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A reproducing kernel method for solving nonlocal functional differential equations with delayed or advanced arguments

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Abstract

This paper discusses an effective approach for solving non-local functional differential equations with delayed or advanced arguments. The reproducing kernel method is utilized to avoid the need for an orthogonalization process. The main objective of this technique is to successfully apply this method to solve singular multi-point boundary value problems with non-local conditions, resulting in an accurate approximate solution and a valid error analysis. This method greatly improves the accuracy of the solutions obtained.

Keywords. Reproducing kernel method, Functional differential equation, Non-local conditions, Error analysis.

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1. Introduction

This paper is concerned with an efficient semi-analytical method to solve non-local functional differential equations with delayed or advanced arguments as follows:

$$\begin{cases} \mathbf{L}(u(\tau)) = u'(\tau) + \rho(\tau)u(\kappa(\tau)) + \varrho(\tau)u(\tau) = \mathbf{N}(u(\tau)) + \mathbf{F}(\tau), & \tau \in [0,1], \\ u(1) = \lambda_1 u(c) - \lambda_2 \int_0^1 su(s)ds, \quad or \quad u(0) = \sum_{i=1}^{m_1} \nu_i u(\zeta_i), \end{cases} \tag{1.1}$$

where $\rho(.)$ and $\varrho(.) \in C[0,1]$, and $\kappa(.) \in C^1[0,1]$, and $\mathbf{L}(u(.))$ is a bounded linear operator, and $\mathbf{N}(u(.))$ is a a continuous nonlinear operator. Additionally, $0 < \zeta_i < 1$, ν_i are constants, $c \in [0,1]$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and m_1 is a constant integer. We suppose $\mathbf{F}(.)$ is given such that Eq. (1.1) satisfies the existence and uniqueness of the solutions.

The existence and uniqueness of solutions for functional differential equations have been extensively studied in [10, 11, 13, 17]. Several authors have proposed different computational methods for solving non-local functional differential equations [2, 5, 14, 16]. X. Li and B. Wu used the general form of the reproducing kernel method (RKM) to solve Eq. (1.1) [22]. This method is very useful, and many researchers use it for solving hard problems, i.e., the system of nonlinear singularly perturbed boundary value problems [1], forced Duffing equations [15], and nonlinear boundary value problems [19]. In our research, we utilize a different implementation of the general form of RKM, as presented by Wang et al. in [25, 26]. This approach, referred to as RKM without the use of the orthogonalization process, is fully explained in their work. There are main factors to increase the accuracy of the approximate solution in the reproducing kernel method: a suitable choice is an inner product in the reproducing kernel space (for short, RKS) because the inner product directly affects the kernel function and, accordingly, the accuracy of the approximation of the solution. The subsequent factor is the selection of the points in the interval [0,1] to construct the basis of the RKS. Indeed, the equidistance points can not provide an appropriate basis, and subsequently, the approximate

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solutions cannot be determined with high accuracy. Before expressing our structure in this study, it is worth noting that approximate solutions must be obtained in a large space. We are trying to calculate the approximate solutions in the large space. This work is structured as follows: Section 2 discusses the main definitions and requirements of the nonlocal functional differential equations procedure and the theory of the RKS. In section 3, the approach is introduced without the Gram-Schmidt orthogonalization process, the convergence is analyzed, and the error for the presented scheme is discussed. Also, several numerical experiments are presented to illustrate the effectiveness of the proposed technique in sections 4 and 5, respectively. Section 6 terminates this article with a brief conclusion.

2. The Construction of the RKS

The reproducing kernel space and the corresponding function are constructed as follows. We consider the Hilbert space $\mathbf{W}^K[0,1]$ [12]

$$\mathbf{W}^K[0,1] = \Big\{u(.)|u^{(K-1)}(.) \text{ is absolutely continuous, and } u^{(K)}(.) \in L^2[0,1]\Big\},$$

which are equipped with the following inner products and norms for $K = 2, 3$:

$$< u_1(.), u_2(.) >_{\mathbf{W}^3[0,1]} = \sum_{i=0}^{2} {u_1}^{(i)}(1){u_2}^{(i)}(1) + \int_0^1 {u_1}^{'''}(\tau){u_2}^{'''}(\tau)d\tau,$$

$$< u_1(.), u_2(.) >_{\mathbf{W}^2[0,1]} = u_1(0)u_2(0) + {u_1}^{'}(1){u_2}^{'}(1) + \int_0^1 {u_1}^{''}(\tau){u_2}^{''}(\tau)d\tau,$$

$$\|u(.)\|_{\mathbf{W}^K[0,1]} = \sqrt{< u, u >_{\mathbf{W}^K[0,1]}}, \qquad u_1(.), u_2(.) \in \mathbf{W}^K[0,1].$$

Theorem 2.1. *The space $\mathbf{W}^K[0,1]$ is a reproducing kernel Hilbert space, and its reproducing kernel is given as follows:*

$$Q_y(\tau) = \begin{cases} Q(\tau, y), & \tau \leq y, \\ Q(y, \tau), & \tau > y, \end{cases}$$

where for $u(0) = 0$, $K = 2$, and $K = 3$ the reproducing kernel $Q(\tau, y)$ is $\frac{\tau^3}{6} + \frac{1}{2}\tau\left(y^2 - 4y\right)$ and

$$\tau^5/120 + (2617\tau y)/2208 - (71\tau^2 y)/138 - (13\tau^5 y)/2208 - (71\tau y^2)/138 + (187\tau^2 y^2)/414 + (\tau^5 y^2)/828$$
$$- (\tau^2 y^3)/12 + (\tau y^4)/24 - (13\tau y^5)/2208 + (\tau^2 y^5)/828 - (\tau^5 y^5)/33120,$$

respectively. Without $u(0) = 0$ for $K = 2$ and $K = 3$, the reproducing kernel $Q(\tau, y)$ is $\frac{\tau^3}{6} + \frac{1}{2}\tau\left(y^2 - 4y\right) - 1$ and $\frac{1}{12}\tau^2\left(-y^3 + 6y^2 - 9y + 4\right) + \frac{1}{24}\tau\left(y^4 - 18y^2 + 56y - 39\right) + \frac{1}{120}\left(-y^5 + 40y^2 - 195y + 276\right)$, respectively; see [12].

