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On the numerical solution of the Bagley-Torvik equation using the Müntz-Legendre wavelet collocation method

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Abstract

The main goals of this work are to solve the Bagley-Torvik (BT) equation using an effective scheme and to find its numerical solution. The scheme uses the collocation method based on the Müntz-Legendre (ML) wavelets. To apply the method, after approximating the unknown solution by mapping it to the wavelet space, we replace it in the desired equation and then obtain the residual using the operational matrices of the derivative and the Caputo fractional derivative (CFD).
Applying the collocation method results in a linear algebraic system. To implement the collocation method, either Chebyshev or Legendre roots serve as collocation points, or uniformly spaced grids are used.
The error analysis is investigated, and some numerical examples are presented to show the scheme's accuracy and effectiveness. Thanks to the flexibility of ML wavelets and the method's structure, we can sometimes obtain the exact solution.

Keywords. Fractional differential equation, Wavelet collocation method, Müntz-Legendre wavelets, Bagley–Torvik equation.

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1. Introduction

Engineering and other disciplines have seen a growing utilization of fractional calculus for explaining various physical phenomena. Various equations involving fractional derivatives have been extensively studied, leading to the emergence of numerous mathematical algorithms to solve them, including the wavelet spectral element [5], the wavelet method [40, 41], Adomian decomposition [8], the finite element-meshfree method [21], multi-step methods [11], implicit integration factor method [44], cubic Hermit spline method [12], Müntz-Legendre Petrov-Galerkin method [33], among others.

In [5], the authors established the necessary and sufficient conditions for the equation

$$\begin{aligned} &{}^{c}\mathcal{D}^{\alpha}_{a^+}(x)(t) = f[t, x(t), {}^{c}\mathcal{D}^{\alpha_1}_{a^+}(x)(t), \cdots, {}^{c}\mathcal{D}^{\alpha_\sigma}_{a^+}(x)(t)], \quad t \in [a, b], \\ &\quad x^{(\kappa)}(a) = b_\kappa, \quad b_\kappa \in \mathbb{R}, \;\; \kappa = 0, 1, \cdots, n-1, \end{aligned}$$

to admit a unique solution. Additionally, an algorithm was presented to reduce the computational cost using the wavelet properties [5]. Bin Jebreen et al. [13] successfully solved the fractional Cauchy problem using the wavelet collocation technique. The study of multi-order fractional differential equations (FDEs) was conducted in [16], where the Galerkin method was applied, using fractional-order Legendre functions as bases. In [9], multi-order FDEs were

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investigated via the fractional-Lucas optimization method. For further studies on numerical method for FDEs, see [1, 2, 38].

The objective of this paper is to introduce an effective algorithm for solving a well-known fractional differential equation, known as the BT equation:

$$ax''(t) + b^c\mathcal{D}_0^\beta(x)(t) + cx(t) = f(t), \quad \beta = 3/2, \quad 0 \le t \le 1, \tag{1.1}$$

with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1, \tag{1.2}$$

where $a \neq 0$, b, and c are constants, and ${}^c\mathcal{D}_0^\beta$ denotes the Caputo fractional derivative

$${}^c\mathcal{D}_0^\beta(x)(t) = \frac{1}{\Gamma(\kappa-\beta)}\int_0^t \frac{x^{(\kappa)}(z)dz}{(t-z)^{\beta-\kappa+1}}, \quad \kappa = -[-\beta]. \tag{1.3}$$

This equation arises in modeling a rigid plate bounded by a Newtonian fluid, first introduced by Torvik and Bagley [42]. They obtained notable results, showing that the interior oscillations of a rigid plate immersed in a Newtonian fluid do not establish a relationship between a retarding force and the velocity. Instead, the retarding force is proportional to the fractional derivative of order 3/2 of the displacement. Their findings suggest that fractional derivatives naturally emerge while in the study of real material behavior, indicating that their use in constitutive relationships is not arbitrary.

