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Infinitely many conservation laws, multi-wave solutions, and interactions for the (2+1)-dimensional complex modified Korteweg-de Vries system of equations

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Abstract

In this paper, we consider the (2+1)-dimensional complex modified Korteweg-de Vries (cmKdV) system of equations. This system of equations is a generalization of the cmKdV equation in the (2+1)-dimension and has great significance in the fields of applied magnetism and nanophysics. On the basis of the Lax pair, infinitely many conservation laws are obtained. In addition, the multi-waves, homoclinic breather, rational, and interactions solutions of this equation are derived with the aid of logarithmic transformation and symbolic computation. For the suitable value of parameters, the 3D surfaces of obtained solutions have been plotted using Mathematica.

Keywords. (2+1)-dimensional cmKdV system of equations, Lax pair, Infinitely many conservation laws, Logarithmic transformation, Multi-wave solutions.

2010 Mathematics Subject Classification. 35C07, 35C08, 35Q55.

1. Introduction

Integrable systems, constructing and studying nonlinear evolution equations (NLEEs) describing natural phenomena, have significant influence on both mathematical and physical fields. The Korteweg-de Vries (KdV) equation and nonlinear Schrodinger (NLS) equation are the most commonly studied integrable systems for nonlinear wave phenomena. A large class of KdV equations are used in fluid dynamics, plasma physics, and other physical fields to describe nonlinear wave propagation [12, 15]. Through the Miura transformation, the KdV equation can lead to the famous modified KdV (mKdV) equation, which is expressed as [10]

$$q_t + q_{xxx} + 6q^2 q_x = 0, \tag{1.1}$$

where $q = q(x,t)$ is a real function. The mKdV equation has been studied by the inverse scattering transformation [32], the Hirota bilinear method [9], and the Darboux transformation method [33, 34]. After being generalized to the complex field, the mKdV forms the famous complex mKdV (cmKdV) equation [11], namely,

$$q_t + q_{xxx} + \beta |q|^2 q_x = 0, \tag{1.2}$$

where $q = q(x,t)$ is a complex function and β is a real constant. The analytical solutions of cmKdV, including soliton, breather, and rogue wave, have been thoroughly studied in [8, 14, 17, 39]. In recent years, besides (1+1)-dimensional NLEEs, more and more attention has been paid to the study of multi-dimensional NLEEs, especially the (2+1)-dimensional ones. One typical form of the (2+1)-dimensional cmKdV equation has been proposed by Myrzakulov [28], which is in the following form:

$$\begin{aligned} q_t + q_{xxy} + iqv + (qw)_x &= 0, \\ v_x + 2i\delta(q^* q_{xy} - q^*_{xy} q) &= 0, \\ w_x - 2\delta(|q|^2)_y &= 0, \end{aligned} \tag{1.3}$$

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where $\delta = \pm 1$. This model is completely integrable and has great significance in applied ferromagnetism and nano-magnetism. The Eq. (1.3) has been studied in several articles by the Darboux transformation (DT) method. In Refs. [35, 36], the 1-soliton and 2-soliton solutions are derived using the DT method. The periodic and breather solutions are extracted by starting with a plane wave seed in [37, 38]. On the other hand, the dark, bright soliton solutions are acquired by using three different methods in [31]. Our aim in this paper is to derive the infinitely many conservation laws based on the Lax pair [2, 41]. Further, we aim to determine the multi-wave, homoclinic breather, rational, and interaction solutions of Eq. (1.3) with the use of logarithmic transformation and symbolic computation without implementing Hirota bilinear forms [3, 4, 30]. Apart from these methods, a lot of powerful techniques have been implemented for extracting the soliton solutions of NLEEs [1, 5–7, 18–27, 29].

2. Lax pair and infinitely many conservation laws

The NLEE has been considered to have the Lax pair or Lax integrability when it can be expressed by the compatibility condition of the two linear differential systems, which agrees that the NLEE satisfies the Lax integrability once its Lax pair can be derived. Lax integrability has been applied to construct the conservation laws describing the conservation of certain physical quantities for the NLEE . Generally, the existence of the infinitely many conservation laws for the NLEEs denotes the complete integrability, i.e., the NLEEs can be solved through the inverse scattering technique and possess multi-soliton solutions [13, 16, 40]. According to the property of the AKNS system [2], the Lax pair for Eq. (1.3) can be defined into the following form [28]:

$$\begin{aligned} &\Psi_x = A\Psi, \\ &\Psi_t = 4\lambda^2 A\Psi_y + B\Psi, \end{aligned} \tag{2.1}$$

where $\Psi = (\Psi_1, \Psi_2)^T$, A and B have the following form,

$$A = -\lambda J_3 + A_0, \; B = \lambda B_1 + B_0, \tag{2.2}$$

with

$$\begin{aligned} &J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \\ &B_1 = \begin{pmatrix} iw & 2iq_y \\ 2ir_y & -iw \end{pmatrix}, \; B_0 = \begin{pmatrix} -iv/2 & -q_{xy} - wq \\ r_{xy} + wr & iv/2 \end{pmatrix}, \end{aligned}$$

