Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-13
DOI:10.22034/cmde.2025.63743.2850

A Reproducing Kernel Method for Solving Nonlocal Functional Differential Equations with Delayed or Advanced Arguments

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Abstract

This paper discusses an effective approach for solving non-local functional differential equations with delayed or advanced arguments. The reproducing kernel method is utilized to avoid the need for an orthogonalization process. The main objective of this technique is to successfully apply this method to solve singular multi-point boundary value problems with non-local conditions, resulting in an accurate approximate solution and a valid error analysis. This method greatly improves the accuracy of the solutions obtained.

Keywords. Reproducing kernel method, Functional differential equation, Non-local conditions, Error analysis.
1991 Mathematics Subject Classification. 65L10, 34K07.

1. Introduction

This paper is concerned with an efficient semi-analytical method to solve non-local functional differential equations with delayed or advanced arguments as follows,

$$\begin{cases} \mathbf{L}(u(\tau)) = u'(\tau) + \rho(\tau)u(\kappa(\tau)) + \varrho(\tau)u(\tau) = \mathbf{N}(u(\tau)) + \mathbf{F}(\tau), & \tau \in [0,1], \\ u(1) = \lambda_1 u(c) - \lambda_2 \int_0^1 s u(s) ds, \quad or \quad u(0) = \sum_{i=1}^{m_1} \nu_i u(\zeta_i), \end{cases} \tag{1.1}$$

where $\rho(.), \varrho(.) \in C[0,1]$ and $\kappa(.) \in C^1[0,1]$ and $\mathbf{L}(u(.))$ is bounded linear operator and $\mathbf{N}(u(.))$ is continuous nonlinear operator and $0 < \zeta_i < 1$, ν_i are constants, $c \in [0,1]$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and m_1 is a constant integer. We suppose, $\mathbf{F}(.)$ is given such that Eq. (1.1) satisfies the existence and uniqueness of the solutions.

The existence and uniqueness of solutions for functional differential equations have been extensively studied in [10, 11, 13, 17]. Several authors have proposed different computational methods for solving non-local functional differential equations [2, 5, 14, 16]. X. Li , B. Wu used the general form of the Reproducing Kernel Method (RKM) to solve Eq. (1.1), [22]. This method is very useful and many researchers use it for solving hard problems, i.e., the system of nonlinear singularly perturbed boundary value problems [1], forced Duffing equations [15], nonlinear boundary value problems [19]. In our research, we utilize a different implementation of the general form of RKM, as presented by Wang et al. in [25, 26]. This approach referred to as RKM without the use of the orthogonalization process, is fully explained in their work. There are main factors to increase the accuracy of the approximate solution in the Reproducing Kernel Method : A suitable choice is an inner product in the reproducing kernel space (for short RKS) because the inner product directly affects the kernel function and accordingly the accuracy of the approximation of the solution. The subsequent factor is the selection of the points in the interval [0,1] to construct the basis of the RKS. Indeed, the equidistance points can not provide an appropriate basis, and subsequently, the approximate solutions cannot be determined with high accuracy. Before expressing our structure in this study, it is worth noting

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Received: 28 September 2024 ; Accepted: 27 February 2025.
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that approximate solutions must be obtained in a large space. We are trying to calculate the approximate solutions in the large space. This work is structured as follows. Section 2 discusses the main definitions and requirements of the nonlocal Functional Differential Equations procedure and the theory of the RKS. In section 3, the approach is introduced without the Gram-Schmidt orthogonalization process the convergence is analyzed, and the error for the presented scheme. Also, several numerical experiments are presented to illustrate the effectiveness of the proposed technique in sections 4, and 5, respectively. Section 6 terminates this article with a brief conclusion.

2. The Construction of the RKS

The reproducing kernel space and the function corresponding to it is constructed as follows. We consider the Hilbert space $\mathbf{W}^K[0,1]$, [12]

$$\mathbf{W}^K[0,1] = \Big\{u(.)|u^{(K-1)}(.) \text{ is absolutely continuous, and } u^{(K)}(.) \in L^2[0,1]\Big\},$$

which are equipped with the following inner products and norms for $K = 2, 3$,

$$< u_1(.), u_2(.) >_{\mathbf{W}^3[0,1]} = \sum_{i=0}^{2} {u_1}^{(i)}(1){u_2}^{(i)}(1) + \int_0^1 {u_1}^{'''}(\tau){u_2}^{'''}(\tau)d\tau,$$

$$< u_1(.), u_2(.) >_{\mathbf{W}^2[0,1]} = u_1(0)u_2(0) + {u_1}^{'}(1){u_2}^{'}(1) + \int_0^1 {u_1}^{''}(\tau){u_2}^{''}(\tau)d\tau,$$

$$\|u(.)\|_{\mathbf{W}^K[0,1]} = \sqrt{< u, u >_{\mathbf{W}^K[0,1]}}, \qquad u_1(.), u_2(.) \in \mathbf{W}^K[0,1].$$