In [27], a thorough discussion is provided on the existence of a unique solution for this equation under homogeneous initial conditions. In this work, the fractional derivative is of the Riemann–Liouville types, while homogeneous initial conditions equip the equation. For nonhomogeneous initial conditions, Diethelm et al. [10] considered the equation with the Caputo fractional derivative (CFD) and proposed a solution using a fractional linear multi-step method, reformulating it into a system of fractional differential equations. The existence and uniqueness of the solution were further examined in [22], while [27] derived the analytical solution as follows:

$$x(t) = \int_0^t K(t-z)f(z)dz, \tag{1.4}$$

where

$$K(t) = 1/a\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{a}{c}\right)^i t^{2i+1}E^{(i)}_{1/2,2+3i/2}\left(\frac{-b}{a}\sqrt{t}\right),$$

and $E_{k,l}(t)$ denotes the Mittag–Leffler function. Notably, for general functions f, evaluating this integral analytically is often impractical, making numerical methods a viable alternative. Yüzbaşi [43] employed the Bessel collocation method to solve the equation with boundary conditions, while [7] introduced a generalized Taylor collocation method. Leszczyński et al. [20] reformulated the BT equation into a system of ordinary differential equations linked to Abel-integral equations, proposing a numerical scheme for its solution. For cases where the source function f is a Heaviside function, the Adomian decomposition method was applied in [31]. Further studies on solving the BT equation can be found in [15, 18, 24, 25].

Recent studies show that wavelets effectively solve differential equations and represent various operators [4]. Wavelet theory comprises two main families: scalar wavelets and multi-wavelets. Unlike scalar wavelets, which rely on a single generator, multi-wavelets employ multiple generators within a multi-resolution analysis framework [14]. This grants them advantages such as symmetry, closed-form expressions, high vanishing moments, and orthogonality.

The Alpert multi-wavelet is a well-known multi-wavelet with various applications in image processing and numerical analysis [4, 34–37]. Another notable class, Müntz-Legendre (ML) wavelets, has recently been applied to fractional optimal control problems [28], multi-order fractional differential equations [13], and fractional pantograph equations [30].

After this brief introduction, the organization of the remaining sections will be as follows: Section 2 introduces the ML wavelets and their properties. Solving the Bagley-Torvik equation using the wavelet collocation method is the objective of section 3. Additionally, we shall present the error analysis for the method. We conduct several numerical experiments to demonstrate the accuracy and usefulness of the method, in section 4. Finally, brief conclusions are included in section 5.

2. Müntz-Legendre wavelets

Suppose that the space $F(\mathcal{B})$ is spanned by a sequence of functions $\{t^{\beta_n}\}_{n=0}^{\infty}$

$$F(\mathcal{B}) := \bigcup_{n=0}^{\infty} F_n(\mathcal{B}) = span\{t^{\beta_n}, n = 0, 1, \ldots\}. \tag{2.1}$$

where $F_n(\mathcal{B}) := span\{t^{\beta_0}, t^{\beta_1}, \ldots, t^{\beta_n}\}$ and $\mathcal{B} = \{0 = \beta_0 < \beta_1 < \ldots\}$ is an increasing sequence [3]. To verify $\overline{F(\mathcal{B})} = C[0,1]$, the condition

$$\sum_{\beta_n > 0} \frac{1 + \log \beta_n}{\beta_n} = \infty, \tag{2.2}$$

is sufficient, and the condition

$$\lim_{n\to\infty} \frac{\beta_n}{n \log n} = 0, \tag{2.3}$$

is necessary.

Thus, $F(\mathcal{B})$ is dense in $C[0,1]$ (the space of continuous functions on $[0,1]$). These criteria were introduced by S. N. Bernstein. It is also worth mentioning that he proposed the existence and uniqueness conditions

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty, \tag{2.4}$$

where $\mathcal{B} = \{0 = \beta_0 < \beta_1 < \ldots\}$. This conjecture was verified two years later by Müntz [26]. Note that the sufficient and necessary conditions for $\overline{F(\mathcal{B})} = L^2(0,1)$ are specified in [39].

Using the functions $\{t^{\beta_n}\}_{n=0}^{\infty}$ as bases is not advisable. Consequently, the Müntz-Legendre (ML) functions are defined to ensure orthogonality and straightforward evaluation.

Assuming that χ represents a simple contour enclosing all zeros of the integrand's denominator, the ML polynomials can be expressed in closed form as [6, 39]:

$$ML_n(t;\mathcal{B}) := \frac{1}{2\pi i} \int_{\chi} \prod_{k=1}^{n-1} \frac{z + \beta_k + 1}{z - \beta_k} \frac{t^k}{z - \beta_n} dz, \tag{2.5}$$

