\, z_r,\, z_r'),\\ &w(t) = \tilde{\Phi}_4(t,\, z_r,\, z_r',\, z_r''),\\ &\hat{l}(t) = \tilde{\Psi}(t,\, z_r,\, z_r',\, z_r'',\, z_r'''), \quad \hat{q}(t) = \tilde{\Phi}_1(t), \quad \hat{s}(t) = \tilde{\Phi}_2(t,\, z_r), \quad \hat{v}(t)\tilde{\Phi}_3(t,\, z_r,\, z_r'),\end{aligned} \tag{5.13}$$

$$\hat{w}(t) = \tilde{\Phi}_4(t,\, z_r,\, z_r',\, z_r''), \quad \hat{\lambda}(t) = \tilde{\Phi}_4(t,\, z_r,\, z_r',\, z_r'',\, z_r''').$$

We solve the linear system (5.5) and (5.8) to obtain the numerical solution for the nonlinear IVPs. To apply the overlapping block method to solve the linear system, we first apply (5.6) and (5.9) to the integration block $[t_0,\, t_2]$. Next, we apply (5.7) and (5.10) to all the subsequent integrating blocks $[t_n,\, t_{n+2}]$ for $n = 2,\, 3,\, \dots,\, N-1$ to obtain the numerical solution for the linear and nonlinear IVPs.

6. Numerical results and discussion

In this section we test the accuracy and efficiency of the method by solving some examples.

Example 6.1. Consider the following nonlinear problem [3]

$$z''' = \frac{1 + 2\sin^2(z)}{\cos^5(z)}, \quad 0 \le t \le \frac{\pi}{4},$$
$$z(0) = 0, \quad z'(0) = 1 \quad z''(0) = 0,$$

with the exact solution given as

$$z(t) = \arcsin(t).$$

In this example,

$$z^{iv} = \frac{z' \sin(z)\left(5 + 2\left(2\cos^2(z) + 5\sin^2(z)\right)\right)}{\cos^6(z)}.$$

Following Equations (5.12) and (5.13),

$$z_{r+1}''' = \frac{1 + 2\sin^2(z_r)}{\cos^5(z_r)},$$
$$z_{r+1}^{iv} = \frac{z' \sin(z_r)\left(5 + 2\left(2\cos^2(z_r) + 5\sin^2(z_r)\right)\right)}{\cos^6(z_r)},$$

are in linear forms. We set

$$\psi = \frac{1 + 2\sin^2(z_r)}{\cos^5(z_r)}, \quad \phi = 0, \quad \chi = 0, \quad \kappa = 0,$$
$$l = \frac{z' \sin(z_r)\left(5 + 2\left(2\cos^2(z_r) + 5\sin^2(z_r)\right)\right)}{\cos^6(z_r)}, \quad q = 0, \quad s = 0, \quad v = 0, \quad w = 0.$$

Table 2 shows the absolute errors, maximum absolute errors and computation time achieved using the non-overlapping and overlapping methods with $h = 1/100$ over twenty iterations. The results of the non-overlapping and overlapping methods were compared to those obtained with ITPB09[3]. The table demonstrates that the overlapping method gives excellent numerical results compared to ITPB09[3] and the non-overlapping method, in terms of absolute and maximum absolute errors. Table 3 displays the maximum absolute errors and computation time with $h = 1/100$ on the closed interval $[0,\, \frac{\pi}{4}]$, obtained using the overlapping adaptive step-size approach across three different tolerances ($Tol = 10^{-18}, 10^{-20}$ and 10^{-22}). The overlapping adaptive step-size approach achieves higher accuracy with smaller tolerances, but it also demands more computation time as the tolerance is reduced. Figure 5 depicts the convergence graph of both the non-overlapping and overlapping methods, showing that the overlapping method converged faster after the fourth iterations.