Using non-local conditions $u(0) = \sum_{i=1}^{m_1} \nu_i u(\zeta_i)$ and $u(1) = \lambda_1 u(c) - \lambda_2 \int_0^1 su(s)ds$ in the form $\gamma_1 u = u(0) - \sum_{i=1}^{m_1} \nu_i u(\zeta_i)$ and $\gamma_2 u = u(1) - \lambda_1 u(c) + \lambda_2 \int_0^1 su(s)ds$ with the same inner products, we define the reproducing kernel space $\hat{\mathbf{W}}^K[0,1]$ as follows.

Definition 2.2. The reproducing kernel space $\hat{\mathbf{W}}^K[0,1]$ is constructed by satisfying the conditions $\gamma_1 u = 0$ or $\gamma_2 u = 0$ and is defined as:

$$\hat{\mathbf{W}}^K[0,1] = \big\{u(.)|u(.) \in \mathbf{W}^K[0,1], \;\; \gamma_1 u = 0 \text{ or } \gamma_2 u = 0\big\}.$$

It is clear that $\hat{\mathbf{W}}^K[0,1]$ is a closed subspace of $\mathbf{W}^K[0,1]$. Hence, $\hat{\mathbf{W}}^K[0,1]$ is also a reproducing kernel space. Taking into account the operator form of the Eq. (1.1), it is easy to demonstrate that $\mathbf{L} : \hat{\mathbf{W}}^2[0,1] \longrightarrow \mathbf{W}^1[0,1]$ is a bounded linear operator.

Theorem 2.3. *[3, 27, 28] If $Q_y(.)$ is the reproducing kernel of the space $\mathbf{W}^2[0,1]$, and $B : \mathbf{W}^2[0,1] \longrightarrow \mathbf{W}^1[0,1]$ is a bounded linear operator, and $q_1(\tau) = B_y(Q_y(\tau))$ and $q_2(\tau) = B_y\big(Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{W^2}}\big)$, then*

$$\|q_1\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s)\big)\big),$$
$$\|q_2\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s) - \frac{q_1(s)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big)\big),$$

where symbols B_τ or B_y indicate that this operator applies to τ or y, respectively.

Theorem 2.4. *If $Q_y(.)$ is the reproducing kernel of the space $\mathbf{W}^2[0,1]$, then*

$$\hat{Q}_y(\tau) = Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}},$$

is the reproducing kernel of the space $\mathbf{H}[0,1] = \big\{u(.)|u(.) \in \mathbf{W}^2[0,1], \;\; B(u) = 0\big\}$.

Proof. First, we show that $\hat{Q}_y(\tau) \in \mathbf{H}[0,1]$, since $q_1(\tau) = B_y(Q_y(\tau))$, $q_1(y) = B_s(Q_y(s))$, and applying Theorem 2.3, we get $\|q_1\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s)\big)\big)$, thus

$$\begin{aligned} B_y(\hat{Q}_y(\tau)) =& B_y(Q_y(\tau)) - \frac{q_1(\tau)B_y(q_1(y))}{\|q_1\|^2_{\mathbf{W}^2}} \\ =& B_y(Q_y(\tau)) - \frac{B_y(Q_y(\tau))B_y(B_s(Q_y(s)))}{B_y\big(B_s\big(Q_y(s)\big)\big)} = 0. \end{aligned}$$

In continuation, we show $\forall u(y) \in \mathbf{H}[0,1]$, $u(\tau) = \big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}}$,

$$\begin{aligned} \big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}} &= \big\langle u(y), Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big\rangle_{\mathbf{H}} \\ &= \big\langle u(y), Q_y(\tau)\big\rangle_{\mathbf{H}} - \big\langle u(y), \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big\rangle_{\mathbf{H}} \\ &= u(\tau) - \frac{q_1(\tau)}{\|q_1\|^2_{\mathbf{W}^2}} B_s\big\langle u(y), Q_y(s)\big\rangle_{\mathbf{H}}, \end{aligned}$$

since $\hat{Q}_y(\tau) \in \mathbf{H}[0,1]$, we follow $B(u) = 0$; hence $B_s(u(s)) = 0$. As a result, $\big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}} = u(\tau)$. For more details refer to [3, 20, 27, 28]. □

3. Main idea

Suppose $r_y(\tau)$ is the reproducing kernel function for space $\mathbf{W}^1[0,1]$ and $\{\tau_i\}_{i=1}^{\infty}$ is a dense set on the domain of Eq. (1.1). We define bases of the reproducing kernel space $\hat{\mathbf{W}}^2[0,1]$ as follow:

$$\xi_i(\tau) = \hat{Q}_y(\tau)|_{y=\tau_i}.$$

Theorem 3.1. *[12] If $\{\tau_i\}_{i=1}^{\infty}$ is a dense set on $[0,1]$, then $\xi_i(\tau) = \hat{Q}_y(\tau)|_{y=\tau_i}$ is a system of complete functions in $\hat{\mathbf{W}}^2[0,1]$.*

Theorem 3.2. *[12] If $\{\tau_i\}_{i=1}^{\infty}$ is a dense set on $[0,1]$, then the analytical solution of Eq. (1.1) is*

$$u_m(\tau) = \sum_{i=1}^{\infty} c_{i,m}\xi_i(\tau), \quad m = 1,2,\dots, \tag{3.1}$$

where $c_{i,m}$ represents the unknown coefficients, they can be determined.