The ML polynomials can then be determined as:

$$L_n(t;\mathcal{B}) = \sum_{k=0}^{n} c_{k,n} t^{\beta_k}, \quad t \in [0,1], \tag{2.6}$$

where the constant β_k are detrmined by $\beta_k := \{k\eta : \eta \in \mathbb{R}, k = 0, \ldots n\}$, and the sequence $\{\beta_k\}_{k=1}^{\infty}$ is ascending. Furthermore, the coefficients $c_{k,n}$ are specified by [6]

$$c_{k,n} := \frac{\prod_{i=0}^{n-1} (\beta_k + \beta_i + 1)}{\prod_{i=0, i\neq k}^{n} (\beta_k - \beta_i)}. \tag{2.7}$$

From Theorem 2.4 in [6], it follows that the ML polynomials are orthogonal satisfying:

$$\int_0^1 L_{n'}(t) L_{n''}(t) dt = \delta_{n',n''}/(\beta_n + \beta_{n''} + 1), \tag{2.8}$$

where $L_n(t) := L_n(t;\mathcal{B})$.

For the multiplicity parameter $\nu \in \mathbb{N}$ and refinement level $s \in \mathbb{N}_0$, there exists a sequence of nested subspaces $\{V_s\}_{s\in\mathbb{N}_0} \subset L^2([0,1])$ spanned by $\phi_{s,a}^n$ (see, multi-resolution analysis (MRA) [23]), i,e.,

$$V_s = span\{\phi_{s,a}^n := \phi^n(2^s - a) : \quad a \in \mathcal{M},\ n \in \mathcal{V}\}, \tag{2.9}$$

in which $\mathcal{V} := \{0, 1, \dots, \nu - 1\}$ and $\mathcal{M} := \{0, 1, \dots, 2^s - 1\}$.

Using ML polynomials $L_n(t)$, the ML wavelets defined by [13] as:

$$\phi_{s,a}^n = \begin{cases} 2^{s/2}\sqrt{2\beta_n + 1} L_n(2^s t - a), & \frac{a}{2^s} \leq t \leq \frac{a+1}{2^s}, \\ 0, & otherwise. \end{cases} \tag{2.10}$$

To approximate a function $x \in L^2[0,1]$, we introduces the projection operator $\mathcal{P}_s : L^2[0,1] \to V_s$ as:

$$x(t) \approx \mathcal{P}_s(x)(t) = \sum_{a=0}^{2^s-1} \sum_{n=0}^{\nu-1} x_{a,n} \phi_{s,a}^n(t) = X^T \mathrm{F}(t) \in V_s, \tag{2.11}$$

where $[\mathrm{F}(t)]_{a\nu+n+1,1} := \phi_{s,a}^n(t)$, and

$$x_{a,n} = \langle x, \phi_{s,a}^n \rangle = \int_0^1 x(t) \phi_{s,a}^n(t) dt. \tag{2.12}$$

The following lemma provides error bounds for the approximation (2.11) [29].

Lemma 2.1. *Let $\nu \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Assume that for any $\omega < \nu$, $x \in H^\omega[0,1]$, then*

$$\|x - \mathcal{P}_s(x)\|_2 \leq c(2^{s-1})^{-\omega} (\nu - 1)^{-\omega} \|x^{(\omega)}\|_2, \tag{2.13}$$

and for $\omega' \geq 1$:

$$\|x - \mathcal{P}_s(x)\|_{H^{\omega'}([0,1])} \leq c(\nu - 1)^{2\omega' - \frac{1}{2} - \omega} (2^{s-1})^{\omega' - \omega} \|x^{(\omega)}\|_2, \tag{2.14}$$

where $H^\omega([0,1])$ denotes the Sobolev space with norm:

$$\|x\|_{H^\omega([0,1])} = \left(\sum_{i=0}^{\omega} \|x^{(i)}\|_2^2 \right)^{1/2}. \tag{2.15}$$

2.1. **Matrix representation of fractional integration.** The objective of this subsection is to derive a square matrix I_β that represent the fractional integral operator (FIO) numerically. The matrix elements are determined by approximating the FIO action on the ML wavelet basis functions:

$$\mathcal{P}_s(\mathcal{I}_0^\beta)(\mathrm{F}(t)) \approx I_\beta(t) \mathrm{F}(t), \tag{2.16}$$

where $\mathcal{I}_0^\beta$ denotes the FIO, i.e.,

$$\mathcal{I}_0^\beta(x)(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - z)^{\beta - 1} x(z) dz, \quad \beta \in \mathbb{R}^+, \quad 0 \leq t \leq 1, \tag{2.17}$$

whit $\Gamma(\beta)$ being the Gamma function.