Example 6.2. Consider the fourth-order nonlinear IVP [4]

$$z^{iv} = (z')^2 - zz''' - 4t^2 + e^t\left(1 - 4t + t^2\right), \quad 0 \le t \le 1,$$
$$z(0) = 1, \quad z'(0) = 1, \quad z''(0) = 3 \quad z'''(0) = 1,$$

with the exact solution given as

$$z(t) = t^2 + e^t.$$

TABLE 2. Absolute error ($|z(t_{n+p_i}) - z_{n+p_i}|$) and maximum absolute error for Example 6.1 with $h = 0.01$, $t \in [0, \frac{\pi}{4}]$.

t	ITPBO9 [3]	Non-Overlapping	Overlapping
0.1	0	2.13268×10^{-25}	9.34272×10^{-26}
0.2	5.55111×10^{-17}	2.20666×10^{-24}	4.05762×10^{-25}
0.3	1.11022×10^{-16}	1.22148×10^{-23}	9.42129×10^{-25}
0.4	3.33067×10^{-16}	6.28176×10^{-23}	1.72996×10^{-24}
0.5	4.44089×10^{-16}	3.81534×10^{-22}	3.01902×10^{-24}
0.6	4.44089×10^{-16}	3.32662×10^{-21}	8.53340×10^{-24}
0.7	5.55111×10^{-16}	5.42307×10^{-20}	1.35741×10^{-22}
MAXERR ($max_i|z(t_{n+p_i}) - z_{n+p_i}|$)		2.79297×10^{-18}	1.42930×10^{-20}
CPU Time		0.2355	0.3183

TABLE 3. Numerical results for Example 6.1 with $t \in [0, \frac{\pi}{4}]$.

h	Tol	Methods	MAXERR	CPU Time
$\frac{1}{100}$	10^{-18}	**Overlapping**	8.95273×10^{-22}	0.4892
	10^{-20}	**Overlapping**	1.11252×10^{-23}	0.6713
	10^{-22}	**Overlapping**	2.28150×10^{-26}	1.1004

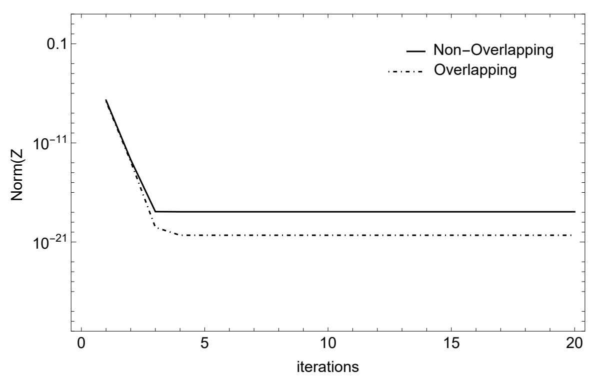

FIGURE 5. Convergence graph for Example 6.1 with $h = \frac{1}{100}$.

The fifth derivative is given by

$$z^v = (2z'' - z''')z' - z\left((z')^2 - zz'''\right) + \left(4t^2 - e^t(1 - 4t + t^2)\right)z - 8t + e^t(-3 - 2t + t^2).$$

Following Equations (5.12) and (5.13),

$$z_{r+1}^{iv} = (z_r')^2 - z_r z_r'' - 4t^2 + e^t\left(1 - 4t + t^2\right),$$

$$z_{r+1}^{v} = (2z_r'' - z_r''')z_r' - z_r\left((z_r')^2 - z_r z_r'''\right) + \left(4t^2 - e^t(1 - 4t + t^2)\right)z_r - 8t + e^t(-3 - 2t + t^2),$$

are in linear forms. We set

$$l = (z_r')^2 - z_r z_r'' - 4t^2 + e^t\left(1 - 4t + t^2\right), \quad q = 0, \quad s = 0, \quad v = 0, \quad w = 0,$$

TABLE 4. Absolute error ($|z(t_{n+p_i}) - z_{n+p_i}|$) and maximum absolute error for Example 6.2 with $h = \frac{1}{320}$, $t \in [0, 1]$.