and $r = \delta q^*$. The compatibility condition of Eq. (2.1) is still

$$A_t - 4\lambda^2 A_y - B_x + [A, B] = 0. \tag{2.3}$$

Next, we will construct the infinitely many conservation laws for Eq. (1.3) with the use of the Lax pair (2.1). Let's define the functions [41] as

$$\Lambda_1 = \frac{\psi_{1,x}}{\psi_1}, \; \Lambda_2 = \frac{\psi_{1,y}}{\psi_1}, \; \Lambda_3 = \frac{\psi_2}{\psi_1}, \tag{2.4}$$

and according to the compatibility condition $\Lambda_{1y} = \Lambda_{2x}$, we have

$$\Lambda_{2,x} = (q\Lambda_3)_y, \tag{2.5}$$

$$\Lambda_{3,x} = -r + 2i\lambda\Lambda_3 - q\Lambda_3^2. \tag{2.6}$$

Then, expanding the expansions of Λ_2 and Λ_3 with respect to λ as follows:

$$\Lambda_2 = \sum_{n=1}^{\infty} \frac{\tau_n}{\lambda^n}, \; \Lambda_3 = \sum_{n=1}^{\infty} \frac{\chi_n}{\lambda^n}. \tag{2.7}$$

Substituting expressions (2.7) into Eq. (2.5) and equating the coefficients of the same power of λ to be zero, we can get the following recursion formula:

$$\chi_1 = -\frac{1}{2}ir, \tag{2.8}$$

$$\chi_2 = -\frac{1}{2}i\chi_{1,x}, \tag{2.9}$$

$$\chi_3 = -\frac{1}{2}i(\chi_{2,x} + q\chi_1^2), \tag{2.10}$$

$$\chi_4 = -\frac{1}{2}i(\chi_{3,x} + 2q\chi_1\chi_2), \tag{2.11}$$

$$\vdots$$

$$\chi_{n+1} = -\frac{1}{2}i(\chi_{n,x} + q\sum_{k=1}^{n-1}\chi_k\chi_{n-k}). \tag{2.12}$$

Substituting expressions (2.7) into Eq. (2.6) and equating the coefficients of the same power of λ to be zero, we can get the following recursion formula:

$$\tau_{1,x} = (q\chi_1)_y, \tag{2.13}$$

$$\tau_{2,x} = (q\chi_2)_y, \tag{2.14}$$

$$\tau_{3,x} = (q\chi_3)_y, \tag{2.15}$$

$$\vdots$$

$$\tau_{n,x} = (q\chi_n)_y, \tag{2.16}$$

where τ_n and χ_n are the functions of x and t. Now, combining the function $\Lambda_4 = \frac{\psi_{1,t}}{\psi_1}$ and functions (2.4), the compatibility conditions $\Lambda_{4,x} = \Lambda_{1,t}$, we have

$$\Lambda_4 = 4\lambda^2\Lambda_2 + A + B\Lambda_3,\ \Lambda_{4,x} = (q\Lambda_3)_t. \tag{2.17}$$

Substituting expressions (2.7) into (2.17) and collecting the coefficients of the same power of λ, we obtain the infinitely many conservation laws for Eq. (1.3) as

$$\frac{\partial\rho_j}{\partial t} + \frac{\partial\Gamma_j}{\partial x} + \frac{\partial\Theta_j}{\partial y} = 0,\ j = 1, 2, 3, ..., \tag{2.18}$$

with

$$\rho_j = -q\chi_j, \tag{2.19}$$

$$\Gamma_j = \sum_{k=0}^{1} B_{j+k-1}\chi_{j+k}, \tag{2.20}$$

$$\Theta_j = 4q\chi_j, \tag{2.21}$$

where ρ_j's denote the conserved densities, and Γ_j's and Θ_j's denote the fluxes along the x and y axes, respectively.

3. Multi-wave soliton and interactional phenomena

Multi-wave solutions have been studied for a long time and are important because they reveal the interactions between the inner waves and the various frequency and velocity components. The whole multi-wave solution, for instance, may sometimes be converted into a single soliton of very high energy that propagates over large regions of space without dispersing.

We analyze the multi-waves method, the homoclinic breather approach, and the interactional solution with the double exp-functions procedure, and their applications for this equation are obtained using logarithmic transformation. To apply the method, let us consider the following complex wave transformation:

$$q=\phi(\zeta)e^{i\theta},\ v=\psi(\zeta),\ w=\chi(\zeta),\ \zeta=\lambda(x+y+\nu t),\ \theta=\alpha x+\beta y+\gamma t, \tag{3.1}$$

here α,β,γ, and ν are the constants. Inserting Eq. (3.1) into Eq. (1.3) and splitting the real and imaginary parts yields

$$(-\alpha(\alpha+2\beta)+\nu)\phi(\xi)+2\delta\phi(\xi)^3+\lambda^2\phi''(\xi)=0, \tag{3.2}$$

$$\psi(\xi)=2(\alpha+\beta)\delta\phi(\xi)^2, \tag{3.3}$$

$$\chi(\xi)=2\delta\phi(\xi)^2, \tag{3.4}$$

and $\gamma=-2\alpha(\alpha+\beta)^2+(2\alpha+\beta)\nu$.

