Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 4, 2025, pp. 1233-1249
DOI:10.22034/cmde.2024.64302.2910

Application of the analytical method for solving the chemical kinetics system

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Abstract

This work introduces an enhanced $tan(\chi/2)$-expansion method to obtain exact solutions for chemical kinetics systems. This technique works directly with the governing equations of chemical kinetics systems. Our work yields many new fundamental traveling wave solutions that combine periodic functions with soliton-like and other trigonometric shapes. To better illustrate our solutions, we show visual representations by assigning specific values to the arbitrary constants. The improved expansion method successfully obtains kink, singular kink, and multiple soliton solutions. The results demonstrate that the method is effective for real-world applications and physics equations. Graphical visualizations support our findings to show the method's accuracy and reliability. Our suggested method effectively solves nonlinear equations and provides useful results for studying complicated wave behavior across multiple scientific disciplines.

Keywords. Improved $\tan(\Phi(\chi)/2)$-expansion method, Generalized (G'/G)-expansion method, Travelling wave, The chemical kinetics system, Solitons, Kink, Periodic and rational solutions.

2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p.

1. Introduction

Nonlinear partial differential equations (PDEs) function as essential tools to investigate nonlinear wave propagation. Nonlinear behavior occurs throughout multiple scientific disciplines like applied science, engineering, and mathematical physics, as well as biosciences, neurosciences, and other areas [2, 19]. Our mathematical models use NLPs because they provide effective solutions for real-world problems. Chemical kinetics, dynamics, and mathematical physics benefit from NLP use to describe both action potential spread and physical processes such as thermal conductivity and wave propagation [25, 30–32]. Researchers in mathematical physics and applied science develop their presentations through mathematical NLP techniques. Our understanding of mathematical physics and applied science has driven the development of numerous nonlinear partial differential equation solutions [22–24]. Since ancient beginnings, NLPs have advanced significantly during the last fifty years [20, 33]. Engineers, scientists, and mathematicians now spend more time studying different NLP problems [12, 27]. Research into nonlinear partial differential equations come first because they drive mathematical physics and applied science. The applications of derivative and integral formulas in chemical kinetics and dynamics help scientists better understand fluid mechanics, control theory, plasma physics, viscoelasticity, and many other real-world phenomena. Nonlinear dynamical systems research now includes mathematical methods to study NLPs [3, 26]. NLPs, which represent advanced DE types, help scientists understand complex wave movement behavior in different systems. Engineers and scientists have discovered wave solutions and simplified descriptions for NLPs during recent years. Our analysis starts with a system of chemical process species differentials that we identify as A, B, and C [5]. The three reactions are as follows

$$A \longrightarrow B, \tag{1.1}$$

$$B + C \longrightarrow A + C, \tag{1.2}$$

Received: 01 November 2024 ; Accepted: 22 February 2025.
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$$B + B \longrightarrow C. \tag{1.3}$$

When we define u_1, u_2, and u_3 as A, B, and C concentrations the reaction rates become k_1, k_2 and k_3. We call the speed of reaction in Equation (1.1) a value of $k1$. When C functions as a catalyst during reaction, Eq. (1.2) to create A from B the reaction rate becomes k_2. The production of C from B depends on the rate constant k_3 alone. Our complete system consists of these integrated elements

$$\begin{aligned} \frac{du_1}{dt} &= -k_1 u_1 + k_2 u_2 u_3, \\ \frac{du_2}{dt} &= k_1 u_1 - k_2 u_2 u_3 - k_3 u_2^2, \\ \frac{du_3}{dt} &= k_3 u_2^2. \end{aligned} \tag{1.4}$$

When three reaction rates remain within small numerical boundaries and show similar size differences, this type of problem can be handled easily [4]. In [18], Khader applied Picard iteration techniques to chemical kinetics problems using Pade' approximation. Different methods exist to solve challenging mathematical models. The improved $\tan(\Phi(\chi)/2)$-expansion method demonstrates its strength as a solution technique for nonlinear differential equations. The improved $\tan(\Phi(\chi)/2)$-expansion method facilitates finding exact soliton solutions while providing new insights into how system parameters affect nerve wave motion. This research uses the expansion method to discover fresh chemical equation wave patterns and examine how order and nonlinearity values control soliton movement. Our research studies how these parameters shape wave profiles and their stability to help us better understand brain signals when healthy and when sick. The research findings can be used across neuroscience, engineering, and applied mathematics with practical applications in mind. The physical meaning of soliton solutions becomes clearer through their study. Research groups have used different approaches to solve the issue, but their methods work only for limited cases. Our main purpose is to develop the chemical equation through $\tan(\Phi(\chi)/2)$-expansion and show new wave solutions. Present studies agree on using the homotopy analysis technique ([7, 8, 29]), variational iteration technique ([9, 13, 17]), the homotopy perturbation technique ([6, 9]), the tanh-coth technique ([15, 16, 21]), the Exp-function method ([10, 11, 14]), the $(G^{'}/G)$-expansion method ([1, 28]) and more. Many engineers study natural language processing because of its practical uses in real-world challenges. The study of wave solutions for NLPs brings together useful scientific knowledge with real-world applications. Researchers work hard to understand wave solutions for NLPs are useful because of their practical applications. This work studies wave solutions for a wide range of NLPs that appear in scientific and engineering fields. Scientists found solitons in the 1960s through the Korteweg-de Vries (KdV) equation work. Scientists discovered that solitons help explain how waves move throughout different physical systems, including natural biological processes. Scientists have increased their research into solitons for chemical systems over the last decade. These models replace traditional integer models with solutions that include information storage. To deliver fresh insights into your study, examine how the improved $\tan(\Phi(\chi)/2)$-expansion technique solves the soliton model. Study how complex wave patterns, including rogue waves and multi-soliton collisions, adjust chemical dynamics to show how nonlinear systems function. ITET serves as a solution method for discovering precise outcomes from nonlinear differential equations. The approach transforms difficult nonlinear equations into a form easier to handle.