Theorem 2.1. *The space $\mathbf{W}^K[0,1]$ is reproducing kernel Hilbert space and its reproducing kernel is given as follows,*

$$Q_y(\tau) = \begin{cases} Q(\tau, y), & \tau \le y, \\ Q(y, \tau), & \tau > y, \end{cases}$$

where for $u(0) = 0$ and $K = 2$ and $K = 3$ the reproducing kernel $Q(\tau, y)$ is $\frac{\tau^3}{6} + \frac{1}{2}\tau\left(y^2 - 4y\right)$ and

$$\tau^5/120 + (2617\tau y)/2208 - (71\tau^2 y)/138 - (13\tau^5 y)/2208 - (71\tau y^2)/138 + (187\tau^2 y^2)/414 + (\tau^5 y^2)/828$$
$$- (\tau^2 y^3)/12 + (\tau y^4)/24 - (13\tau y^5)/2208 + (\tau^2 y^5)/828 - (\tau^5 y^5)/33120,$$

respectively. Without $u(0) = 0$ for $K = 2$ and $K = 3$ the reproducing kernel $Q(\tau, y)$ is $\frac{\tau^3}{6} + \frac{1}{2}\tau\left(y^2 - 4y\right) - 1$ and $\frac{1}{12}\tau^2\left(-y^3 + 6y^2 - 9y + 4\right) + \frac{1}{24}\tau\left(y^4 - 18y^2 + 56y - 39\right) + \frac{1}{120}\left(-y^5 + 40y^2 - 195y + 276\right)$, respectively, see [12].

Using non-local conditions $u(0) = \sum_{i=1}^{m_1} \nu_i u(\zeta_i)$ and $u(1) = \lambda_1 u(c) - \lambda_2 \int_0^1 su(s)ds$ in the form $\gamma_1 u = u(0) - \sum_{i=1}^{m_1} \nu_i u(\zeta_i)$ and $\gamma_2 u = u(1) - \lambda_1 u(c) + \lambda_2 \int_0^1 su(s)ds$ with the same inner products, we define the reproducing kernel space $\hat{\mathbf{W}}^K[0,1]$ as follow.

Definition 2.2. The reproducing kernel space $\hat{\mathbf{W}}^K[0,1]$ is constructed by satisfying the conditions $\gamma_1 u = 0$, or $\gamma_2 u = 0$ and is defined as:

$$\hat{\mathbf{W}}^K[0,1] = \big\{u(.)|u(.) \in \mathbf{W}^K[0,1], \ \ \gamma_1 u = 0, \text{ or } \gamma_2 u = 0\big\}.$$

It is clear that $\hat{\mathbf{W}}^K[0,1]$ is a closed subspace of $\mathbf{W}^K[0,1]$. Hence, $\hat{\mathbf{W}}^K[0,1]$ is also a reproducing kernel space. Taking account into the operator form of the Eq. (1.1), it is easy to demonstrate that $\mathbf{L} : \hat{\mathbf{W}}^2[0,1] \longrightarrow \mathbf{W}^1[0,1]$ is bounded linear operator.

Theorem 2.3. *[3, 27, 28] If $Q_y(.)$ is reproducing kernel of the space $\mathbf{W}^2[0,1]$, and $B : \mathbf{W}^2[0,1] \longrightarrow \mathbf{W}^1[0,1]$ is a bounded linear operator, and $q_1(\tau) = B_y(Q_y(\tau))$ and $q_2(\tau) = B_y\big(Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{W^2}}\big)$, then*

$$\|q_1\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s)\big)\big),$$

$$\|q_2\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s) - \frac{q_1(s)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big)\big),$$

where, symbols B_τ or B_y indicate that this operator applies on τ or y, respectively.

Theorem 2.4. *If $Q_y(.)$ is reproducing kernel of the space $\mathbf{W}^2[0,1]$, then*

$$\hat{Q}_y(\tau) = Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}},$$

is reproducing kernel of the space $\mathbf{H}[0,1] = \big\{u(.)|u(.) \in \mathbf{W}^2[0,1],\ \ B(u) = 0\big\}$.

Proof. First, we show that $\hat{Q}_y(\tau) \in \mathbf{H}[0,1]$, since $q_1(\tau) = B_y(Q_y(\tau))$, $q_1(y) = B_s(Q_y(s))$ and applying Theorem 2.3, we get $\|q_1\|^2_{\mathbf{W}^2} = B_y\big(B_s\big(Q_y(s)\big)\big)$, thus

$$\begin{aligned} B_y(\hat{Q}_y(\tau)) =& B_y(Q_y(\tau)) - \frac{q_1(\tau)B_y(q_1(y))}{\|q_1\|^2_{\mathbf{W}^2}} \\ =& B_y(Q_y(\tau)) - \frac{B_y(Q_y(\tau))B_y(B_s(Q_y(s)))}{B_y\big(B_s\big(Q_y(s)\big)\big)} = 0. \end{aligned}$$

In continuation we show $\forall u(y) \in \mathbf{H}[0,1]$, $u(\tau) = \big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}}$,

$$\begin{aligned} \big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}} &= \big\langle u(y), Q_y(\tau) - \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big\rangle_{\mathbf{H}} \\ &= \big\langle u(y), Q_y(\tau)\big\rangle_{\mathbf{H}} - \big\langle u(y), \frac{q_1(\tau)q_1(y)}{\|q_1\|^2_{\mathbf{W}^2}}\big\rangle_{\mathbf{H}} \\ &= u(\tau) - \frac{q_1(\tau)}{\|q_1\|^2_{\mathbf{W}^2}} B_s\big\langle u(y), Q_y(s)\big\rangle_{\mathbf{H}}, \end{aligned}$$

since $\hat{Q}_y(\tau) \in \mathbf{H}[0,1]$ we follow $B(u) = 0$, hence $B_s(u(s)) = 0$, as a result $\big\langle u(y), \hat{Q}_y(\tau)\big\rangle_{\mathbf{H}} = u(\tau)$. For more details refer to [3, 20, 27, 28]. ☐