t	I2PBDO6 [4]	Non-Overlapping	Overlapping
0.103125	0	4.27643×10^{-40}	9.19949×10^{-41}
0.206250	3.90992×10^{-16}	6.74423×10^{-39}	7.42356×10^{-40}
0.306250	7.24419×10^{-16}	3.25745×10^{-38}	2.39449×10^{-39}
0.406250	9.99369×10^{-16}	1.00346×10^{-37}	5.47436×10^{-39}
0.506250	1.29479×10^{-15}	2.40730×10^{-37}	1.03515×10^{-38}
0.603125	0	4.82338×10^{-37}	1.71014×10^{-38}
0.703125	1.51633×10^{-15}	8.85567×10^{-37}	2.64519×10^{-38}
0.803125	2.86323×10^{-15}	1.49761×10^{-36}	3.85039×10^{-38}
0.903125	4.56227×10^{-15}	2.37844×10^{-36}	5.35387×10^{-38}
MAXERR ($max_i|z(t_{n+p_i}) - z_{n+p_i}|$)		3.59487×10^{-36}	7.18511×10^{-38}
CPU Time		1.2767	1.9243

TABLE 5. Numerical results for Example 6.2 with $t \in [0, 1]$.

h	Tol	Methods	MAXERR	CPU Time
0.2		**Overlapping**	1.29631×10^{-16}	0.0479
	10^{-10}	**ODE45**	9.78817×10^{-12}	0.0142
		ODE23	2.41496×10^{-10}	0.0780
		Overlapping	6.04135×10^{-20}	0.0863
	10^{-12}	**ODE45**	1.18128×10^{-13}	0.0205
		ODE23	2.42339×10^{-12}	0.1561

$$\hat{l} = (2z_r'' - z_r''')z' - z_r\left((z_r')^2 - z_r z_r'''\right) - 8t + e^t(-3 - 2t + t^2), \quad \hat{q} = 4t^2 - e^t(1 - 4t + t^2),$$
$$\hat{s} = 0, \quad \hat{v} = 0, \quad \hat{w} = 0, \quad \hat{\lambda} = 0.$$

Table 4 presents the computation time, absolute errors and maximum absolute errors for the non-overlapping and overlapping methods, with $h = 1/320$ over twenty iterations. These results are compared with those of I2PBDO6 [4], which uses the same step-size. The overlapping method outperformed the two methods across all metrics. Its absolute errors and maximum absolute errors are smaller than those of the non-overlapping method and I2PBDO6 [4], demonstrating the advantages of the overlapping method in terms of accuracy and stability. Table 5 shows the maximum absolute errors and computation time achieved with the overlapping adaptive step-size approach. When compared with ODE23 and ODE45, the overlapping method performs effectively in terms of accuracy, with maximum absolute errors improving significantly as the tolerance (Tol) is reduced. However, this enhanced precision requires higher computation time. Figure 6 illustrates the convergence graph for the non-overlapping and overlapping methods. Both methods converge at the twelfth iterations, highlighting the good performance of the overlapping method.

Example 6.3. Consider a third-order linear system [3] given as

$$z_1''' = \frac{1}{68}\left(817z_1 + 1393z_2 + 448z_3\right), \quad z_1(0) = 2, \quad z_1'(0) = -12, \quad z_1''(0) = 20,$$

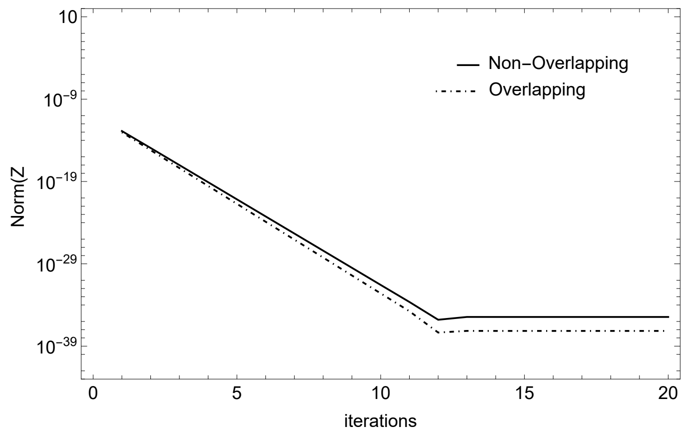

FIGURE 6. Convergence graph for Example 6.2 with $h = \frac{1}{320}$.