By using the logarithmic transformation-based approaches in combination with symbolic structures of exponential functions [3, 4, 30]

$$\phi(\xi)=2\log(f(\xi))_\xi, \tag{3.5}$$

Eq. (3.2) has the form

$$2\left(4\delta+\lambda^2\right)f'(\xi)^3-3\lambda^2 f(\xi)f'(\xi)f''(\xi)+f(\xi)^2\left((-\alpha(\alpha+2\beta)+\nu)f'(\xi)+\lambda^2 f^{(3)}(\xi)\right)=0. \tag{3.6}$$

Now, our concern is to treat Eq. (3.6) to find the different forms of the solutions.

3.1. **Three waves hypothesis.** The three-waves hypothesis is a well-known phenomenon in nonlinear science that describes the interaction of three waves that satisfy certain resonance conditions. Let's consider the three-wave hypothesis of the form [3, 4, 30]:

$$f(\xi)=b_0\cosh\left(a_1\xi+a_2\right)+b_1\cos\left(a_3\xi+a_4\right)+b_2\cosh\left(a_5\xi+a_6\right), \tag{3.7}$$

where $a_1,\ldots,a_6$ are constants that will be discovered later. The algebraic equations are simply found by plugging Eq. (3.7) into Eq. (3.6) and gathering all similar power coefficients of trigonometric and hyperbolic functions to zero. Solving the algebraic equations by Mathematica gives the following sets:

Set 1:

$$b_1=0,\ a_1=a_5,\ \nu=\alpha(\alpha+2\beta)-8\delta a_5^2,\ \lambda=2i\sqrt{\delta}. \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$f(\xi)=\cosh\left(a_2+\xi a_5\right)b_0+\cosh\left(\xi a_5+a_6\right)b_2. \tag{3.9}$$

Since $\phi(\xi)=2\log(f(\xi))_\xi$, we have

$$\phi(\xi)=\frac{2a_5\left(\sinh\left(a_2+\xi a_5\right)b_0+\sinh\left(\xi a_5+a_6\right)b_2\right)}{\cosh\left(a_2+\xi a_5\right)b_0+\cosh\left(\xi a_5+a_6\right)b_2}, \tag{3.10}$$

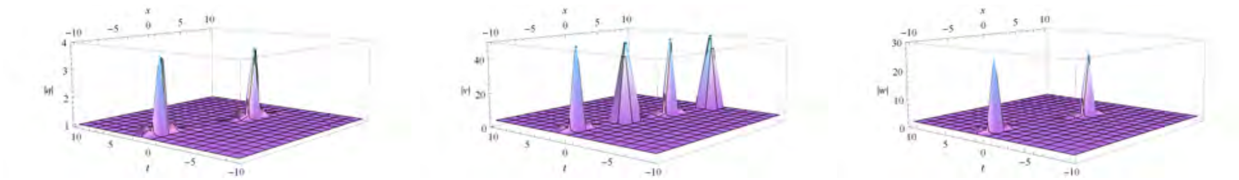
and hence, we attain the following multi-wave interaction solutions:

$$q(x,y,t)=\frac{2a_5\left(\sinh\left(a_2+\xi a_5\right)b_0+\sinh\left(\xi a_5+a_6\right)b_2\right)}{\cosh\left(a_2+\xi a_5\right)b_0+\cosh\left(\xi a_5+a_6\right)b_2}\,e^{i(x\alpha+y\beta+t\gamma)},$$

$$v(x,y,t)=\frac{8(\alpha+\beta)\delta a_5^2\left(\sinh\left(a_2+\xi a_5\right)b_0+\sinh\left(\xi a_5+a_6\right)b_2\right)^2}{\left(\cosh\left(a_2+\xi a_5\right)b_0+\cosh\left(\xi a_5+a_6\right)b_2\right)^2},$$

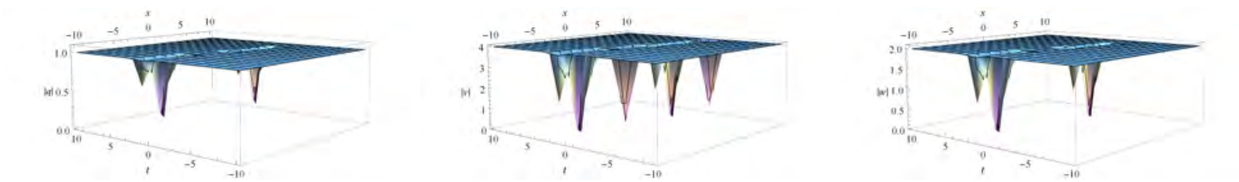
$$w(x,y,t)=\frac{8\delta a_5^2\left(\sinh\left(a_2+\xi a_5\right)b_0+\sinh\left(\xi a_5+a_6\right)b_2\right)^2}{\left(\cosh\left(a_2+\xi a_5\right)b_0+\cosh\left(\xi a_5+a_6\right)b_2\right)^2}, \tag{3.11}$$

where $\xi=2i\sqrt{\delta}(x+y+\nu t)$ and $\delta<0$. Figure 2 represents the dynamical behavior of the solution (3.11).