We denoted the approximate solution of Eq. (1.1) as $u_{n,m}(\tau)$. We solve the following system of algebraic equations for $m = 1, 2, \ldots,$ to determine the unknown coefficients $c_{i,m}$:

$$\sum_{i=1}^{n} c_{i,m}\mathbf{L}\xi_i(\tau)|_{\tau=\tau_j} = \mathbf{N}(u_{n,m-1}(\tau))|_{\tau=\tau_j} + \mathbf{F}(\tau_j), \quad m = 1, 2, \ldots, \quad j = 1, 2, \ldots, n. \tag{3.2}$$

We have Eq. (3.2) in matrix form,

$$\mathbf{AC} = \mathbf{B},$$

where

$$\mathbf{A} = \mathbf{L}\xi_i(\tau)|_{\tau=\tau_j} = \begin{bmatrix} \mathbf{L}\xi_1(\tau_1) & \mathbf{L}\xi_2(\tau_1) & \mathbf{L}\xi_3(\tau_1) & \ldots & \mathbf{L}\xi_{n-1}(\tau_1) & \mathbf{L}\xi_n(\tau_1) \\ \mathbf{L}\xi_1(\tau_2) & \mathbf{L}\xi_2(\tau_2) & \mathbf{L}\xi_3(\tau_2) & \ldots & \mathbf{L}\xi_{n-1}(\tau_2) & \mathbf{L}\xi_n(\tau_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{L}\xi_1(\tau_{n-1}) & \mathbf{L}\xi_2(\tau_{n-1}) & \mathbf{L}\xi_3(\tau_{n-1}) & \ldots & \mathbf{L}\xi_{n-1}(\tau_{n-1}) & \mathbf{L}\xi_n(t_{n-1}) \\ \mathbf{L}\xi_1(\tau_n) & \mathbf{L}\xi_2(\tau_n) & \mathbf{L}\xi_3(\tau_n) & \ldots & \mathbf{L}\xi_{n-1}(\tau_n) & \mathbf{L}\xi_n(\tau_n) \end{bmatrix}, \tag{3.3}$$

$$\mathbf{B} = \mathbf{N}(u_{n,m-1}(\tau))|_{\tau=\tau_j} + \mathbf{F}(\tau_j) = \begin{bmatrix} \mathbf{N}(u_{n,m-1}(\tau_1)) + \mathbf{F}(\tau_1) \\ \mathbf{N}(u_{n,m-1}(\tau_2)) + \mathbf{F}(\tau_2) \\ \vdots \\ \mathbf{N}(u_{n,m-1}(\tau_{n-1})) + \mathbf{F}(\tau_{n-1}) \\ \mathbf{N}(u_{n,m-1}(\tau_n)) + \mathbf{F}(\tau_n) \end{bmatrix}, \tag{3.4}$$

and

$$\mathbf{C} = \begin{bmatrix} c_{1,m} \\ c_{2,m} \\ \vdots \\ c_{n-1,m} \\ c_{n,m} \end{bmatrix}, \tag{3.5}$$

whit $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$, and from our assumptions, $\mathbf{A}^{-1}$ exists and is unique, see [4, 6, 8].

Remark 3.3. We can define another basis for the reproducing kernel space $\hat{\mathbf{W}}^2[0,1]$ as follow:

$$\eta_i(\tau) = \mathbf{L}_y\hat{Q}_y(\tau)|_{y=\tau_i} = \frac{\partial \hat{Q}_y(\tau)}{\partial y}|_{y=\tau_i} + \rho(y)\hat{Q}_y(\kappa(\tau))|_{y=\tau_i} + \varrho(y)\hat{Q}_y(\tau)|_{y=\tau_i}.$$

4. Convergence Analysis and Error Bound

Theorem 4.1. *[26] The approximate solution (3.2) and its derivative are uniformly convergent to the exact solution of Eq. (1.1).*

Corollary 4.2. *[12] The approximate solution $u^{(l)}_{n,m}(\tau)$ is uniformly convergent to $u^{(l)}(\tau)$ in space $\hat{\mathbf{W}}^K[0,1]$ for $K = 2,3$ and $l = 0,1$.*

Theorem 4.3. *[9, 18, 21] In the space $\hat{\mathbf{W}}^K[0,1]$, if $u^{(K+1)}(\tau) \in C[0,1]$ and $\|u^{(K+1)}_{n,m}(\tau)\|_\infty$ is bounded, then error bounds are given as follows:*

$$\|u_{n,m} - u\|_\infty = max_{\tau\in[0,1]}|u_{n,m}(\tau) - u(\tau)| \le \theta_1 h^{K+1},$$
$$\|u'_{n,m} - u'\|_\infty = max_{\tau\in[0,1]}|u'_{n,m}(\tau) - u'(\tau)| \le \theta_2 h^{K},$$

where $u_{n,m}(\tau)$ is the approximate solution and $u(\tau)$ is the exact solution of the problem (1.1), and $h = max|\tau_{i+1} - \tau_i|$, $i = 1, 2, \ldots, n$, and θ_1 and θ_2 are positive constants.

Remark 4.4. The stability of the solution to Eq. (1.1) is defined in the kernel space $\hat{\mathbf{W}}^K[0,1]$. Let $u(.)$ be a solution of Eq. (1.1). The approximate method, which computes $u_{n,m}(.)$ using the right-hand side $\mathbf{F}_n(.)$, is called stable in $\hat{\mathbf{W}}^K[0,1]$ if the following is true: whenever $\lim_{n\to\infty}\|\mathbf{F}-\mathbf{F}_n\|_{\hat{\mathbf{W}}^K}=0$, it follows that

$$\lim_{n\to\infty}\|u-u_{n,m}\|_{\hat{\mathbf{W}}^K}=0.$$

To investigate the stability of the proposed method for the solution of problem (1.1). We add a perturbation ε in the right-hand side. On the other hand, we demonstrate the variation of the obtained solution from the proposed method is bounded by a constant multiple of ε. In other words, the obtained solution depends continuously on the right-hand side.

Theorem 4.5. *The present method is stable in the reproducing kernel space* $\hat{\mathbf{W}}^K[0,1]$.