In the sequel, the piecewise Taylor functions of fractional order (FPTFs) will be introduced to help us calculate the entries of I_β, i.e.,

$$p_{s,a}^n(t) = \begin{cases} t^{\beta_n}, & \frac{a}{2^s} \leq t \leq \frac{a+1}{2^s}, \\ 0, & otherwise, \end{cases} \quad a \in \mathcal{M}, n \in \mathcal{V}, s \in \mathbb{N}_0. \tag{2.18}$$

Let $P(t)$ be the vector function whose $(a\nu+n+1)$-th component is $p_{s,a}^n(t)$. The connection between the ML wavelets $\phi_{s,a}^n(t)$ and FPTFs is given by:

$$\mathrm{F}(t) = \Upsilon^{-1} P(t), \tag{2.19}$$

where the transformation matrix Υ has entries:

$$\Upsilon_{i,j} = \langle P_j(t), \mathrm{F}_i(t) \rangle = \int_0^1 \mathrm{F}_i(t) P_j(t) dt, \quad i, j = 1 : N, \qquad N = 2^s \nu. \tag{2.20}$$

Let W be a ν-dimensional vector whit components $\{t^{\beta_n}\}_{n=1}^{\nu}$. We observe that:

$$P(t) = [W, \dots, W]^T. \tag{2.21}$$

From the definition of the fractional integral operator and known results [17], we have:

$$\mathcal{I}_0^{\beta}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{2.22}$$

Thus, it follows from (2.18) and (2.22) that

$$\mathcal{I}_0^{\beta}(P_i)(t) = \frac{\Gamma(\beta_i+1)}{\Gamma(\beta_i+\beta+1)} x^{\beta_i+\beta}, \quad i = 1 : N. \tag{2.23}$$

This leads to the matrix representation:

$$\mathcal{I}_0^{\beta}(P)(t) = I_{P,\beta}(t)P(t), \tag{2.24}$$

where

$$I_{P,\beta}(t) = \begin{pmatrix} B_\beta(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_\beta(t) \end{pmatrix}, \tag{2.25}$$

whit $B_\beta(t) := t^{\beta} K$, (satisfying $\mathcal{I}_0^{\beta}(W)(t) = B_\beta(t)W(t)$) and

$$\begin{cases} (K)_{i,j} = \Gamma(\beta_i+1)/\Gamma(\beta_i+\beta+1), & i = j, \\ (K)_{i,j} = 0, & i \neq j, \\ i, j = 1 : N. \end{cases} \tag{2.26}$$

The fractional integration matrix $I_\beta(t)$ is obtained through:

$$\begin{aligned} \mathcal{P}_s(\mathcal{I}_0^{\beta})(\mathrm{F}(t)) &= \mathcal{P}_s(\mathcal{I}_0^{\beta})(\Upsilon^{-1}P(t)) \\ &= \Upsilon^{-1} I_{P,\beta}(t)P(t) \\ &= \Upsilon^{-1} I_{P,\beta}(t)\Upsilon \mathrm{F}(t). \end{aligned} \tag{2.27}$$

Thus, we derive:

$$I_\beta(t) := \Upsilon^{-1} I_{P,\beta}(t)\Upsilon. \tag{2.28}$$

2.2. **Matrix representation of CFD.** Given $\beta \in \mathbb{R}^{+}$, assume that $AC^{\beta}([0,1])$ denotes the space of functions such that

$$AC^{\beta}[0,1] = \{x : [0,1] \to \mathbb{C}, \quad \& \quad \mathcal{D}^{(\beta-1)}(x) \in AC[0,1]\},$$

where $\mathcal{D} = \frac{d}{dt}$ denotes the derivative operator.

For any $x(t) \in AC^{\beta}([0,1])$, we have

$$^{c}\mathcal{D}_0^{\beta}(x)(t) = \frac{1}{\Gamma(\kappa-\beta)} \int_0^t \frac{x^{(\kappa)}(z)dz}{(t-z)^{\beta-\kappa+1}} =: \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^{\kappa}(x)(t), \tag{2.29}$$

where $\kappa = -[-\beta]$. Consequently,

$$^{c}\mathcal{D}_0^{\beta}(z^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta}, \quad (\alpha > \kappa). \tag{2.30}$$

The goal is to construct a square matrix D_β satisfying

$$^{c}\mathcal{D}_0^{\beta}(\mathrm{F}(t)) \approx D_\beta \mathrm{F}(t). \tag{2.31}$$