(A) (B) (C)

FIGURE 1. 3D graph of solution (3.11) with $b_1 = -0.1, b_2 = a_4 = a_6 = 0.1, a_5 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 2$.



(A) (B) (C)

FIGURE 2. 3D graph of solution (3.11) with $b_1 = b_2 = a_4 = a_6 = 0.2, a_5 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 2$.

Set 2:

$$b_2 = 0,\ a_1 = -ia_3,\ \nu = \alpha(\alpha + 2\beta) + 8\delta a_3^2,\ \lambda = 2i\sqrt{\delta}. \tag{3.12}$$

Substituting (3.12) into (3.7), we get

$$f(\xi) = \cosh(a_2 - i\xi a_3)\, b_0 + \cos(\xi a_3 + a_4)\, b_1. \tag{3.13}$$

Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = \frac{2a_3\left(-i\sinh(a_2 - i\xi a_3)\, b_0 - \sin(\xi a_3 + a_4)\, b_1\right)}{\cosh(a_2 - i\xi a_3)\, b_0 + \cos(\xi a_3 + a_4)\, b_1}, \tag{3.14}$$

and hence, we attain the following multi-wave interaction solutions:

$$q(x,y,t) = \frac{2a_3\left(-i\sinh(a_2 - i\xi a_3)\, b_0 - \sin(\xi a_3 + a_4)\, b_1\right)}{\cosh(a_2 - i\xi a_3)\, b_0 + \cos(\xi a_3 + a_4)\, b_1}\, e^{i(x\alpha + y\beta + t\gamma)},$$

$$v(x,y,t) = -\frac{8(\alpha+\beta)\delta a_3^2\left(\sinh(a_2 - i\xi a_3)\, b_0 - i\sin(\xi a_3 + a_4)\, b_1\right)^2}{\left(\cosh(a_2 - i\xi a_3)\, b_0 + \cos(\xi a_3 + a_4)\, b_1\right)^2},$$

$$w(x,y,t) = -\frac{8\delta a_3^2\left(\sinh(a_2 - i\xi a_3)\, b_0 - i\sin(\xi a_3 + a_4)\, b_1\right)^2}{\left(\cosh(a_2 - i\xi a_3)\, b_0 + \cos(\xi a_3 + a_4)\, b_1\right)^2}, \tag{3.15}$$

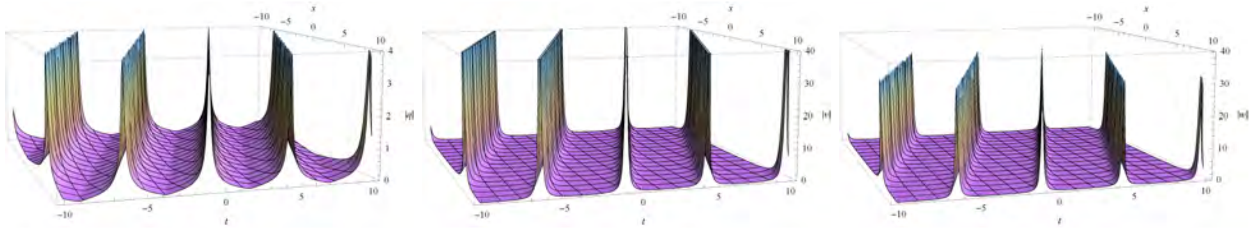
where $\xi = 2i\sqrt{\delta}(x + y + \nu t)$ and $\delta < 0$. Figure 3 represents the dynamical behavior of the solution (3.15).

Set 3:

$$b_0 = 0,\ a_3 = ia_5,\ \nu = \alpha(\alpha + 2\beta) - 8\delta a_5^2,\ \lambda = 2i\sqrt{\delta}. \tag{3.16}$$

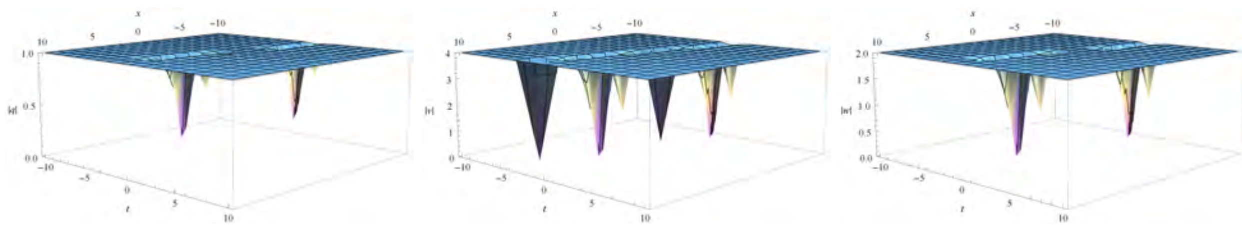
Substituting (3.16) into (3.7), we get

$$f(\xi) = \cos(a_4 - i\xi a_5)\, b_1 + \cosh(\xi a_5 + a_6)\, b_2. \tag{3.17}$$



(A) (B) (C)

FIGURE 3. 3D graph of solution (3.15) with $b_0 = b_1 = a_2 = a_3 = a_4 = 0.1, \delta = -1$, and $\alpha = \beta = 1$ at $y = 1$.