Proof. Suppose the Eq. (1.1) has solution $u(\tau)$ and let $\mathbf{L}(u_{n,m}(\tau))=\mathbf{F}_n(\tau)$ and $\mathbf{F}(\tau)=\mathbf{F}_n(\tau)+\varepsilon_n(\tau)$, where $\varepsilon_n(.)$ is a perturbation and $\varepsilon_n(.)\xrightarrow{\hat{\mathbf{W}}^K}0$ $(n\to\infty)$. From the Equations (3.1) and (3.2), we get

$$u(\tau)=\sum_{i=1}^{\infty}c_i\xi_i(\tau),\qquad u_{n,m}(\tau)=\sum_{i=1}^{n}c_i\xi_i(\tau),$$

for $\mathbf{F}(\tau),\mathbf{F}_n(\tau)\in\mathbf{W}^1[0,1]$, we have

$$\mathbf{L}(u(\tau)-u_{n,m}(\tau))=\mathbf{F}(\tau)-\mathbf{F}_n(\tau)=\varepsilon_n(\tau).$$

According to the properties of the operator L in Eq. (1.1), we follow the existence and uniqueness of the solution, namely, the operator $\mathbf{L}^{-1}$ exists. Therefore,

$$u(\tau)-u_{n,m}(\tau)=\mathbf{L}^{-1}\varepsilon_n(\tau).$$

Since $\mathbf{L}^{-1}$ is continuous, it is bounded, and $\varepsilon_n(.)\xrightarrow{\hat{\mathbf{W}}^K}0$ $(n\to\infty)$, we have

$$\lim_{n\to\infty}\|u-u_{n,m}\|_{\hat{\mathbf{W}}^K}\le\|\mathbf{L}^{-1}\|\|\varepsilon_n\|_{\hat{\mathbf{W}}^K}=0.$$

□

5. Numerical Examples

In this section we used software package Mathematica 12.1, and absolute errors are used to show numerical example results. The convergence orders for the approximate solutions are calculated using the $C_r=Log_2^{E_n/E_{2n}}$ where $E_n=Max_{\tau\in[0,1]}\left|u(\tau)-u_{n,m}(\tau)\right|$ and $E'_n=Max_{\tau\in[0,1]}\left|u'(\tau)-u'_{n,m}(\tau)\right|$. Comparing the accuracy of the present method with that of the method used in [22] (E_n,E'_n) are given in Tables 1, 4, 7, and 8 for Examples 5.1, 5.2, and 5.3, specifically in the spaces $\mathbf{W}^3[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$. The convergence orders for Examples 5.1, 5.2, and 5.3 with different numbers of collocation points $(n=5,10,20,40)$ are calculated and presented in Tables 2, 3, 5, 6, 9, and 10. These results serve as evidence for the accuracy of error analysis in Theorem 4.3. The graphs of the absolute errors for the approximate solutions and their derivatives with $n=11$ and $n=51$ for Examples 5.1, 5.2, and 5.3 in the spaces $\hat{\mathbf{W}}^2[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$ are given in Figures 1–6.

Example 5.1. [22–24] Consider the nonlocal functional differential equation with advanced argument as follows:

$$\begin{cases}u'(\tau)+u(\tau)-sin(\sqrt{\tau})u(\frac{\tau^2}{2})=\mathbf{F}(\tau), & \tau\in(0,1),\\ u(0)-u(\frac{1}{8})-u(\frac{1}{2})+a_0u(\frac{1}{8})=0,\end{cases}$$

where $a_0=5.15793$is considered such that the exact solution is $u(\tau)=sinh(\tau)$.

Example 5.2. [22–24] Consider the functional differential equation with integral condition of the following form:

$$\begin{cases}u'(\tau)+500e^{\tau}u(\sqrt{\tau})+2000u(\tau)=\mathbf{F}(\tau), & \tau\in(0,1),\\ u(1)=5\int_0^1 su(s)ds,\end{cases}$$

where the exact solution is $u(\tau)=\tau^3$.

TABLE 1. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ and $\mathbf{W}^3[0,1]$ for Example 5.1.

PM $\hat{\mathbf{W}}^3$				[22] $\mathbf{W}^3$	
E_{11}	E_{51}	E'_{11}	E'_{51}	E_{11}	E_{51}
2.50×10^{-6}	1.25×10^{-7}	1.40×10^{-4}	2.00×10^{-7}	1.60×10^{-4}	1.00×10^{-6}

TABLE 2. Convergence orders for Example 5.1 in space $\hat{\mathbf{W}}^2[0,1]$.

E_5	E_{10}	$Log2\,\frac{E_5}{E_{10}}$	E_{20}	$Log2\,\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\,\frac{E_{20}}{E_{40}}$
1.20×10^{-3}	3.00×10^{-4}	2.00	7.00×10^{-5}	2.09954	1.50×10^{-5}	2.22239
E'_5	E'_{10}	$Log2\,\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\,\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\,\frac{E'_{20}}{E'_{40}}$
3.00×10^{-2}	8.00×10^{-3}	1.90689	2.00×10^{-3}	2.00	4.00×10^{-4}	2.32193

TABLE 3. Convergence orders for Example 5.1 in space $\hat{\mathbf{W}}^3[0,1]$.

E_5	E_{10}	$Log2\,\frac{E_5}{E_{10}}$	E_{20}	$Log2\,\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\,\frac{E_{20}}{E_{40}}$
2.50×10^{-4}	5.00×10^{-6}	5.64386	3.00×10^{-7}	4.05889	1.25×10^{-7}	1.26303
E'_5	E'_{10}	$Log2\,\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\,\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\,\frac{E'_{20}}{E'_{40}}$
2.50×10^{-3}	2.00×10^{-4}	3.64386	1.20×10^{-5}	4.05889	8.00×10^{-7}	3.90689

Example 5.3. [22–24] Consider the following functional differential equation with proportional delay (the pantograph equation):

$$\begin{cases} u'(\tau) + \frac{1}{10}u(\frac{\tau}{5}) + u(\tau) = \mathbf{F}(\tau), & \tau \in (0,1), \\ u(0) = 1, \end{cases}$$

where the exact solution of this equation is $u(\tau) = e^{-\tau}$.

The numerical results obtained using the RKS method are presented in the tables and all figures, including the convergence order, maximum absolute error for an approximate solution, and maximum absolute error for the derivative of an approximate solution. More details are given in the introduction of this section. Further, the numerical results obtained by applying the proposed method are in agreement with the theoretical results and order of convergence.