To compute the elements of D_β, we use $^{c}\mathcal{D}_0^{\beta} = \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^{\kappa}$ (due to the CFD definition) as follows:

$$\begin{aligned} ^{c}\mathcal{D}_0^{\beta}(\mathrm{F}(t)) = \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^{\kappa}(\mathrm{F}(t)) &\approx \mathcal{I}_0^{\kappa-\beta}(D^{\kappa}\mathrm{F}(t)) \\ &= D^{\kappa}\mathcal{I}_0^{\kappa-\beta}(\mathrm{F}(t)) \approx D^{\kappa} I_{\kappa-\beta}(\mathrm{F}(t)). \end{aligned}$$

Here D represents the matrix form of $\mathcal{D}$ (see, e.g., [33]). Thus,

$$D_\beta := D^\kappa I_{\kappa-\beta}. \tag{2.32}$$

This approach reduces computational cost by avoiding direct CFD operations on ML wavelets.

3. Wavelet collocation method for the BT equation

The BT equation is:

$$ax''(t) + b^c\mathcal{D}_0^{3/2}(x)(t) + cx(t) = f(t), \quad 0 \le t \le 1, \tag{3.1}$$

with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1. \tag{3.2}$$

To apply the collocation method, the unknown solution is first mapped into the approximation space V_s by using the operator $\mathcal{P}_s$, i,e.,

$$x(t) \approx \mathcal{P}_s(x)(t) = X^T\mathrm{F}(t) := x_N(t). \tag{3.3}$$

The N-dimensional vector X holds the unknowns that require specification.
Inserting $x_N(t)$ into the Equation (3.1) gives rise to equation

$$ax''_N(t) + b^c\mathcal{D}_0^{3/2}(x_N)(t) + cx_N(t) = f(t), \tag{3.4}$$

Using operational matrices $D_{3/2}$ and D, we approximate the functions $x''_N(t)$, ${}^c\mathcal{D}_0^{3/2}(x_N)(t)$, and $f(t)$ through the projection operator $\mathcal{P}_s$ as follows:

$$\begin{aligned}
&x''_N(t) \approx \mathcal{P}_s(x''_N)(t) = X^T D^2\mathrm{F}(t),\\
&{}^c\mathcal{D}_0^{3/2}(x_N)(t) \approx \mathcal{P}_s({}^c\mathcal{D}_0^{3/2}(x_N)(z))(t) = X^T D_{3/2}\mathrm{F}(t),\\
&f(t) \approx \mathcal{P}_s(f)(t) = F^T\mathrm{F}(t).
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.3) yields the residual:

$$r_s(t) := \left(aX^TD^2 + bX^TD_{3/2} + cX^T - F^T\right)\mathrm{F}(t). \tag{3.6}$$

The collocation method aims to minimize $r_s(t)$. By selecting collocation points $\{t_n\}_{n=1}^N \in [0,1]$, the method leads to a linear system that satisfies $r_s(t_n) = 0$. This is equivalent to solving the system

$$X^T\left(aD^2 + bD_{3/2} + cI\right) = F^T, \tag{3.7}$$

and finding the unknown coefficients X. In other words, there is a linear system

$$AX = F, \tag{3.8}$$

where $A := (aD^2 + bD_{3/2} + cI)^T$, that must be solved to gain the unknowns. To use the initial conditions, we replace the first and the second elements of $r_s(t_i)$ by

$$r_s(t_1) := X^T\mathrm{F}(0) - x_0, \quad r_s(t_2) := X^T D\mathrm{F}(0) - x_1,$$

respectively.

It is to be noted here that in this study, the collocation points are either chosen as the roots of Chebyshev or Legendre polynomials, or as uniform points from $[0,1]$.

More abstractly, let $C([0,1]) \to V_s$, there is a projection operator Q_N. To be more precise, the projection $Q_N(x)$ maps w into V_S such that interpolates it at the points $\{t_N\}_{N=1}^N \in [0,1]$. As a result, $Q_N r_s = 0$ can be used instead of $r_s(t_n)$. Equivalently, one can write

$$Q_N\left(\left(aX^TD^2 + bX^TD_{3/2} + cX^T\right)\mathrm{F}(t)\right) = Q_N\left(F^T\mathrm{F}(t)\right). \tag{3.9}$$

3.1. **Error analysis.**

Lemma 3.1. *([17]). For $\beta > 0$ and $1 \leq q \leq \infty$, the operator $\mathcal{I}_0^\beta$ is bounded in $L^q([0,1])$:*

$$\|\mathcal{I}_0^\beta(x)\|_q \leq \frac{1}{\Gamma(\beta+1)}\|x\|_q. \tag{3.10}$$

Theorem 3.2. *Let r_s be the residual defined in (3.6). The error of the presented method for solving the BT equation (3.1) satisfies:*

$$\|x - x_N\| \leq C\|r_s^{(N+1)}(t)\|, \tag{3.11}$$

where x_N is the approximate solution and C is a constant.