$$z_2''' = -\frac{1}{68}\left(1141z_1 + 2837z_2 + 896z_3\right), \quad z_2(0) = -2, \quad z_2'(0) = 28, \quad z_2''(0) = -52,$$
$$z_3''' = \frac{1}{136}\left(3059z_1 + 4319z_2 + 1592z_3\right), \quad z_3(0) = -12, \quad z_3'(0) = -33, \quad z_3''(0) = 5,$$

with the exact solution given as

$$z_1(t) = e^t - 2e^{2t} + 3e^{-3t},$$
$$z_2(t) = 3e^t + 2e^{2t} - 7e^{-3t},$$
$$z_3(t) = -11e^t - 5e^{2t} + 4e^{-3t}.$$

Table 6 compares the maximum absolute errors and computation time for different step-sizes ($h = 1/10, 1/40, 1/80$) over ten iterations on the closed interval [0,2] using the non-overlapping and overlapping methods. The results confirm the good accuracy of the overlapping method achieving smaller errors for all functions z_1, z_2, and z_3, despite its slightly higher computational cost. Table 7 presents the maximum absolute errors and computation time using the overlapping adaptive step-size approach for $h = 0.2$ and $h = 0.5$ with different tolerances (Tol). The results indicate that the overlapping method achieves the best accuracy. However, in terms of computation time, it is slightly slower than ODE45 but faster than ODE23. Figures 7, 8, and 9 display the convergence graphs for z_1, z_2, and z_3, respectively, showing that the overlapping method converged faster after the third iterations. Figure 10 depicts the exact and numerical solutions of the overlapping method for z_1, z_2, and z_3 with $h = 1/10$, demonstrating that the numerical and exact solutions coincide, emphasizing the efficiency of the overlapping method.

Example 6.4. Consider a fourth-order linear system [4]

$$z_1^{iv} = -z_2'', \quad z_1(0) = 1, \quad z_1'(0) = 1, \quad z_1''(0) = 1, \quad z_1'''(0) = 1,$$
$$z_2^{iv} = -z_1'', \quad z_2(0) = -1, \quad z_2'(0) = -1, \quad z_2''(0) = -1, \quad z_2'''(0) = -1,$$
$$z_3^{iv} = -z_3'' - z_3 - \cos(t), \quad z_3(0) = -1, \quad z_3'(0) = 0, \quad z_3''(0) = 1, \quad z_3'''(0) = 0,$$
$$z_4^{iv} = -z_4'' - z_4 - 2\cos(t), \quad z_4(0) = -2, \quad z_4'(0) = 0, \quad z_4''(0) = 2, \quad z_4''''(0) = 0,$$

with the exact solution given as

$$z_1(t) = e^t,$$
$$z_2(t) = -e^t,$$
$$z_3(t) = -\cos(t),$$
$$z_4(t) = -2\cos(t).$$

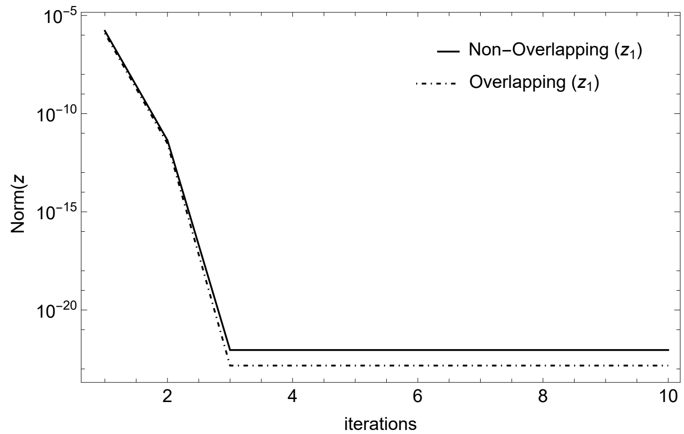

FIGURE 7. Convergence graph for z_1 in Example 6.3 with $h = \frac{1}{80}$.