Table 4. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ and $\mathbf{W}^3[0,1]$ for Example 5.2.

PM $\hat{\mathbf{W}}^3$				[22] $\mathbf{W}^3$	
E_{11}	E_{51}	E'_{11}	E'_{51}	E_{11}	E_{51}
4.00×10^{-5}	8.00×10^{-9}	4.00×10^{-3}	8.00×10^{-6}	1.50×10^{-4}	1.75×10^{-6}

Table 5. Convergence orders for Example 5.2 in space $\hat{\mathbf{W}}^2[0,1]$.

E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
1.50×10^{-2}	1.20×10^{-3}	3.64386	8.00×10^{-5}	3.90689	5.00×10^{-6}	4.00
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
6.00×10^{-1}	1.50×10^{-1}	2.00	4.00×10^{-2}	1.90689	1.00×10^{-2}	2.00

Table 6. Convergence orders for Example 5.2 in space $\hat{\mathbf{W}}^3[0,1]$.

E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
3.50×10^{-3}	6.00×10^{-5}	5.86625	1.40×10^{-6}	5.42146	2.50×10^{-8}	5.80735
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
8.00×10^{-2}	6.00×10^{-3}	3.73697	3.00×10^{-4}	4.32193	1.50×10^{-5}	4.32193

Table 7. Comparison of the errors in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.3.

τ	[22] $\mathbf{W}^3$ $(n=11)$	PM $\hat{\mathbf{W}}^2$ $(n=11)$	[22] $\mathbf{W}^3$ $(n=51)$	PM $\hat{\mathbf{W}}^2$ $(n=51)$
2^{-1}	1.59×10^{-6}	6.09×10^{-7}	3.33×10^{-11}	1.13×10^{-9}
2^{-2}	1.87×10^{-6}	2.60×10^{-7}	4.13×10^{-11}	2.03×10^{-10}
2^{-3}	2.71×10^{-6}	1.60×10^{-7}	4.62×10^{-11}	1.38×10^{-10}
2^{-4}	2.41×10^{-6}	2.45×10^{-8}	4.89×10^{-11}	1.46×10^{-10}
2^{-5}	1.00×10^{-6}	2.07×10^{-11}	5.05×10^{-11}	1.07×10^{-11}
2^{-6}	3.51×10^{-7}	6.71×10^{-9}	4.74×10^{-11}	9.69×10^{-12}

TABLE 8. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.3.

τ	[22] $\mathbf{W}^3$ $(n=11)$	PM $\hat{\mathbf{W}}^3$ $(n=11)$	[22] $\mathbf{W}^3$ $(n=51)$	PM $\hat{\mathbf{W}}^3$ $(n=51)$
2^{-1}	1.59×10^{-6}	5.26×10^{-7}	3.33×10^{-11}	2.50×10^{-10}
2^{-2}	1.87×10^{-6}	1.07×10^{-8}	4.13×10^{-11}	3.58×10^{-11}
2^{-3}	2.71×10^{-6}	1.57×10^{-7}	4.62×10^{-11}	7.70×10^{-12}
2^{-4}	2.41×10^{-6}	3.64×10^{-7}	4.89×10^{-11}	4.17×10^{-11}
2^{-5}	1.00×10^{-6}	6.39×10^{-7}	5.05×10^{-11}	1.69×10^{-11}
2^{-6}	3.51×10^{-7}	3.99×10^{-7}	4.74×10^{-11}	1.915×10^{-11}

TABLE 9. Convergence orders for Example 5.3 in space $\hat{\mathbf{W}}^2[0,1]$.

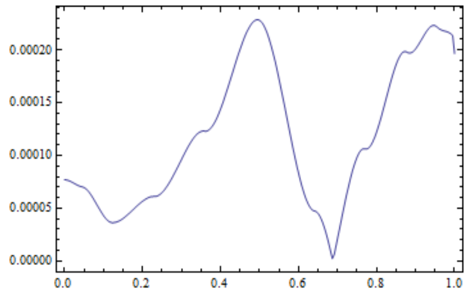
E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
2.50×10^{-5}	1.20×10^{-6}	4.38082	6.0×10^{-8}	4.32193	4.00×10^{-9}	3.90689
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
3.00×10^{-4}	2.50×10^{-5}	3.58496	2.50×10^{-6}	3.32193	3.00×10^{-7}	3.05889

TABLE 10. Convergence orders for Example 5.3 in space $\hat{\mathbf{W}}^3[0,1]$.

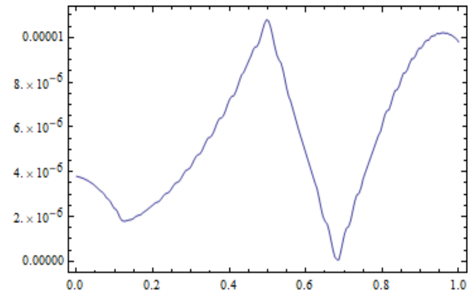
E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
6.00×10^{-5}	1.20×10^{-6}	5.64386	3.50×10^{-8}	5.09954	8.00×10^{-10}	5.45121
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
8.00×10^{-4}	5.00×10^{-5}	4.00	2.50×10^{-6}	4.32193	7.00×10^{-8}	5.15843

6. Conclusion

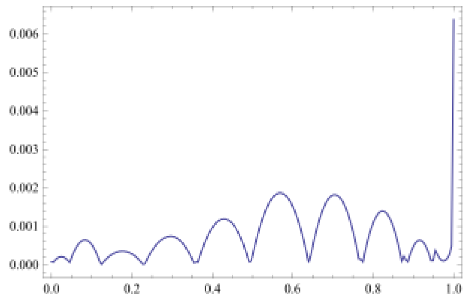
In this paper, we have successfully solved non-local functional differential equations with delayed or advanced arguments using various implementations of the RKM. This approach eliminates the need for the Gram-Schmidt orthogonalization process. Our method allows for the straightforward incorporation of non-local conditions into the reproducing kernel of the spaces $W^2[0,1]$ and $W^3[0,1]$, resulting in the creation of new spaces $\hat{\mathbf{W}}^2[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$ for solving the problem. After comparing the tables and figures related to absolute errors, it can be concluded that the present method has a faster convergence rate for both the approximate solution and its derivative compared to the method used in [22]. When comparing the convergence orders in Tables 2, 3, 5, 6, 9, and 10, it is clear that the convergence rates for Examples 5.1, 5.2, and 5.3 are $O(h^3)$ and $O(h^2)$ in the space $\hat{\mathbf{W}}^2[0,1]$, and $O(h^4)$ and $O(h^3)$ in the space $\hat{\mathbf{W}}^3[0,1]$, respectively.