3. MAIN IDEA

Suppose $r_y(\tau)$ is the reproducing kernel function for space $\mathbf{W}^1[0,1]$ and $\{\tau_i\}_{i=1}^{\infty}$ are dense set on domain of Eq. (1.1). We define bases of the reproducing kernel space $\hat{\mathbf{W}}^2[0,1]$ as follow,

$$\xi_i(\tau) = \hat{Q}_y(\tau)|_{y=\tau_i}.$$

Theorem 3.1. *[12] If $\{\tau_i\}_{i=1}^{\infty}$ are dense set on $[0,1]$ then $\xi_i(\tau) = \hat{Q}_y(\tau)|_{y=\tau_i}$ is a system of complete functions in $\hat{\mathbf{W}}^2[0,1]$.*

Theorem 3.2. *[12] If $\{\tau_i\}_{i=1}^{\infty}$ are dense set on $[0,1]$, then analytical solution of the Eq. (1.1) is*

$$u_m(\tau) = \sum_{i=1}^{\infty} c_{i,m}\xi_i(\tau), \quad m = 1, 2, \ldots, \tag{3.1}$$

where $c_{i,m}$ represents the unknown coefficients, they can be determined.

We called approximation solution of Eq. (1.1) with $u_{n,m}(\tau)$. We solve following system of algebraic equations for $m = 1, 2, \ldots$ to determine the unknown coefficients $c_{i,m}$,

$$\sum_{i=1}^{n} c_{i,m}\mathbf{L}\xi_i(\tau)|_{\tau=\tau_j} = \mathbf{N}(u_{n,m-1}(\tau))|_{\tau=\tau_j} + \mathbf{F}(\tau_j), \quad m = 1, 2, \ldots \quad j = 1, 2, \ldots, n. \tag{3.2}$$

We have Eq. (3.2) as matrix form,

$$\mathbf{AC} = \mathbf{B},$$

where

$$\mathbf{A} = \mathbf{L}\xi_i(\tau)|_{\tau=\tau_j} = \begin{bmatrix} \mathbf{L}\xi_1(\tau_1) & \mathbf{L}\xi_2(\tau_1) & \mathbf{L}\xi_3(\tau_1) & \dots & \mathbf{L}\xi_{n-1}(\tau_1) & \mathbf{L}\xi_n(\tau_1) \\ \mathbf{L}\xi_1(\tau_2) & \mathbf{L}\xi_2(\tau_2) & \mathbf{L}\xi_3(\tau_2) & \dots & \mathbf{L}\xi_{n-1}(\tau_2) & \mathbf{L}\xi_n(\tau_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{L}\xi_1(\tau_{n-1}) & \mathbf{L}\xi_2(\tau_{n-1}) & \mathbf{L}\xi_3(\tau_{n-1}) & \dots & \mathbf{L}\xi_{n-1}(\tau_{n-1}) & \mathbf{L}\xi_n(t_{n-1}) \\ \mathbf{L}\xi_1(\tau_n) & \mathbf{L}\xi_2(\tau_n) & \mathbf{L}\xi_3(\tau_n) & \dots & \mathbf{L}\xi_{n-1}(\tau_n) & \mathbf{L}\xi_n(\tau_n) \end{bmatrix}, \tag{3.3}$$

and

$$\mathbf{B} = \mathbf{N}(u_{n,m-1}(\tau))|_{\tau=\tau_j} + \mathbf{F}(\tau_j) = \begin{bmatrix} \mathbf{N}(u_{n,m-1}(\tau_1)) + \mathbf{F}(\tau_1) \\ \mathbf{N}(u_{n,m-1}(\tau_2)) + \mathbf{F}(\tau_2) \\ \vdots \\ \mathbf{N}(u_{n,m-1}(\tau_{n-1})) + \mathbf{F}(\tau_{n-1}) \\ \mathbf{N}(u_{n,m-1}(\tau_n)) + \mathbf{F}(\tau_n) \end{bmatrix}, \tag{3.4}$$

and

$$\mathbf{C} = \begin{bmatrix} c_{1,m} \\ c_{2,m} \\ \vdots \\ c_{n-1,m} \\ c_{n,m} \end{bmatrix}, \tag{3.5}$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$, and from our assumptions, the $\mathbf{A}^{-1}$ exists and unique, see [4, 6, 8].

Remark 3.3. We can define another bases for the reproducing kernel space $\hat{\mathbf{W}}^2[0,1]$ as follow,

$$\eta_i(\tau) = \mathbf{L}_y \hat{Q}_y(\tau)|_{y=\tau_i} = \frac{\partial \hat{Q}_y(\tau)}{\partial y}|_{y=\tau_i} + \rho(y)\hat{Q}_y(\kappa(\tau))|_{y=\tau_i} + \varrho(y)\hat{Q}_y(\tau)|_{y=\tau_i}.$$

4. CONVERGENCE ANALYSIS AND ERROR BOUND

Theorem 4.1. *[26] Approximate solution (3.2) and its derivative are uniformly convergent to exact solution of Eq. (1.1).*

Corollary 4.2. *[12] The approximate solution $u^{(l)}_{n,m}(\tau)$ uniformly convergent to $u^{(l)}(\tau)$ in space $\hat{\mathbf{W}}^K[0,1]$ for $K = 2, 3$ and $l = 0, 1$.*

Theorem 4.3. *[9, 18, 21] In the space $\hat{\mathbf{W}}^K[0,1]$, if $u^{(K+1)}(\tau) \in C[0,1]$ and $\|u^{(K+1)}_{n,m}(\tau)\|_\infty$ is bounded, then errors bounds are given as follows,*

$$\|u_{n,m} - u\|_\infty = max_{\tau\in[0,1]}|u_{n,m}(\tau) - u(\tau)| \leq \theta_1 h^{K+1},$$
$$\|u'_{n,m} - u'\|_\infty = max_{\tau\in[0,1]}|u'_{n,m}(\tau) - u'(\tau)| \leq \theta_2 h^{K},$$

where $u_{n,m}(\tau)$ is approximate solution and $u(\tau)$ is exact solution of the problem (1.1) and $h = max|\tau_{i+1} - \tau_i|$, $i = 1, 2, \dots, n$ and θ_1, θ_2 are positive constants.