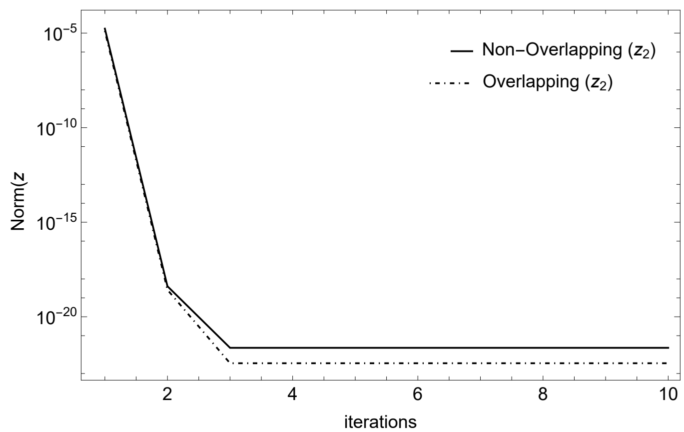

FIGURE 8. Convergence graph for z_2 in Example 6.3 with $h = \frac{1}{80}$.

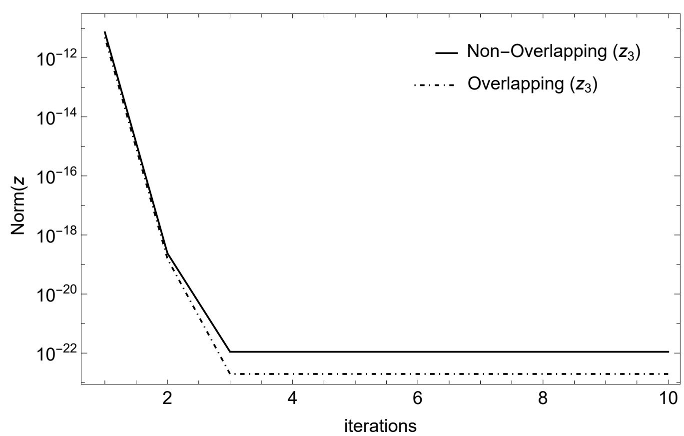

FIGURE 9. Convergence graph for z_3 in Example 6.3 with $h = \frac{1}{80}$.

TABLE 6. MAXERR ($max_i|\, z(t_{n+p_i}) - z_{n+p_i}\,|$) for z_1, z_2 and z_3 in Example 6.3 with $t \in [0,\, 2]$.

h	z	**Non-Overlapping**	**Overlapping**
0.1	z_1	1.21534×10^{-13}	9.47693×10^{-14}
	z_2	3.05899×10^{-13}	2.23912×10^{-13}
	z_3	1.42510×10^{-13}	1.23923×10^{-13}
CPU Time		0.1274	0.1381
0.025	z_1	1.04563×10^{-19}	3.01961×10^{-20}
	z_2	2.59635×10^{-19}	7.09723×10^{-20}
	z_3	1.25715×10^{-19}	3.98110×10^{-20}
CPU Time		0.3669	0.5986
0.0125	z_1	9.95333×10^{-23}	1.54919×10^{-23}
	z_2	2.46764×10^{-22}	3.63886×10^{-23}
	z_3	1.20002×10^{-22}	2.04450×10^{-23}
CPU Time		0.7585	1.1557

TABLE 7. Numerical results for Example 6.3 with $t \in [0,\, 2]$.

h	**Tol**	**Methods**	**MAXERR**	**CPU Time**
0.2	10^{-10}	**Overlapping**	1.41077×10^{-12}	0.1065
		ODE45	4.47552×10^{-9}	0.0338
		ODE23	2.66006×10^{-8}	0.2482
0.5	10^{-7}	**Overlapping**	2.20466×10^{-7}	0.0250
		ODE45	3.73085×10^{-6}	0.0135
		ODE23	2.62768×10^{-5}	0.0355

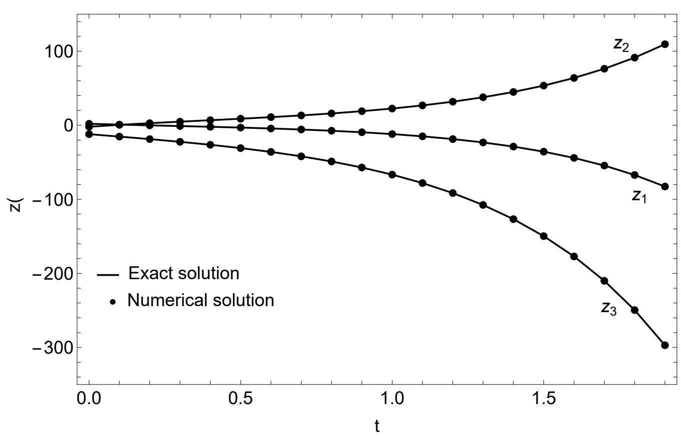

FIGURE 10. Numerical vs Exact solution for z_1, z_2 and z_3 in Example 6.3 with $h = \frac{1}{10}$, $t \in [0,\, 2]$.