(A) (B) (C)

FIGURE 4. 3D graph of solution (3.19) with $b_0 = b_2 = a_2 = a_6 = 0.1, a_5 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 2$.

Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = \frac{2a_5\left(i\sin\left(a_4 - i\xi a_5\right)b_1 + \sinh\left(\xi a_5 + a_6\right)b_2\right)}{\cos\left(a_4 - i\xi a_5\right)b_1 + \cosh\left(\xi a_5 + a_6\right)b_2}, \tag{3.18}$$

and hence we attain the following multi-waves interaction solutions

$$q(x,y,t) = \frac{2a_5\left(i\sin\left(a_4 - i\xi a_5\right)b_1 + \sinh\left(\xi a_5 + a_6\right)b_2\right)}{\cos\left(a_4 - i\xi a_5\right)b_1 + \cosh\left(\xi a_5 + a_6\right)b_2} e^{i(x\alpha + y\beta + t\gamma)},$$

$$v(x,y,t) = \frac{8(\alpha+\beta)\delta a_5^2\left(i\sin\left(a_4 - i\xi a_5\right)b_1 + \sinh\left(\xi a_5 + a_6\right)b_2\right)^2}{\left(\cos\left(a_4 - i\xi a_5\right)b_1 + \cosh\left(\xi a_5 + a_6\right)b_2\right)^2},$$

$$w(x,y,t) = \frac{8\delta a_5^2\left(i\sin\left(a_4 - i\xi a_5\right)b_1 + \sinh\left(\xi a_5 + a_6\right)b_2\right)^2}{\left(\cos\left(a_4 - i\xi a_5\right)b_1 + \cosh\left(\xi a_5 + a_6\right)b_2\right)^2}, \tag{3.19}$$

where $\xi = 2i\sqrt{\delta}(x + y + \nu t)$ and $\delta < 0$. Figure 4 represents the dynamical behaviour of the solution (3.19).

Set 4:

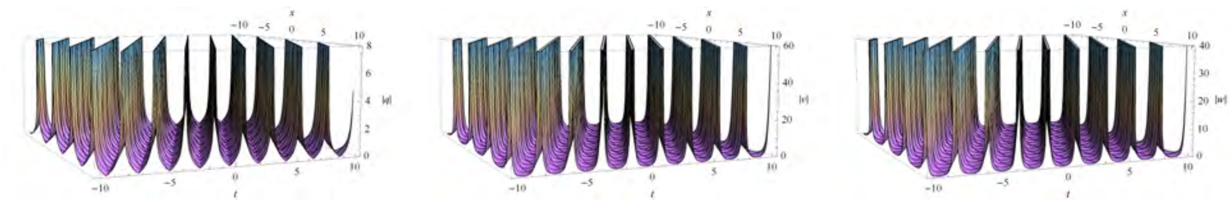
$$a_1 = ia_3,\ a_5 = ia_3,\ \nu = \alpha(\alpha + 2\beta) - 8\delta a_1^2,\ \lambda = -2i\sqrt{\delta}. \tag{3.20}$$

Substituting (3.20) into (3.7), we get

$$f(\xi) = \cosh\left(a_2 + i\xi a_3\right)b_0 + \cos\left(\xi a_3 + a_4\right)b_1 + \cos\left(\xi a_3 - ia_6\right)b_2. \tag{3.21}$$

Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = \frac{2a_3\left(i\sinh\left(a_2 + i\xi a_3\right)b_0 - \sin\left(\xi a_3 + a_4\right)b_1 - \sin\left(\xi a_3 - ia_6\right)b_2\right)}{\cosh\left(a_2 + i\xi a_3\right)b_0 + \cos\left(\xi a_3 + a_4\right)b_1 + \cos\left(\xi a_3 - ia_6\right)b_2}, \tag{3.22}$$



(A) (B) (C)

FIGURE 5. 3D graph of solution (3.23) with $b_0 = b_1 = b_2 = a_2 = a_4 = a_6 = 0.1, a_3 = 0.3, \delta = -1$, and $\alpha = \beta = 1$ at $y = 1$.