Remark 4.4. The stability of the solution for Eq. (1.1) is defined in the kernel space $\hat{\mathbf{W}}^K[0,1]$. Let $u(\tau)$ be a solution of Eq. (1.1). It is called that the approximate method on the solution $u(\tau)$ from $u_{n,m}(\tau)$ with the right hand side $\mathbf{F}_n(\tau)$ is stable in $\hat{\mathbf{W}}^K[0,1]$, if $\lim_{n\to\infty} \|F - F_n\|_{\hat{\mathbf{W}}^K} = 0$, then $\lim_{n\to\infty} \|u - u_{n,m}\|_{\hat{\mathbf{W}}^K} = 0$.

To investigate the stability of the proposed method for the solution of problem (1.1). We add a perturbation ε in the right-hand side. On the other hand, we demonstrate variation of the obtained solution from the proposed method is bounded by a constant multiple of ε. In other words, the obtained solution depends continuously on the right-hand side.

Theorem 4.5. *The present method is stable in the reproducing kernel space* $\hat{\mathbf{W}}^K[0,1]$.

Proof. Suppose the Eq. (1.1) has solution $u(\tau)$ and let $\mathbf{L}(u_{n,m}(\tau)) = \mathbf{F}_n(\tau)$ and $\mathbf{F}(\tau) = \mathbf{F}_n(\tau) + \varepsilon_n(\tau)$, where $\varepsilon_n(.)$ is a perturbation and $\varepsilon_n(.) \xrightarrow{\hat{\mathbf{W}}^K} 0 \ (n \to \infty)$. From the Equations (3.1) and (3.2)

$$u(\tau) = \sum_{i=1}^{\infty} c_i \xi_i(\tau), \qquad u_{n,m}(\tau) = \sum_{i=1}^{n} c_i \xi_i(\tau),$$

for $\mathbf{F}(\tau), \mathbf{F}_n(\tau) \in \mathbf{W}^1[0,1]$, we have,

$$\mathbf{L}(u(\tau) - u_{n,m}(\tau)) = \mathbf{F}(\tau) - \mathbf{F}_n(\tau) = \varepsilon_n(\tau).$$

According to the properties of the operator L in Eq. (1.1), we follow the existence and uniqueness of the solution, namely, the operator $\mathbf{L}^{-1}$ exists. therefore,

$$u(\tau) - u_{n,m}(\tau) = \mathbf{L}^{-1}\varepsilon_n(\tau),$$

since $\mathbf{L}^{-1}$ is continuous, it is bounded and $\varepsilon_n(.) \xrightarrow{\hat{\mathbf{W}}^K} 0 \ (n \to \infty)$, we have

$$\lim_{n\to\infty} \|u - u_{n,m}\|_{\hat{\mathbf{W}}^K} \leq \|\mathbf{L}^{-1}\| \|\varepsilon_n\|_{\hat{\mathbf{W}}^K} = 0.$$

□

5. NUMERICAL EXAMPLES

In this section we used software package Mathematica 12.1 and absolute errors are used to show numerical examples results. The convergence order for the approximate solutions are calculated using the $C_r = Log_2^{E_n/E_{2n}}$ where $E_n = Max_{\tau\in[0,1]}\left|u(\tau) - u_{n,m}(\tau)\right|$ and $E'_n = Max_{\tau\in[0,1]}\left|u'(\tau) - u'_{n,m}(\tau)\right|$. Comparing the accuracy of the present method with the method [22] (E_n, E'_n) are given in the Tables 1, 4, 7, and 8 for Examples 5.1, 5.2, and 5.3 in the spaces $\mathbf{W}^3[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$. The convergence orders for Examples 5.1, 5.2, and 5.3 with different numbers of collocation points $(n = 5, 10, 20, 40)$ are calculated and presented in Tables 2, 3, 5, 6, 9, and 10. These results serve as evidence for the accuracy of error analysis Theorem 4.3. The graph of the absolute errors for the approximate solutions and their derivatives with $n = 11$ and $n = 51$ for Examples 5.1, 5.2, and 5.3 in the spaces $\hat{\mathbf{W}}^2[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$ are given in the Figures 1, 2, 3, 4, 5, and 6.

Example 5.1. [22–24] Consider the nonlocal functional differential equation with advanced argument as follows:

$$\begin{cases} u'(\tau) + u(\tau) - sin(\sqrt{\tau})u(\frac{\tau^2}{2}) = \mathbf{F}(\tau), & \tau \in (0,1), \\ u(0) - u(\frac{1}{8}) - u(\frac{1}{2}) + a_0 u(\frac{1}{8}) = 0, \end{cases}$$

where $a_0 = 5.15793$is considered such that exact solution is $u(\tau) = sinh(\tau)$.

Example 5.2. [22–24] Consider the functional differential equation with integral condition of the form:

$$\begin{cases} u'(\tau) + 500e^{\tau}u(\sqrt{\tau}) + 2000u(\tau) = \mathbf{F}(\tau), & \tau \in (0,1), \\ u(1) = 5\int_0^1 su(s)ds, \end{cases}$$

the exact solution by $u(\tau) = \tau^3$.