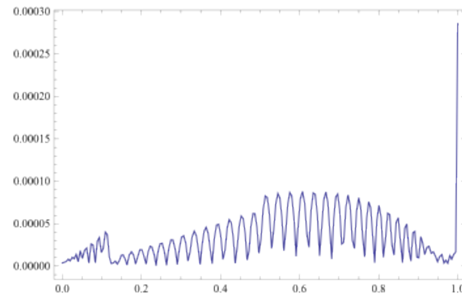
(a) $|u(.) - u_{11}(.)|$.



(b) $|u(.) - u_{51}(.)|$.

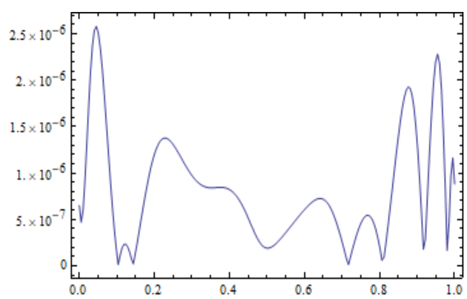


(c) $|u'(.) - u'_{11}(.)|$.

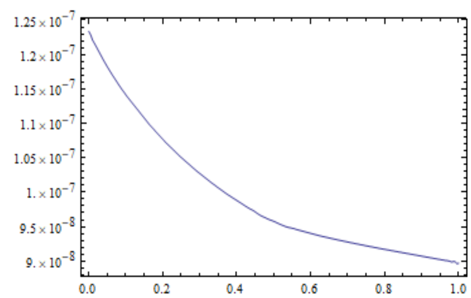


(d) $|u'(.) - u'_{51}(.)|$.

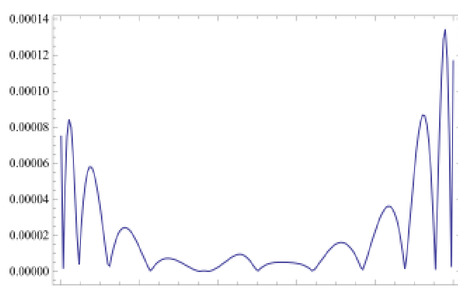
FIGURE 1. The graphs of the absolute errors in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.1.



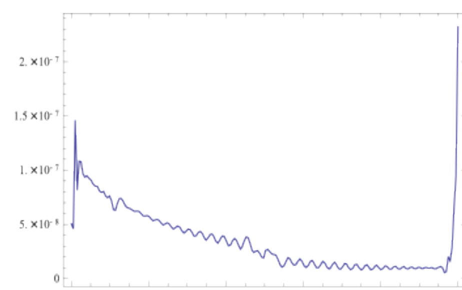
(a) $|u(.) - u_{11}(.)|$.



(b) $|u(.) - u_{51}(.)|$.

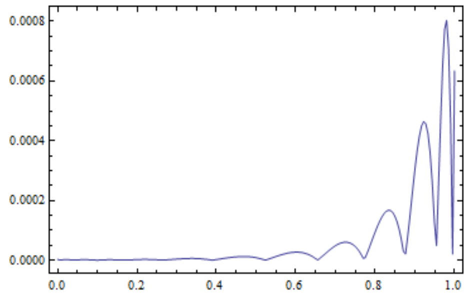


(c) $|u'(.) - u'_{11}(.)|$.

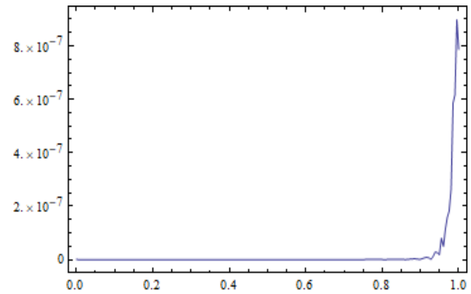


(d) $|u'(.) - u'_{51}(.)|)$.

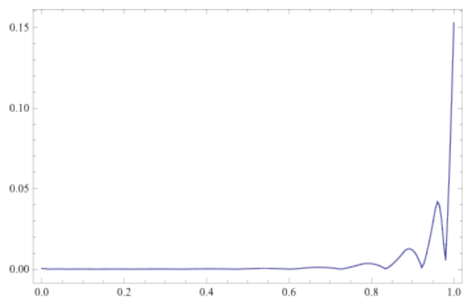
FIGURE 2. The graphs of the absolute errors in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.1.



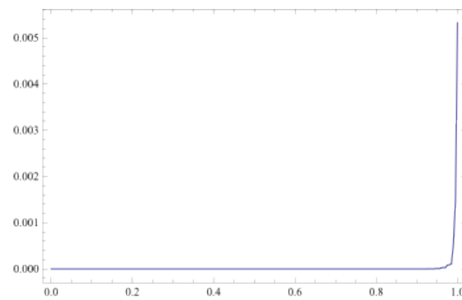
(a) $|u(.) - u_{11}(.)|$.



(b) $|u(.) - u_{51}(.)|$.

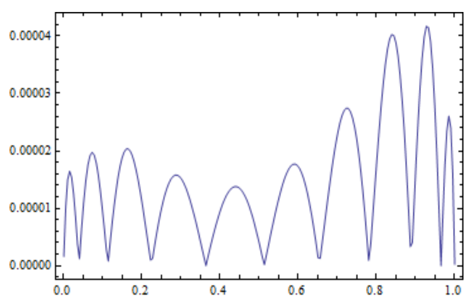


(c) $|u'(.) - u'_{11}(.)|$.