Table 8 displays the maximum absolute errors and computation time, using the non-overlapping and overlapping methods. The overlapping method outperforms the non-overlapping method in terms of accuracy, yielding smaller maximum absolute errors across all step sizes. This confirms the good performance of the overlapping method. Table 9 presents the maximum absolute errors and computation time obtained using the overlapping adaptive step-size approach with $h = 0.2$ and $h = 0.5$ over ten iterations with specific tolerances (Tol). The overlapping adaptive step-size method achieves the best accuracy overall, particularly for $h = 0.2$, where its error is significantly lower than

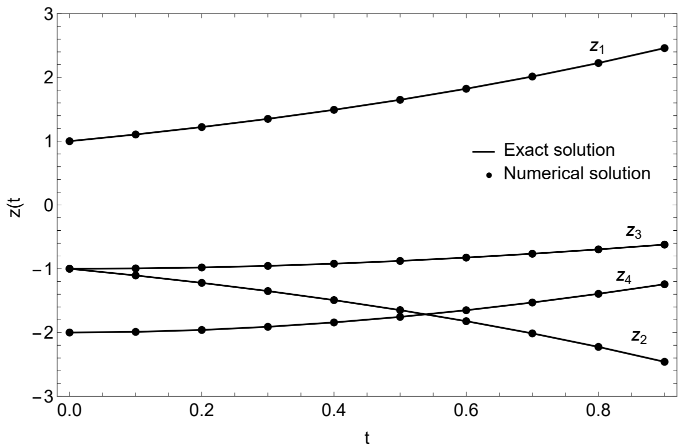

FIGURE 11. Numerical vs Exact solution for z_1, z_2, z_3 and z_4 in Example 6.4 with $h = \frac{1}{10}$, $t \in [0, 1]$.

TABLE 8. MAXERR $(max_i |\, z(t_{n+p_i}) - z_{n+p_i}\, |)$ for z_1, z_2, z_3 and z_4 in Example 6.4 with $t \in [0, 1]$.

h	z	**Non-Overlapping**	**Overlapping**
0.1	z_1	9.09267×10^{-21}	4.31883×10^{-21}
	z_2	9.09267×10^{-21}	4.31883×10^{-21}
	z_3	5.94868×10^{-21}	3.34199×10^{-21}
	z_4	1.18974×10^{-20}	6.68398×10^{-21}
CPU Time		0.1031	0.1223
0.025	z_1	5.80881×10^{-27}	8.75993×10^{-28}
	z_2	5.80881×10^{-27}	8.75993×10^{-28}
	z_3	4.13562×10^{-27}	7.62448×10^{-28}
	z_4	8.27125×10^{-27}	1.52490×10^{-27}
CPU Time		0.4166	0.6116
0.0125	z_1	5.29115×10^{-30}	4.16465×10^{-31}
	z_2	5.29115×10^{-30}	4.16465×10^{-31}
	z_3	3.81784×10^{-30}	3.69405×10^{-31}
	z_4	7.63567×10^{-30}	7.38810×10^{-31}
CPU Time		0.8461	1.3089

that of ODE45 and ODE23. For $h = 0.5$, the overlapping method demonstrates competitive performance in both accuracy and computation time. Figure 11 illustrates the exact and numerical solutions for z_1, z_2, z_3, and z_4 obtained through the overlapping method with $h = 1/10$. The results demonstrate a strong agreement between the numerical and exact solutions, emphasizing the efficiency and accuracy of the overlapping method.