Proof. Subtracting (3.9) from (3.1) leads to

$$ax''(t) - aQ_N(x''_N)(t) + b^c\mathcal{D}_0^{3/2}(x)(t) - bQ_N(^c\mathcal{D}_0^{3/2}(x_N))(t) + cx(t) - cQ_N(x_N)(t) = f(t) - Q_N(f)(t). \tag{3.12}$$

Equivalently, one can write

$$\begin{aligned} &ax''(t) - ax''_N + ax''_N - aQ_N(x''_N)(t) + b^c\mathcal{D}_0^{3/2}(x)(t) - b^c\mathcal{D}_0^{3/2}(x_N)(t) + b^c\mathcal{D}_0^{3/2}(x_N)(t) \\ &\quad - bQ_N(^c\mathcal{D}_0^{3/2}(x_N))(t) + cx(t) - cx_N(t) + cx_N(t) - cQ_N(x_N)(t) = f(t) - Q_N(f)(t). \end{aligned} \tag{3.13}$$

Given $e_N = x - x_N$, it is easy to gain the following relation

$$\begin{aligned} &ae''_N(t) + a\,(I - Q_N)\,(e''_N)(t) + b^c\mathcal{D}_0^{3/2}(e_N)(t) + b\,(I - Q_N)\,(^c\mathcal{D}_0^{3/2}(e_N))(t) + ce_N(t) \\ &\qquad + c\,(I - Q_N)\,(e_N)(t) = (I - Q_N)\,(f)(t). \end{aligned} \tag{3.14}$$

Considering

$$r_s(x) = ax_N(t) + b^c\mathcal{D}_0^{3/2}(x_N)(t) + cx_N(t) - f(t), \tag{3.15}$$

and making some simplifications in Equation (3.14), it is straightforward to reach the relation

$$ae''_N(t) + b^c\mathcal{D}_0^{3/2}(e_N)(t) + ce_N(t) = (I - Q_N)\,(r_s)(t). \tag{3.16}$$

It follows from (3.16) that

$$e''_N(t) = \frac{1}{a}\,(I - Q_N)\,(r_s)(t) - \frac{b}{a}{}^c\mathcal{D}_0^{3/2}(e_N)(t) - \frac{c}{a}e_N(t). \tag{3.17}$$

Taking norms and applying Lemma 3.1:

$$\|e''_N(t)\| \leq \frac{c}{a}\|e_N(t)\| + \frac{1}{a}\|\,(I - Q_N)\,(r_s)(t)\| + \frac{b}{a\Gamma(3/2)}\|e''_N(t)\|. \tag{3.18}$$

Letting $\rho := 1 - \frac{b}{a\Gamma(3/2)} \geq 0$, we can rewrite (3.18) as follows;

$$\|e''_N(t)\| \leq \frac{c}{a\rho}\|e_N(t)\| + \frac{1}{a\rho}\|\,(I - Q_N)\,(r_s)(t)\|. \tag{3.19}$$

On the other hand, using the Cauchy formula for repeated integration, we have

$$e_N(t) = \int_0^t (t - x)e''_N(x)dx. \tag{3.20}$$

Taking the norm, one can show that

$$\|e_N(t)\| \leq M\|e''_N(x)\|. \tag{3.21}$$

Substituting (3.19) into (3.21), leads to

$$\|e_N(t)\| \leq \frac{cM}{a\rho}\|e_N(t)\| + \frac{M}{a\rho}\|\,(I - Q_N)\,(r_s)(t)\|. \tag{3.22}$$

Assuming $\sigma := 1 - \frac{cM}{a\rho} \geq 0$ and using the interpolation error, one can obtain

$$\|e_N(t)\| \leq C\|r_s^{(N+1)}(t)\|, \tag{3.23}$$

where $C = \frac{M}{a\rho\sigma}$. □

4. Numerical simulations and results

This section includes several illustrative examples to demonstrate the efficacy of the presented scheme. To provide a comprehensive overview of the method's effectiveness, absolute errors

$$e_N = |x(t) - x_N(t)|,$$

and L_2 error

$$L^2 - error = \left(\int_0^1 |x(t) - x_N(t)|^2\right)^{1/2}.$$

may be reported in tables or figures.