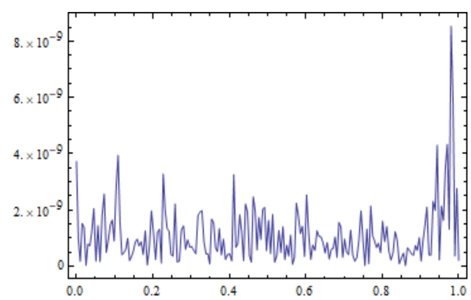


(d) $|u'(.) - u'_{51}(.)|$.

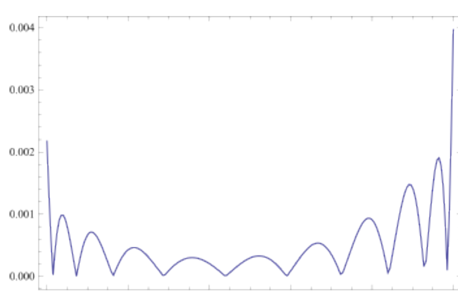
FIGURE 3. The graphs of the absolute errors in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.2.



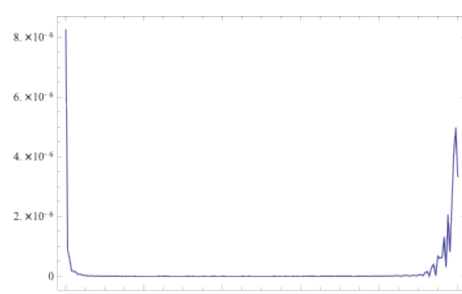
(a) $|u(.) - u_{11}(.)|$.



(b) $|u(.) - u_{51}(.)|$.

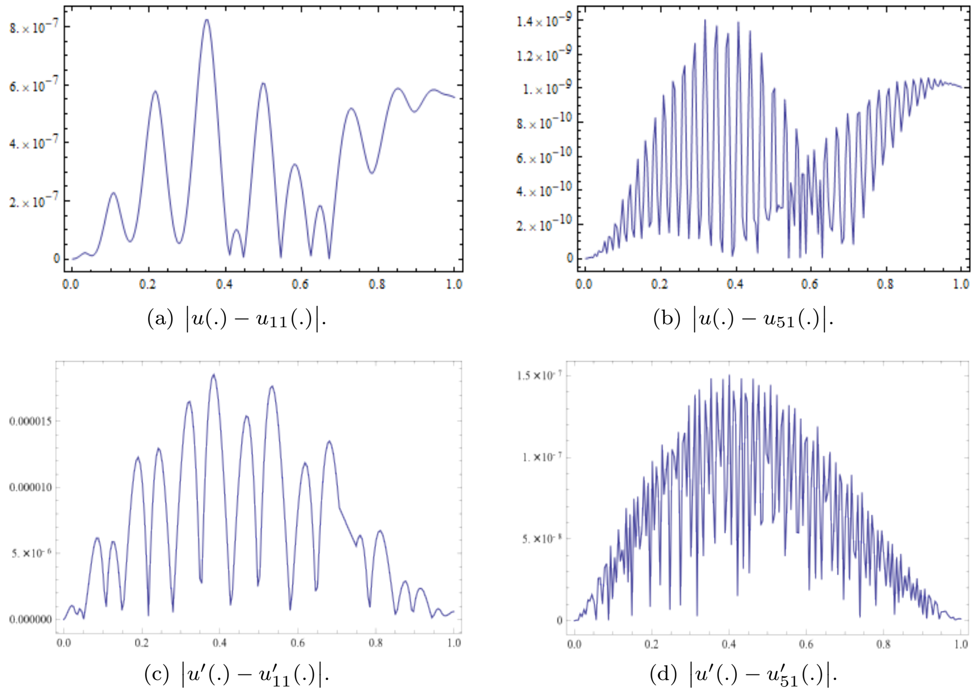


(c) $|u'(.) - u'_{11}(.)|$.



(d) $|u'(.) - u'_{51}(.)|$.

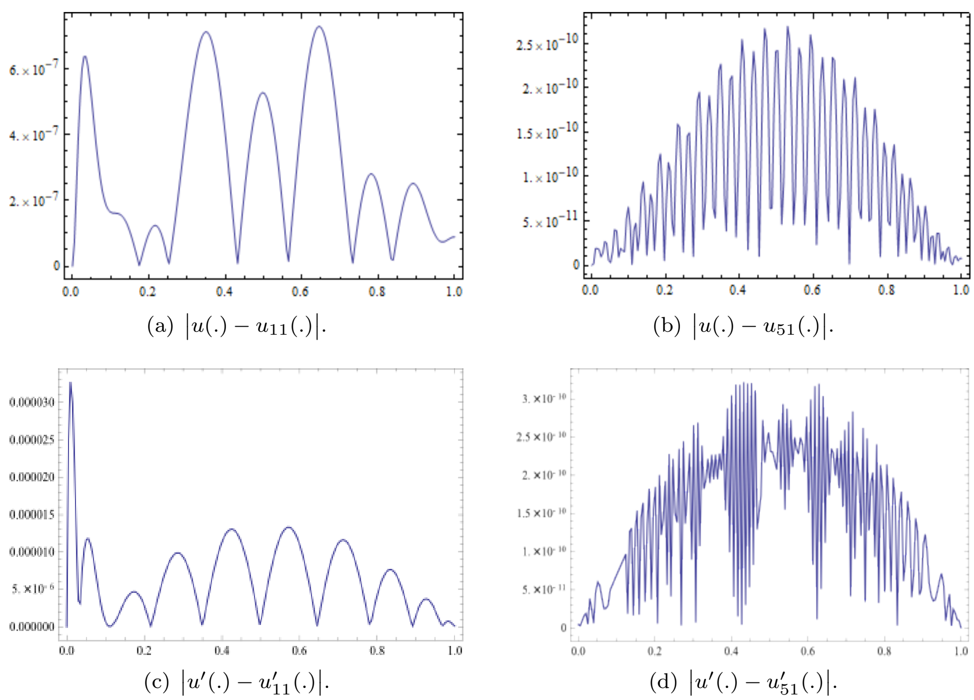
FIGURE 4. The graphs of the absolute errors in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.2.