Example 6.5. Consider a third-order nonlinear system [18] given as

$$z_1''' = \frac{1}{2}\, z_3\, z_2'\, e^{4t}, \quad z_1(0) = 1, \quad z_1'(0) = -1, \quad z_1''(0) = 1,$$
$$z_2''' = \frac{8}{3}\, z_1\, z_3'\, e^{2t}, \quad z_2(0) = 1, \quad z_2'(0) = -2, \quad z_2''(0) = 4,$$
$$z_3''' = 27\, z_2\, z_1', \quad z_3(0) = 1, \quad z_3'(0) = -3, \quad z_3''(0) = 9,$$

TABLE 9. Numerical results for Example 6.4 with $t \in [0, 1]$.

h	Tol	Methods	MAXERR	CPU Time
0.2		**Overlapping**	2.79064×10^{-17}	0.1464
	10^{-15}	**ODE45**	1.68754×10^{-14}	0.0595
		ODE23	1.98064×10^{-13}	1.7617
0.5		**Overlapping**	4.08033×10^{-13}	0.0528
	10^{-13}	**ODE45**	1.89182×10^{-13}	0.0299
		ODE23	8.96172×10^{-13}	0.7089

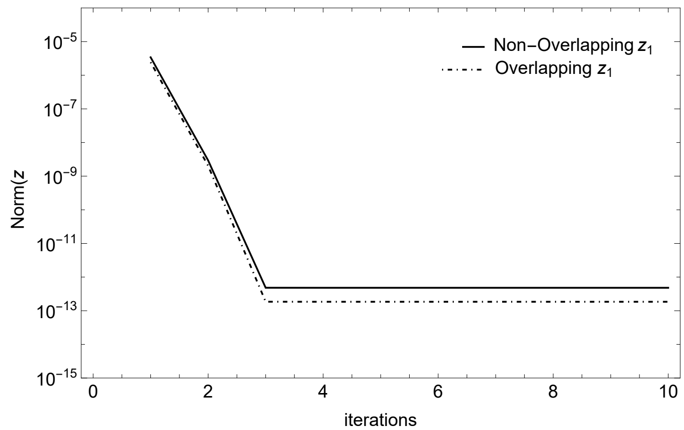

FIGURE 12. Convergence graph for z_1 in Example 6.5 with $h = \frac{1}{10}$, $t \in [0, 3]$.

with the exact solution given as

$$z_1(t) = e^{-t},$$
$$z_2(t) = e^{-2t},$$
$$z_3(t) = e^{-3t}.$$

Table 10 presents the maximum absolute errors and computation time obtained using $h = 1/10$ and ten iterations on the closed interval $[0, \hat{b}]$, with different values of $\hat{b}$ for tolerances $Tol = 10^{-8}$ and 10^{-12}. The overlapping adaptive step-size approach, FDTHBS [18] and Matlab solvers ODE45 and ODE23, were compared. For tolerance $Tol = 10^{-8}$ and $\hat{b} = 3$, the overlapping method yields much smaller maximum errors (10^{-13}). At $Tol = 10^{-12}$ and $\hat{b} = 6$, the overlapping method maintains its accuracy advantage over ODE45, ODE23 and FDTHBS [18]. Despite the slightly longer computation time for the overlapping method, its superior accuracy is clear. Figures 12, 13, and 14 depict the convergence graphs for z_1, z_2 and z_3 obtained using the overlapping and non-overlapping methods on the closed interval $[0, 3]$ with $h = 1/10$. The graph shows that the overlapping converges faster than the non-overlapping method at the third iterations.

7. CONCLUSIONS

In this study, an overlapping adaptive step-size multi-derivative HBM was developed for higher-order IVPs. The method utilized the second-to-last intra-step point from the previous step in each integrating block. Within a two-step block, three intra-step points were taken into consideration. In addition, the method incorporated an adaptive step-size formulation to enhance its effectiveness. For solving nonlinear IVPs, the modified-Picard iteration technique was applied to linearize the problems. The effectiveness of the method was validated via numerical experiments, which were compared with similar non-overlapping and other established methods as presented in Tables 2–10 and Figures 5–14.