To achieve greater accuracy, we elevate precision above 50 digits. All examples were run using Maple and Matlab softwares (version 2022).

Example 4.1. First, we implement the present method for Equation (3.1) with $a = 1$, $b = 8/17$, $c = 13/51$, $x_0 = 0$, and $x_1 = 27/125$. Also, we have

$$f(t) = \frac{t^{-1/2}}{89250\sqrt{\pi}}\left(48u(t) + 7\sqrt{\pi}tv(t)\right), \quad 0 < t \leq 1,$$

in which

$$v(t) = 3250t^5 - 9425t^4 + 264880t^3 - 448107t^2 + 233262t - 34578,$$
$$u(t) = 16000t^4 - 32480t^3 + 21280t^2 - 4746t.$$

The exact solution mentioned in [18] is equal to

$$x(t) = x^5 - \frac{29}{10}t^4 + \frac{76}{25}t^3 - \frac{339}{250}t^2 + \frac{27}{125}t.$$

To solve this example using the presented method, we can obtain the exact solution by selecting $\eta = 1$, $N = 6$, and all three collocation points, including Chebyshev roots, Legendre roots, and uniform grids. Figure 1 shows the accuracy of the present method using $\eta = 1$, $N = 6$, and Legendre roots.

For comparison with existing methods, Table 1 presents absolute errors at different time points. The proposed scheme provides a highly accurate approximate solution compared to other options.

Example 4.2. For the second example, we consider the Bagley-Torvik Equation (3.1) with:

$$x''(t) + \frac{1}{2}{}^c\mathcal{D}_0^{3/2}(x)(t) + \frac{1}{2}x(t) = \begin{cases} 8, & 0 \leq t \leq 1, \\ 0, & t > 1, \end{cases} \quad 0 \leq t \leq 1, \tag{4.1}$$

with conditions

$$x(0) = 1, \quad x'(0) = 0. \tag{4.2}$$

For this example, the exact solution can be obtained using (1.4).

To compare the performance of different methods, Table 2 presents absolute errors at various time point. The results demonstrate that our method achieves higher accuracy than existing approaches. Figure 2 illustrates the method's precision, showing how the error decreases as N increases. These convergence properties are further confirmed in Table 3, which includes corresponding CPU times.

TABLE 1. The results obtained for Example 4.1 compared to other methods.

	Proposed method	[43]	[32]	
$t\backslash N$	6	8	8	16
0.1	4.00×10^{-52}	1.08×10^{-02}	3.60×10^{-04}	9.13×10^{-05}
0.2	3.16×10^{-51}	8.96×10^{-03}	1.58×10^{-03}	1.33×10^{-04}
0.3	8.70×10^{-51}	3.78×10^{-03}	1.79×10^{-03}	1.22×10^{-04}
0.4	1.58×10^{-50}	1.44×10^{-07}	1.63×10^{-03}	8.99×10^{-04}
0.5	2.16×10^{-50}	1.00×10^{-03}	1.16×10^{-03}	1.77×10^{-05}
0.6	2.36×10^{-50}	6.62×10^{-08}	5.84×10^{-04}	5.36×10^{-05}
0.7	2.03×10^{-50}	1.26×10^{-03}	1.27×10^{-04}	2.15×10^{-05}
0.8	1.20×10^{-50}	1.28×10^{-03}	1.20×10^{-04}	1.65×10^{-06}
0.9	2.60×10^{-51}	2.07×10^{-08}	5.54×10^{-04}	1.42×10^{-06}