(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

FIGURE 5. The graphs of the absolute errors in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.3.



(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

FIGURE 6. The graphs of the absolute errors in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.3.

Declarations

Author Contributions: I confirm that all authors listed on the title page have contributed significantly to the work, have read the manuscript, attest to the validity and legitimacy of the data and its interpretation, and agree to its submission.

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References

[1] S. Abbasbandy, H. Sahihi, and T. Allahviranloo, *Combining the reproducing kernel method with a practical technique to solve the system of nonlinear singularly perturbed boundary value problems*, Comput. Methods Differ. Equ., *10*(4) (2022), 942–953.

[2] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, *Fractional order differential systems involving right Caputo and left Riemann–Liouville fractional derivatives with nonlocal coupled conditions*, Bound. Value Probl., *109* (2019), 2019.

[3] T. Allahviranloo and H. Sahihi, *Reproducing kernel method to solve parabolic partial differential equations with nonlocal conditions*, Numer. Method. Partial Diff. Equ., *36* (2020), 1758–1772.

[4] T. Allahviranloo and H. Sahihi, *Reproducing kernel method to solve fractional delay differential equations*, Appl. Math. Comput., *400* (2021), 126095.

[5] M. M. Alsuyuti, E. H. Doha, S. S. Ezz-Eldien, and I. K. Youssef, *Spectral Galerkin schemes for a class of multi-order fractional pantograph equations*, J. Comput. Appl. Math., *384* (2021), 113157.

[6] N. Aronszajn, *Theory of reproducing kernel*, Trans. Amer. Math. Soc., *68* (1950), 337–404.

[7] K. Atkinson and W. Han, *Theoretical Numerical Analysis A Functional Analysis Framework*, Third Edition. Springer Science, 2009.

[8] E. Babolian and D. Hamedzadeh, *A splitting iterative method for solving second kind integral equations in reproducing kernel spaces*, J. Comput. Appl. Math., *326* (2017), 204–216.

[9] E. Babolian, S. Javadi, and E. Moradi, *Error analysis of reproducing kernel Hilbert space method for solving functional integral equations*, J. Comput. Appl. Math., *300* (2016), 300–311.

[10] J. Caballero, L. Plociniczak, and K. Sadarangani, *Existence and uniqueness of solutions in the Lipschitz space of a functional equation and its application to the behavior of the paradise fish*, Appl. Math. Comput., *477* (2024), 128798.

[11] J. Čermák and L. Nechvátal, *On stability of linear differential equations with commensurate delayed arguments*, Appl. Math. Lett., *125* (2022), 107750.

[12] M. Cui and Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science, Hauppauge, New York, United States, 2009.

[13] J. Diblik , *Novel criterion for the existence of solutions with positive coordinates to a system of linear delayed differential equations with multiple delays*, Appl. Math. Lett., *152* (2024), 109032.

[14] S. S. Ezz-Eldien, *On solving systems of multi-pantograph equations via spectral tau method*, Appl. Math. Comput., *321* (2018), 63–73.

[15] A. Ghasemi and A. Saadatmandi, *A new Bernstein-reproducing kernel method for solving forced Duffing equations with integral boundary conditions*, Comput. Methods Differ. Equ., *12* (2024), 329–337.

[16] C. S. Goodrich, *Pointwise conditions in discrete boundary value problems with nonlocal boundary conditions*, Appl. Math. Comput., *67* (2017), 7–15.
[17] S. Guo and S. Li, *On the stability of reaction–diffusion models with nonlocal delay effect and nonlinear boundary condition*, Appl. Math. Lett., *103* (2020), 106197.
[18] R. Ketabchi, R. Mokhtari, and E. Babolian, *Some error estimates for solving Volterra integral equations by using the reproducing kernel method*, J. Comput. Appl. Math., *273* (2015), 245–250.
[19] X. Li, Y. Gao, and B. Wu, *Mixed reproducing kernel-based iterative approach for nonlinear boundary value problems with nonlocal conditions*, Comput. Methods Differ. Equ., *9* (2021), 649–658.
[20] Z. Y. Li, Y. L. Wang, F. G. Tan, X. H. Wan, H. Yu, and J. S. Duan, *Solving a class of linear nonlocal boundary value problems using the reproducing kernel*, Appl. Math. Comput., *265* (2015), 1098–1105.
[21] X. Y. Li and B. Y. Wu, *Error estimation for the reproducing kernel method to solve linear boundary value problems*, J. Comput. Appl. Math., *243* (2013), 10–15.
[22] X. Y. Li and B. Y. Wu, *A continuous method for nonlocal functional differential equations with delayed or advanced arguments*, J. Math. Anal. Appl., *409* (2014), 485–493.
[23] Y. Muroya, E. Ishiwata, and H. Brunner, *On the attainable order of collocation methods for pantograph integro-differential equations*, J. Comput. Appl. Math., *152* (2003), 347–366.
[24] M. Sezer, S. Yalçinbaş, and M. Gülsu, *A Taylor polynomial approach for solving generalized pantograph equations with nonhomogeneous term*, Int. J. Comput. Math., *85* (2008), 1055–1063.
[25] Y. Wang, T. Chaolu, and P. Jing, *New algorithm for second-order boundary value problems of integro-differential equation*, J. Comput. Appl. Math., *229* (2009), 1–6.
[26] Y. Wang, T. Chaolu, and Z. Chen, *Using reproducing kernel for solving a class of singular weakly nonlinear boundary value problems*, Int. J. Comput. Math., *87* (2010), 367–380.
[27] Y. Wang, X. Cao, and X. Li, *A new method for solving singular fourth-order boundary value problems with mixed boundary conditions*, Appl. Math. Comput., *217* (2011), 7385–7390.
[28] Y. Wang, M. Du, F. Tan, Z. Li, and T. Nie, *Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions*, Appl. Math. Comput., *219* (2013), 5918–5925.