Example 5.3. [22–24] Consider the following functional differential equation with proportional delay (pantograph equation) :

$$\begin{cases} u'(\tau) + \frac{1}{10}u(\frac{\tau}{5}) + u(\tau) = \mathbf{F}(\tau), & \tau \in (0,1), \\ u(0) = 1, \end{cases}$$

the exact solution of this equation is $u(\tau) = e^{-\tau}$.

TABLE 1. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ and $\mathbf{W}^3[0,1]$ for Example 5.1.

PM $\hat{\mathbf{W}}^3$				[22] $\mathbf{W}^3$	
E_{11}	E_{51}	E'_{11}	E'_{51}	E_{11}	E_{51}
2.50×10^{-6}	1.25×10^{-7}	1.40×10^{-4}	2.00×10^{-7}	1.60×10^{-4}	1.00×10^{-6}

TABLE 2. Convergence order for Example 5.1 in space $\hat{\mathbf{W}}^2[0,1]$.

E_5	E_{10}	$Log2\frac{E_5}{E_{10}}$	E_{20}	$Log2\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\frac{E_{20}}{E_{40}}$
1.20×10^{-3}	3.00×10^{-4}	2.00	7.00×10^{-5}	2.09954	1.50×10^{-5}	2.22239
E'_5	E'_{10}	$Log2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\frac{E'_{20}}{E'_{40}}$
3.00×10^{-2}	8.00×10^{-3}	1.90689	2.00×10^{-3}	2.00	4.00×10^{-4}	2.32193

TABLE 3. Convergence order for Example 5.1 in space $\hat{\mathbf{W}}^3[0,1]$.

E_5	E_{10}	$Log2\frac{E_5}{E_{10}}$	E_{20}	$Log2\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\frac{E_{20}}{E_{40}}$
2.50×10^{-4}	5.00×10^{-6}	5.64386	3.00×10^{-7}	4.05889	1.25×10^{-7}	1.26303
E'_5	E'_{10}	$Log2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\frac{E'_{20}}{E'_{40}}$
2.50×10^{-3}	2.00×10^{-4}	3.64386	1.20×10^{-5}	4.05889	8.00×10^{-7}	3.90689

TABLE 4. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ and $\mathbf{W}^3[0,1]$ for Example 5.2.

PM $\hat{\mathbf{W}}^3$				[22] $\mathbf{W}^3$	
E_{11}	E_{51}	E'_{11}	E'_{51}	E_{11}	E_{51}
4.00×10^{-5}	8.00×10^{-9}	4.00×10^{-3}	8.00×10^{-6}	1.50×10^{-4}	1.75×10^{-6}

The numerical results obtained using the RKS method are presented in the tables and all figures, including the convergence order, maximum absolute error for an approximate solution, and maximum absolute error for the derivative of an approximate solution, more details are given in the introduction of this section. Further, the numerical results obtained by applying the proposed method are in agreement with the theoretical results and order of convergence.

Table 5. Convergence order for Example 5.2 in space $\hat{\mathbf{W}}^2[0,1]$.

E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
1.50×10^{-2}	1.20×10^{-3}	3.64386	8.00×10^{-5}	3.90689	5.00×10^{-6}	4.00
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
6.00×10^{-1}	1.50×10^{-1}	2.00	4.00×10^{-2}	1.90689	1.00×10^{-2}	2.00

Table 6. Convergence order for Example 5.2 in space $\hat{\mathbf{W}}^3[0,1]$.

E_5	E_{10}	$Log_2\frac{E_5}{E_{10}}$	E_{20}	$Log_2\frac{E_{10}}{E_{20}}$	E_{40}	$Log_2\frac{E_{20}}{E_{40}}$
3.50×10^{-3}	6.00×10^{-5}	5.86625	1.40×10^{-6}	5.42146	2.50×10^{-8}	5.80735
E'_5	E'_{10}	$Log_2\frac{E'_5}{E'_{10}}$	E'_{20}	$Log_2\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2\frac{E'_{20}}{E'_{40}}$
8.00×10^{-2}	6.00×10^{-3}	3.73697	3.00×10^{-4}	4.32193	1.50×10^{-5}	4.32193

Table 7. Comparison of the errors in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.3.

τ	[22]	PM	[22]	PM
	$\mathbf{W}^3$	$\hat{\mathbf{W}}^2$	$\mathbf{W}^3$	$\hat{\mathbf{W}}^2$
	$(n=11)$	$(n=11)$	$(n=51)$	$(n=51)$
2^{-1}	1.59×10^{-6}	6.09×10^{-7}	3.33×10^{-11}	1.13×10^{-9}
2^{-2}	1.87×10^{-6}	2.60×10^{-7}	4.13×10^{-11}	2.03×10^{-10}
2^{-3}	2.71×10^{-6}	1.60×10^{-7}	4.62×10^{-11}	1.38×10^{-10}
2^{-4}	2.41×10^{-6}	2.45×10^{-8}	4.89×10^{-11}	1.46×10^{-10}
2^{-5}	1.00×10^{-6}	2.07×10^{-11}	5.05×10^{-11}	1.07×10^{-11}
2^{-6}	3.51×10^{-7}	6.71×10^{-9}	4.74×10^{-11}	9.69×10^{-12}

Table 8. Comparison of the errors in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.3.