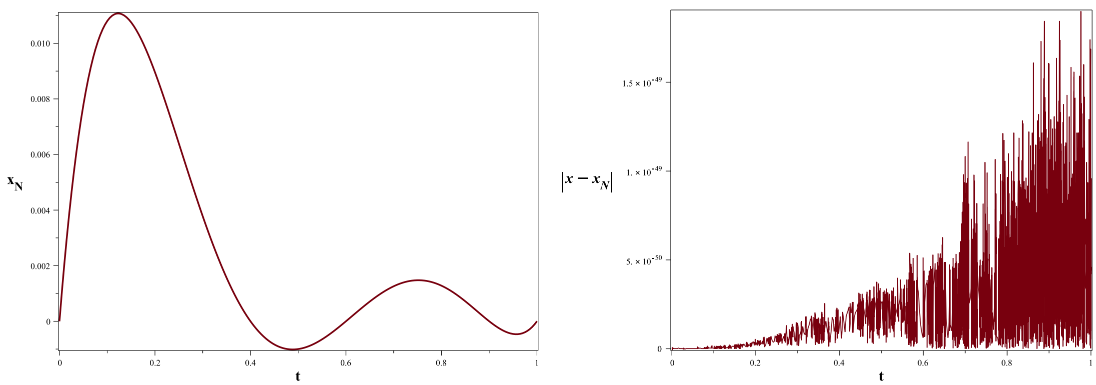

FIGURE 1. The obtained numerical solution and corresponding absolute error for Example 4.1.

TABLE 2. Assessing the accuracy of the method compared to other methods (Example 4.2).

	Presented method	[18]	[7]
t	$\eta = 0.5, N = 16$	$\eta = 0.5, N = 16$	$N = 16$
0.1	2.74×10^{-12}	7.60×10^{-10}	1.93×10^{-06}
0.2	4.16×10^{-12}	1.00×10^{-10}	4.90×10^{-06}
0.3	4.98×10^{-12}	1.00×10^{-10}	8.40×10^{-06}
0.4	5.27×10^{-12}	2.00×10^{-10}	1.28×10^{-05}
0.5	5.16×10^{-12}	7.00×10^{-10}	2.13×10^{-05}
0.6	4.63×10^{-12}	6.00×10^{-09}	3.16×10^{-05}
0.7	3.83×10^{-12}	1.70×10^{-08}	4.42×10^{-05}
0.8	2.77×10^{-12}	4.30×10^{-08}	5.43×10^{-05}
0.9	1.49×10^{-12}	1.01×10^{-07}	1.22×10^{-04}

5. CONCLUSIONS

This paper, a well-known fractional equation called the BT equation is solved using the collocation method. To implement the scheme, we first map the unknown solution into the wavelet space using ML wavelets. Then, by

TABLE 3. The L^2-error for $\eta = 0.5$ (Example 4.2).

		$N=8$	$N=10$	$N=12$	$N=14$	$N=16$
Chebyshev nodes	L^2-error	3.02×10^{-05}	2.64×10^{-06}	3.51×10^{-08}	1.47×10^{-10}	5.28×10^{-13}
	CPU time	2.531	3.391	4.547	8.718	22.016
Legendre nodes	L^2-error	6.68×10^{-06}	1.07×10^{-06}	1.18×10^{-07}	4.73×10^{-11}	3.88×10^{-12}
	CPU time	2.547	3.046	4.547	9.125	19.016
Uniform meshes	L^2-error	4.00×10^{-05}	2.15×10^{-06}	2.07×10^{-07}	8.60×10^{-10}	1.54×10^{-10}
	CPU time	2.688	3.110	4.609	8.172	21.062

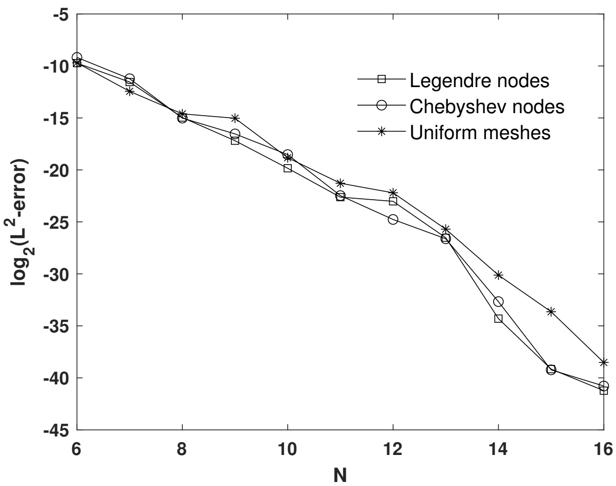

FIGURE 2. The obtained L^2–error via different values of N for Example 4.2.

employing the operational matrices of CFD and derivative, we reduced the problem to a linear system of algebraic equations through the collocation method. We consider various collocation points and solve the problem effectively and accurately. The illustrated examples show that when the parameter η is selected correctly, the method yields the exact solution. Otherwise, we can obtain the expected results by increasing the parameter N. Due to the flexibility of ML wavelets in selecting their basis power and the structure of the presented algorithm, this method has high potential for solving various fractional equations.

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