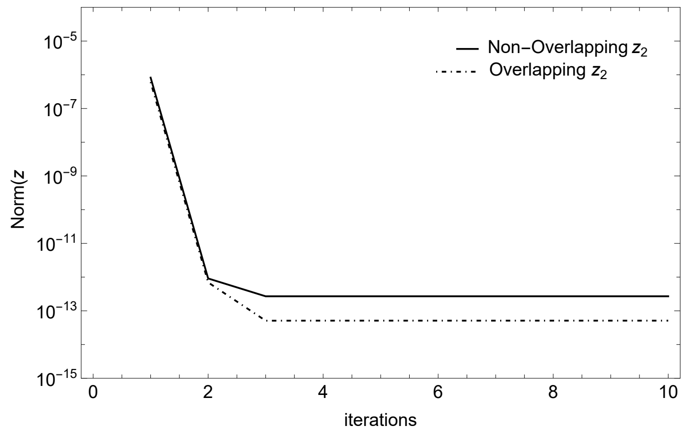

FIGURE 13. Convergence graph for z_2 in Example 6.5 with $h = \frac{1}{10}$, $t \in [0, 3]$.

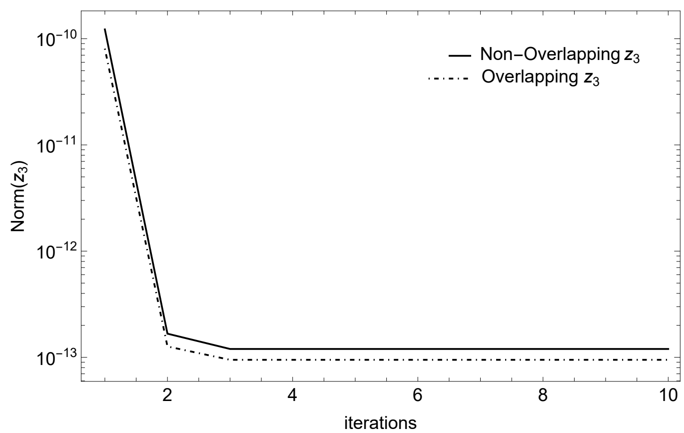

FIGURE 14. Convergence graph for z_3 in Example 6.5 with $h = \frac{1}{10}$, $t \in [0, 3]$.

TABLE 10. Numerical results for Example 6.5 with $h = \frac{1}{10}$, $t \in [0, \hat{b}]$.

Tol	$\hat{b}$	Methods	MAXERR(z_1)	MAXERR(z_2)	MAXERR(z_3)	CPU Time
10^{-8}	3	**Overlapping**	1.9141×10^{-13}	1.9928×10^{-13}	1.5908×10^{-13}	0.1850
		ODE45	1.8466×10^{-8}	2.5643×10^{-9}	1.7200×10^{-9}	0.0152
		ODE23	1.7418×10^{-7}	2.9474×10^{-8}	6.0908×10^{-9}	0.0464
		FDTHBS [18]	4.3654×10^{-8}	3.0841×10^{-8}	2.4332×10^{-8}	0.1726
10^{-12}	6	**Overlapping**	5.9266×10^{-14}	2.8141×10^{-15}	2.5966×10^{-17}	1.3578
		ODE45	6.1205×10^{-8}	3.6520×10^{-9}	5.5952×10^{-11}	0.0913
		ODE23	6.1882×10^{-8}	4.3340×10^{-9}	8.6448×10^{-11}	0.8425
		FDTHBS [18]	4.5391×10^{-7}	2.8727×10^{-8}	4.5412×10^{-10}	0.5646

These experiments confirmed that the overlapping adaptive step-size multi-derivative HBM performs much better in terms of accuracy than the Matlab solvers ODE23 and ODE45 as well as the non-overlapping methods in the literature. The estimation error and adaptive step-size strategy presented in this study successfully overcome the limitations of existing numerical methods for solving IVPs, which often use small fixed step-sizes that require unnecessarily small steps across the entire integration interval. The adaptive strategy used in the method introduces specific challenges, such as the need to estimate local errors accurately, determine whether to accept or reject a step, and decide on a new step size. More research is needed to extend the applicability of the overlapping adaptive step-size technique for boundary value problems (BVPs), bivariate overlapping HBM for PDEs and generalized multi-point overlapping HBM for IVPs and BVPs.

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