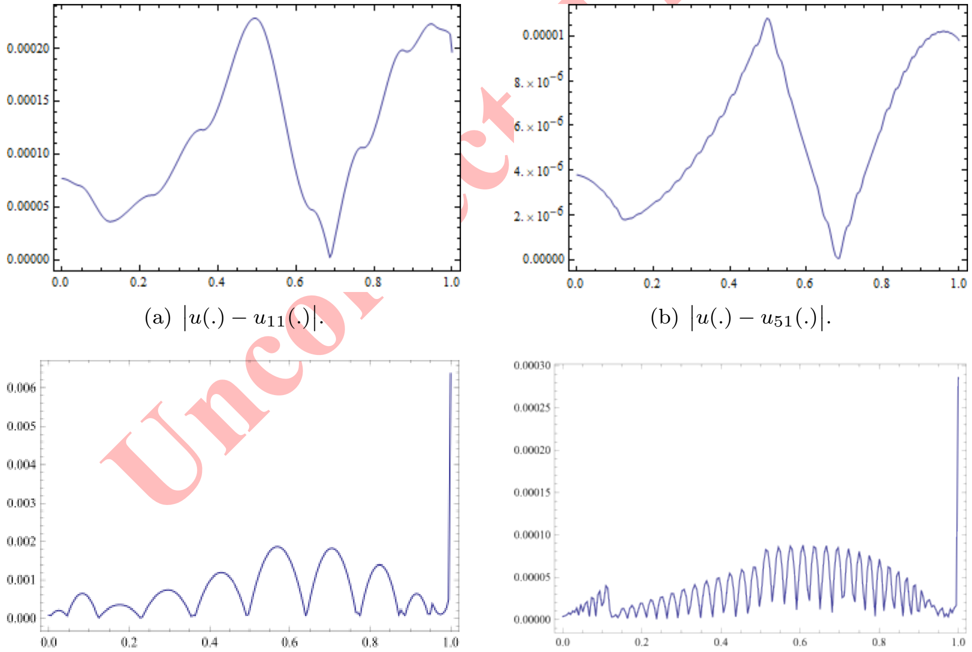
τ	[22]	PM	[22]	PM
	$\mathbf{W}^3$	$\hat{\mathbf{W}}^3$	$\mathbf{W}^3$	$\hat{\mathbf{W}}^3$
	$(n=11)$	$(n=11)$	$(n=51)$	$(n=51)$
2^{-1}	1.59×10^{-6}	5.26×10^{-7}	3.33×10^{-11}	2.50×10^{-10}
2^{-2}	1.87×10^{-6}	1.07×10^{-8}	4.13×10^{-11}	3.58×10^{-11}
2^{-3}	2.71×10^{-6}	1.57×10^{-7}	4.62×10^{-11}	7.70×10^{-12}
2^{-4}	2.41×10^{-6}	3.64×10^{-7}	4.89×10^{-11}	4.17×10^{-11}
2^{-5}	1.00×10^{-6}	6.39×10^{-7}	5.05×10^{-11}	1.69×10^{-11}
2^{-6}	3.51×10^{-7}	3.99×10^{-7}	4.74×10^{-11}	1.915×10^{-11}

TABLE 9. Convergence order for Example 5.3 in space $\hat{\mathbf{W}}^2[0,1]$.

E_5	E_{10}	$Log2\,\frac{E_5}{E_{10}}$	E_{20}	$Log2\,\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\,\frac{E_{20}}{E_{40}}$
2.50×10^{-5}	1.20×10^{-6}	4.38082	6.0×10^{-8}	4.32193	4.00×10^{-9}	3.90689
E'_5	E'_{10}	$Log2\,\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\,\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\,\frac{E'_{20}}{E'_{40}}$
3.00×10^{-4}	2.50×10^{-5}	3.58496	2.50×10^{-6}	3.32193	3.00×10^{-7}	3.05889

TABLE 10. Convergence order for Example 5.3 in space $\hat{\mathbf{W}}^3[0,1]$.

E_5	E_{10}	$Log2\,\frac{E_5}{E_{10}}$	E_{20}	$Log2\,\frac{E_{10}}{E_{20}}$	E_{40}	$Log2\,\frac{E_{20}}{E_{40}}$
6.00×10^{-5}	1.20×10^{-6}	5.64386	3.50×10^{-8}	5.09954	8.00×10^{-10}	5.45121
E'_5	E'_{10}	$Log2\,\frac{E'_5}{E'_{10}}$	E'_{20}	$Log2\,\frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log2\,\frac{E'_{20}}{E'_{40}}$
8.00×10^{-4}	5.00×10^{-5}	4.00	2.50×10^{-6}	4.32193	7.00×10^{-8}	5.15843



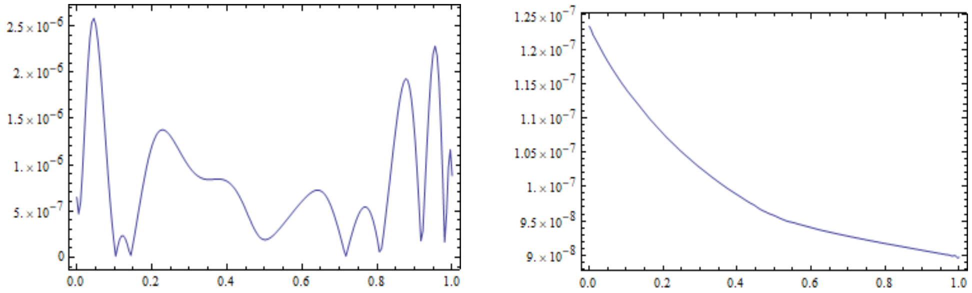
(a) $|u(.) - u_{11}(.)|$.

(b) $|u(.) - u_{51}(.)|$.