and hence, we attain the following multi-wave interaction solutions:

$$q(x,y,t) = \frac{2a_3\left(i\sinh\left(a_2+i\xi a_3\right)b_0 - \sin\left(\xi a_3+a_4\right)b_1 - \sin\left(\xi a_3 - ia_6\right)b_2\right)}{\cosh\left(a_2+i\xi a_3\right)b_0 + \cos\left(\xi a_3+a_4\right)b_1 + \cos\left(\xi a_3-ia_6\right)b_2}\, e^{i(x\alpha+y\beta+t\gamma)},$$

$$v(x,y,t) = -\frac{8(\alpha+\beta)\delta a_3^2\left(\sinh\left(a_2+i\xi a_3\right)b_0 + i\left(\sin\left(\xi a_3+a_4\right)b_1 + \sin\left(\xi a_3-ia_6\right)b_2\right)\right)^2}{\left(\cosh\left(a_2+i\xi a_3\right)b_0 + \cos\left(\xi a_3+a_4\right)b_1 + \cos\left(\xi a_3-ia_6\right)b_2\right)^2},$$

$$w(x,y,t) = -\frac{8\delta a_3^2\left(\sinh\left(a_2+i\xi a_3\right)b_0 + i\left(\sin\left(\xi a_3+a_4\right)b_1 + \sin\left(\xi a_3-ia_6\right)b_2\right)\right)^2}{\left(\cosh\left(a_2+i\xi a_3\right)b_0 + \cos\left(\xi a_3+a_4\right)b_1 + \cos\left(\xi a_3-ia_6\right)b_2\right)^2}, \tag{3.23}$$

where $\xi = 2i\sqrt{\delta}(x+y+\nu t)$ and $\delta < 0$. Figure 5 represents the dynamical behavior of the solution (3.23).

3.2. **Interactional phenomena and exponential form.** Multi-wave solutions are important as they reveal the interactions between the inner waves and the various frequency and velocity components. For this purpose, we construct the following double exponential function assumption to seek interactional solutions:

$$f(\xi) = b_1 e^{\xi a_1 + a_2} + b_2 e^{\xi a_3 + a_4}, \tag{3.24}$$

where $a_i (i = 1, ..., 4)$ are constants. By inserting (3.24) into (3.6) and setting the coefficients of all powers of exponential functions to zero, we obtain a system of equations. By solving this system, we get

$$a_1 = -a_3,\ \nu = \alpha(\alpha+2\beta) - 8\delta a_3^2,\ \lambda = 2i\sqrt{\delta}. \tag{3.25}$$

Substituting (3.25) into (3.7), we get

$$f(\xi) = b_1 e^{a_2 - \xi a_3} + b_2 e^{\xi a_3 + a_4}. \tag{3.26}$$

Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = -\frac{2a_3\left(e^{a_2}b_1 - e^{2\xi a_3+a_4}b_2\right)}{e^{a_2}b_1 + e^{2\xi a_3+a_4}b_2}, \tag{3.27}$$

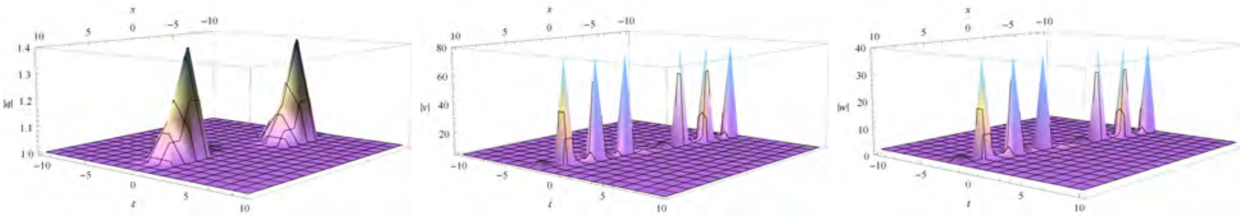
and hence, we attain the following multi-wave interaction solutions:

$$q(x,y,t) = -\frac{2a_3\left(e^{a_2}b_1 - e^{2\xi a_3+a_4}b_2\right)}{e^{a_2}b_1 + e^{2\xi a_3+a_4}b_2}\, e^{i(x\alpha+y\beta+t\gamma)},$$

$$v(x,y,t) = \frac{8(\alpha+\beta)\delta a_3^2\left(e^{a_2}b_1 - e^{2\xi a_3+a_4}b_2\right)^2}{\left(e^{a_2}b_1 + e^{2\xi a_3+a_4}b_2\right)^2},$$

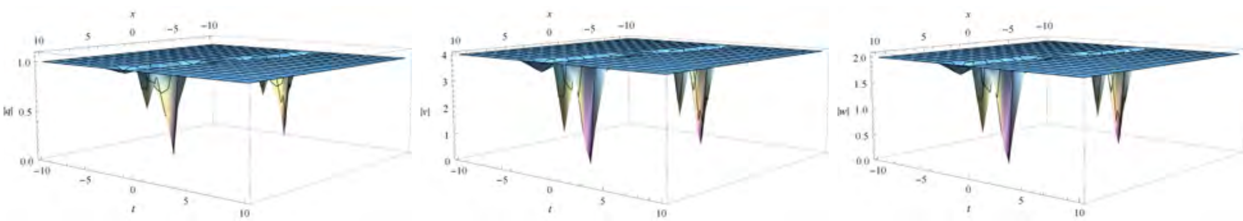
$$w(x,y,t) = \frac{8\delta a_3^2\left(e^{a_2}b_1 - e^{2\xi a_3+a_4}b_2\right)^2}{\left(e^{a_2}b_1 + e^{2\xi a_3+a_4}b_2\right)^2}, \tag{3.28}$$

where $\xi = 2i\sqrt{\delta}(x+y+\nu t)$ and $\delta < 0$. Figure 7 represents the dynamical behavior of the solution (3.28).