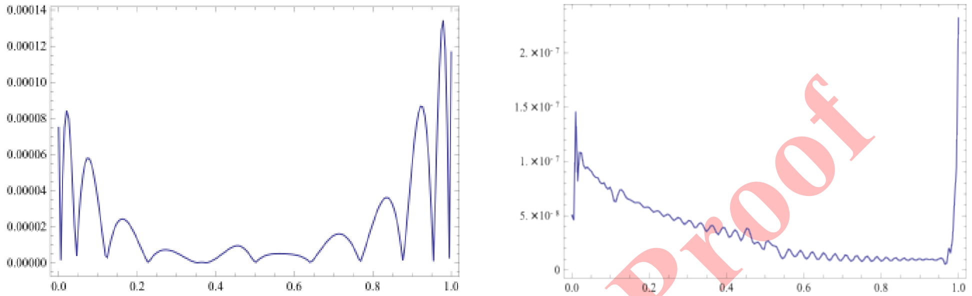
(c) $|u'(.) - u'_{11}(.)|$.

(d) $|u'(.) - u'_{51}(.)|$.

FIGURE 1. Graphs of the absolute error in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.1.

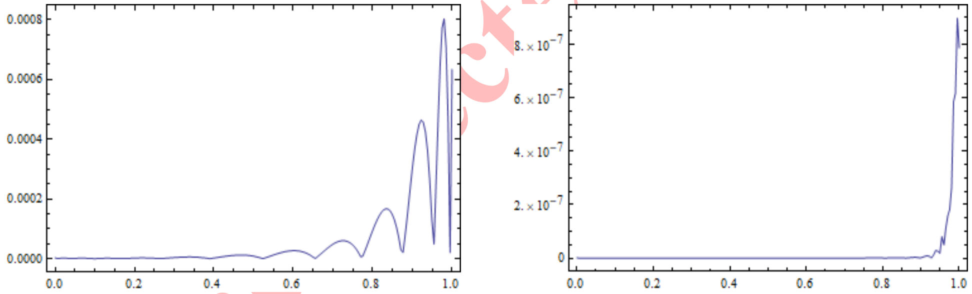


(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

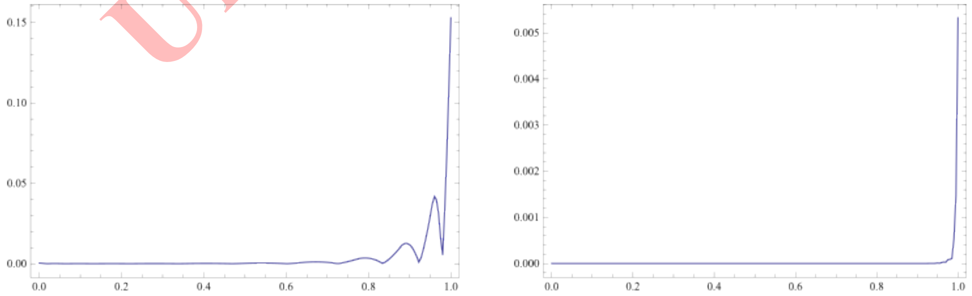


(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$).

FIGURE 2. Graphs of the absolute error in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.1.

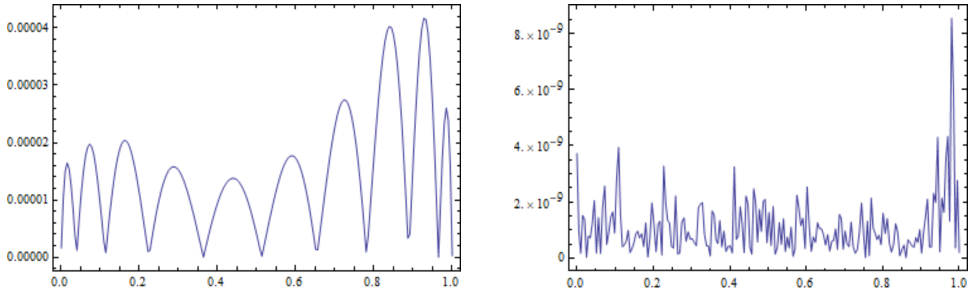


(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

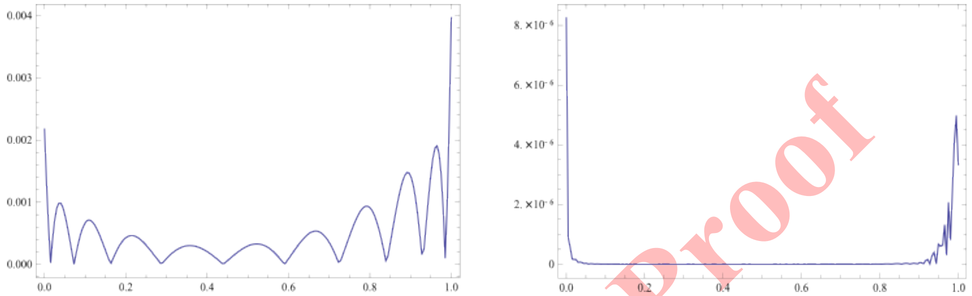


(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

FIGURE 3. Graphs of the absolute error in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.2.

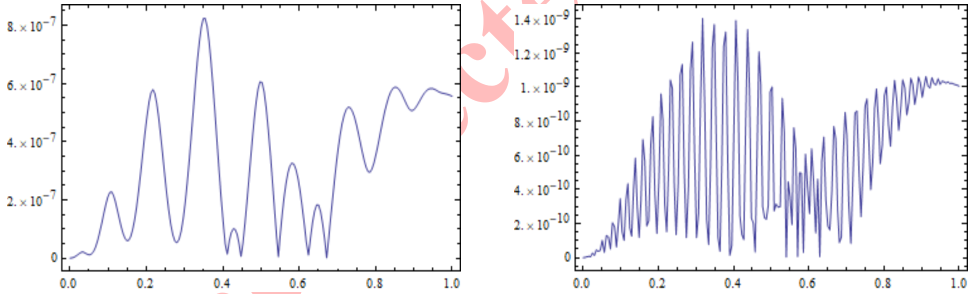


(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

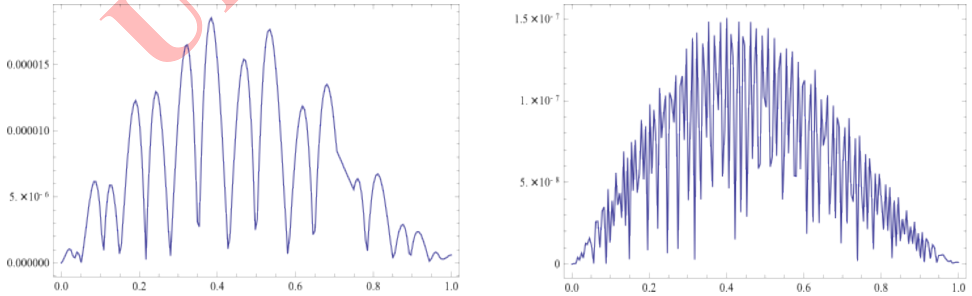


(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

FIGURE 4. Graphs of the absolute error in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.2.

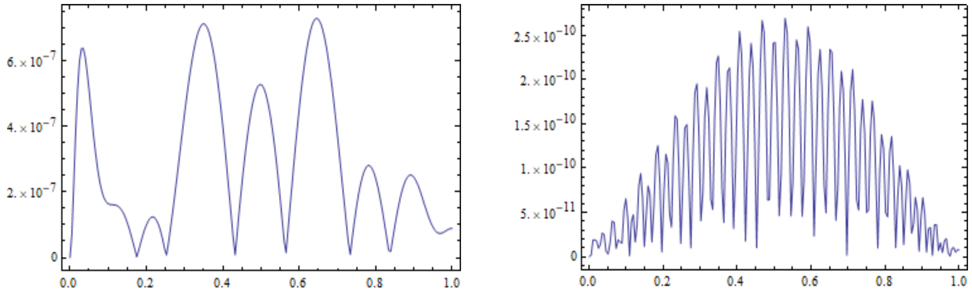


(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.

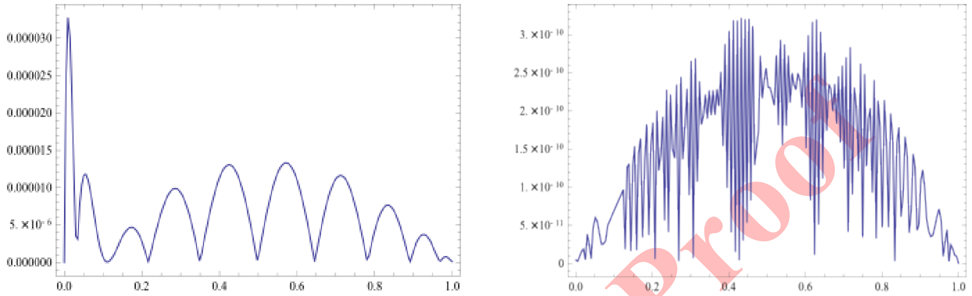


(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

FIGURE 5. Graphs of the absolute error in space $\hat{\mathbf{W}}^2[0,1]$ for Example 5.3.



(a) $|u(.) - u_{11}(.)|$. (b) $|u(.) - u_{51}(.)|$.



(c) $|u'(.) - u'_{11}(.)|$. (d) $|u'(.) - u'_{51}(.)|$.

Figure 6. Graphs of the absolute error in space $\hat{\mathbf{W}}^3[0,1]$ for Example 5.3.

6. Conclusion

In this paper, we have successfully solved non-local functional differential equations with delayed or advanced arguments using various implementations of the RKM. This approach eliminates the need for the Gram-Schmidt orthogonalization process. Our method allows for the straightforward incorporation of non-local conditions into the reproducing kernel of the spaces $W^2[0,1]$ and $W^3[0,1]$, resulting in the creation of new spaces $\hat{\mathbf{W}}^2[0,1]$ and $\hat{\mathbf{W}}^3[0,1]$ for solving the problem. After comparing the tables and figures related to absolute errors, it can be concluded that the present method has a faster convergence rate for both the approximate solution and its derivative compared to the method used in [22]. When comparing the convergence order tables 2,3,5,6,9 and 10, it is clear that the convergence rates for Examples 5.1, 5.2, and 5.3 are $O(h^3)$, $O(h^2)$ in the space $\hat{\mathbf{W}}^2[0,1]$, and $O(h^4)$ and $O(h^3)$ in the space $\hat{\mathbf{W}}^3[0,1]$, respectively.

Declarations

Author Contributions: I confirm that all authors listed on the title page have contributed significantly to the work, have read the manuscript, attest to the validity and legitimacy of the data and its interpretation, and agree to its submission.

Funding: The authors declare that this research received no grant from any funding agency in the public, commercial, or not-for-profit sectors.

Competing interests: The authors declare that they have no conflict of interest.

Data availability: All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

We thank the anonymous reviewers for helpful comments, which lead to definite improvement in the manuscript.

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