(A) (B) (C)

FIGURE 6. 3D graph of solution (3.28) with $b_1 = -0.2, b_2 = a_2 = a_4 = 0.1, a_3 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 1$.



(A) (B) (C)

FIGURE 7. 3D graph of solution (3.28) with $b_1 = 0.1, b_2 = a_2 = a_4 = 0.3, a_3 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 1$.

3.3. **Homoclinic breather approach.** Breathers are special solitons with periodic structures localized in space. It is often used to explain the generation of rogue waves and the nonlinear stage of modulation instability. Breathers are mostly restricted in space and time. For breather solutions, we construct the following double exponential function assumption with a cosine function:

$$f(\xi) = e^{-\tau_1(\xi a_1 + a_2)} + b_0 \cos\left(\tau_2\left(\xi a_5 + a_6\right)\right) + b_1 e^{\tau_1(\xi a_3 + a_4)}, \tag{3.29}$$

where $a_i (i = 1, ..., 6)$ are constants. By inserting (3.29) into (3.6) and setting the coefficients of all powers of exponential functions to zero, we obtain a system of equations. By solving this system, we get

$$a_1 = -a_3,\ a_5 = \frac{i a_3 \tau_1}{\tau_2},\ \nu = \alpha(\alpha + 2\beta) - 8\delta a_3^2 \tau_1^2,\ \lambda = 2i\sqrt{\delta}, \tag{3.30}$$

Substituting (3.30) into (3.7), we get

$$f(\xi) = e^{-(a_2 - \xi a_3)\tau_1} + \cosh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(\xi a_3 + a_4)\tau_1} b_1. \tag{3.31}$$

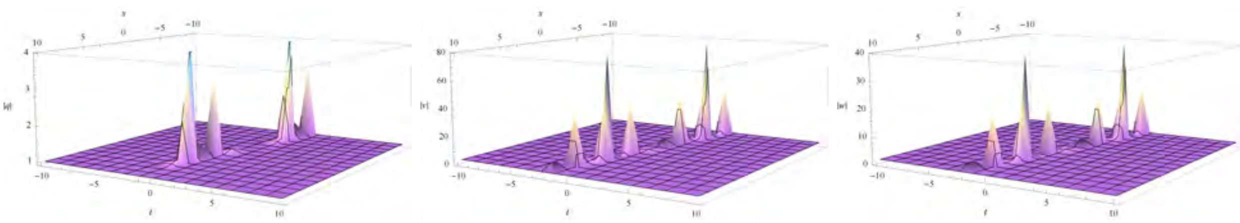
Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = \frac{2a_3\left(1 + e^{(a_2 - \xi a_3)\tau_1} \sinh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1\right)\tau_1}{1 + e^{(a_2 - \xi a_3)\tau_1} \cosh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1}, \tag{3.32}$$

and hence, we attain the following multi-wave interaction solutions:

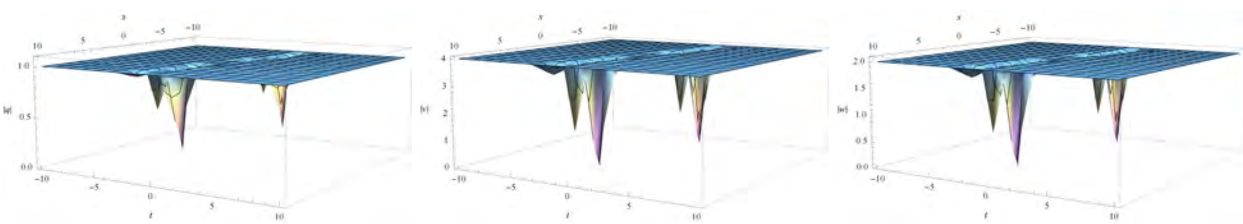
$$q(x, y, t) = \frac{2a_3\left(1 + e^{(a_2 - \xi a_3)\tau_1} \sinh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1\right)\tau_1}{1 + e^{(a_2 - \xi a_3)\tau_1} \cosh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1}\ e^{i(x\alpha + y\beta + t\gamma)},$$

$$v(x, y, t) = \frac{8(\alpha + \beta)\delta a_3^2\left(1 + e^{(a_2 - \xi a_3)\tau_1} \sinh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1\right)^2 \tau_1^2}{\left(1 + e^{(a_2 - \xi a_3)\tau_1} \cosh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2 + a_4)\tau_1} b_1\right)^2},$$



(A) (B) (C)

FIGURE 8. 3D graph of solution (3.33) with $\tau_1 = \tau_2 = 1, b_0 = -0.5, b_1 = a_2 = a_4 = a_6 = 0.1, a_3 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 2$.



(A) (B) (C)

FIGURE 9. 3D graph of solution (3.33) with $\tau_1 = \tau_2 = 1, b_0 = 0.1, b_1 = a_2 = a_4 = a_6 = 0.3, a_3 = 0.5, \delta = -1$, and $\alpha = \beta = 1$ at $y = 2$.

$$w(x,y,t) = \frac{8\delta a_3^2 \left(1 + e^{(a_2-\xi a_3)\tau_1} \sinh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2+a_4)\tau_1} b_1\right)^2 \tau_1^2}{\left(1 + e^{(a_2-\xi a_3)\tau_1} \cosh\left(\xi a_3 \tau_1 - i a_6 \tau_2\right) b_0 + e^{(a_2+a_4)\tau_1} b_1\right)^2}, \tag{3.33}$$

where $\xi = 2i\sqrt{\delta}(x + y + \nu t)$ and $\delta < 0$. Figure 9 represents the dynamical behavior of the solution (3.33).

3.4. **Peregrine-like rational solitons.** In general, rogue waves are frequently termed "rational solitons on a finite background", for instance, the so-called "Peregrine soliton" actually represents the archetypal rogue wave form. In order to obtain the peregrine-like rational solutions, we assume the following ansatz:

$$f(\xi) = \left(a_1\xi + a_2\right)^2 + \left(a_3\xi + a_4\right)^2 + a_5, \tag{3.34}$$

where $a_i (i = 1, ..., 6)$ are constants. By inserting (3.34) into (3.6) and setting the coefficients of the quadratic function to be zero, we get a system of algebraic equations. By solving this system, we get

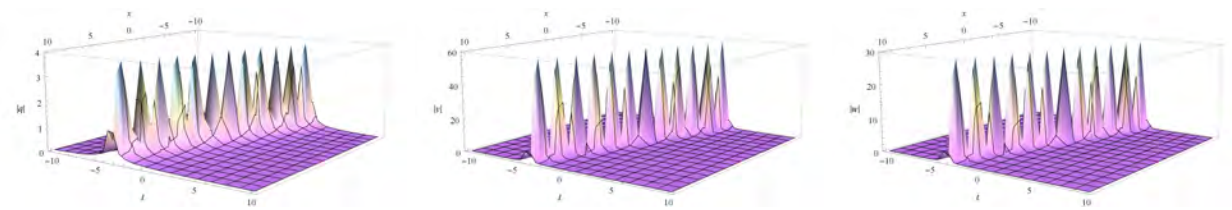
$$a_5 = -\frac{\left(-a_2 a_3 + a_1 a_4\right)^2}{a_1^2 + a_3^2}, \ \nu = \alpha(\alpha + 2\beta), \ \lambda = 4i\sqrt{\delta}. \tag{3.35}$$

Substituting (3.35) into (3.7), we get

$$f(\xi) = \frac{\left(\xi a_1^2 + a_1 a_2 + a_3\left(\xi a_3 + a_4\right)\right)^2}{a_1^2 + a_3^2}. \tag{3.36}$$

Since $\phi(\xi) = 2\log(f(\xi))_\xi$, we have

$$\phi(\xi) = \frac{4\left(a_1^2 + a_3^2\right)}{\xi a_1^2 + a_1 a_2 + a_3\left(\xi a_3 + a_4\right)}, \tag{3.37}$$



(A) (B) (C)

FIGURE 10. 3D graph of solution (3.38) with $a_1 = a_2 = a_4 = 0.1, a_3 = 0.2, \delta = -1$, and $\alpha = \beta = 1$ at $y = 1$.

and hence, we attain the following multi-wave interaction solutions:

$$q(x,y,t) = \frac{4\left(a_1^2 + a_3^2\right)}{\xi a_1^2 + a_1 a_2 + a_3\left(\xi a_3 + a_4\right)} e^{i(x\alpha + y\beta + t\gamma)},$$

$$v(x,y,t) = \frac{32(\alpha+\beta)\delta\left(a_1^2 + a_3^2\right)^2}{\left(\xi a_1^2 + a_1 a_2 + a_3\left(\xi a_3 + a_4\right)\right)^2},$$

$$w(x,y,t) = \frac{32\delta\left(a_1^2 + a_3^2\right)^2}{\left(\xi a_1^2 + a_1 a_2 + a_3\left(\xi a_3 + a_4\right)\right)^2}, \tag{3.38}$$

where $\xi = 4i\sqrt{\delta}(x + y + \nu t)$ and $\delta < 0$. Figure 10 represents the dynamical behavior of the solution (3.38).

4. Conclusions

In this study, the (2+1)-dimensional cmKdV system of equations has been investigated. Based on the Lax Pair, the infinitely many conservation laws have been obtained. By implementing the logarithmic transformation and symbolic computation with the ansatz functions technique, the multi-waves, homoclinic breather, and rational solutions are obtained for the (2+1)-dimensional cmKdV system of equations. Meanwhile, the double exponential and interactional phenomena are also investigated. The dynamical behaviors of obtained solutions are represented in three-dimensional figures. The parameter selections for graphical presentations were chosen as a result of obtaining a meaningful graphic in terms of soliton representation. These plots giving novel multi-wave soliton solutions are made to reveal important wave characteristics.

Compliance with ethical standards.

- **Conflict of interest:** The authors declare that they have no conflict of interest with regard to the publication of this manuscript.
- **Data Availability Statements:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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