The paper is managed as follows: In section 2, we describe the ITET. Also, in section 3, we describe the improved $\tan(\Phi(\chi)/2)$-expansion technique. In section 4, we examine the chemical kinetics system with two methods expressed in sections 2 and 3. Moreover, the conclusion and advantages are pointed out in section 5.

2. Steps of improved $\tan(\Phi(\chi)/2)$-expansion technique

Step 1. Suppose that the given nonlinear partial differential equation for $\mathrm{u(x,t)}$ is in the following

$$\mathcal{N}(\mathrm{u}, \mathrm{u_x}, \mathrm{u_t}, \mathrm{u_{xx}}, \mathrm{u_{tt}}, ...) = 0, \tag{2.1}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u'', \mu^2 u'', ...) = 0. \tag{2.2}$$

Convert the original nonlinear differential equation into a new form by adding a new variable or expanding the series to make it easier to solve. Match the top nonlinear term with the top derivative term to discover a basic solution shape.

Step 2. Choose your solution type from the available options after balancing.

$$u(\chi) = S(\Phi) = \sum_{k=0}^{m} A_k \left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^k + \sum_{k=1}^{m} B_k \left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^{-k}, \tag{2.3}$$

where $A_k (0 \le k \le m)$ and $B_k (1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\Phi = \Phi(\chi)$ satisfies the following ordinary differential equation:

$$\Phi'(\chi) = r_1 \sin(\Phi(\chi)) + r_2 \cos(\Phi(\chi)) + r_3. \tag{2.4}$$

The following unique solutions to Equation (2.4) will be examined:

Family 1: **When** $r_1^2 + r_2^2 - r_3^2 < 0$ **and** $r_2 - r_3 \neq 0$**,** then

$\Phi(\chi) = -2\arctan\left[-\frac{r_1}{r_2 - r_3} + \frac{\sqrt{-(r_1^2+r_2^2-r_3^2)}}{r_2 - r_3}\tan\left(\frac{\sqrt{-(r_1^2+r_2^2-r_3^2)}}{2}(\chi+\omega)\right)\right]$.

Family 2: **When** $r_1^2 + r_2^2 - r_3^2 > 0$ **and** $r_2 - r_3 \neq 0$**,** then

$\Phi(\chi) = -2\arctan\left[-\frac{r_1}{r_2 - r_3} - \frac{\sqrt{-(r_1^2+r_2^2-r_3^2)}}{-(r_1^2+r_2^2-r_3^2)}\tanh\left(\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{2}(\chi+\omega)\right)\right]$.

Family 3: **When** $r_1^2 + r_2^2 - r_3^2 > 0$**,** $r_2 \neq 0$ **and** $r_3 = 0$, then

$\Phi(\chi) = 2\arctan\left[\frac{r_1}{r_2} + \frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{r_1^2+r_2^2}}{2}(\chi+\omega)\right)\right]$.

Family 4: **When** $r_1^2 + r_2^2 - r_3^2 < 0$**,** $r_3 \neq 0$ **and** $r_2 = 0$, then

$\Phi(\chi) = 2\arctan\left[-\frac{r_1}{r_3} + \frac{\sqrt{r_3^2 - r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right]$.

Family 5: **When** $r_1^2 + r_2^2 - r_3^2 > 0$**,** $r_2 - r_3 \neq 0$ **and** $r_1 = 0$, then

$\Phi(\chi) = 2\arctan\left[\sqrt{\frac{r_2+r_3}{r_2-r_3}}\tanh\left(\frac{\sqrt{r_2^2-r_3^2}}{2}(\chi+\omega)\right)\right]$.

Family 6: **When** $r_1 = 0$ **and** $r_3 = 0$, then $\Phi(\chi) = \arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1}, \frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]$.

Family 7: **When** $r_2 = 0$ **and** $r_3 = 0$, then $\Phi(\chi) = \arctan\left[\frac{2e^{r_1(\chi+\omega)}}{e^{2r_1(\chi+\omega)}+1}, \frac{e^{2r_1(\chi+\omega)}-1}{e^{2r_1(\chi+\omega)}+1}\right]$.

Family 8: **When** $r_1^2 + r_2^2 = r_3^2$**,** then $\Phi(\chi) = -2\arctan\left[\frac{(r_2 + r_3(r_1(\chi+\omega)+2)}{r_1^2(\chi+\omega)}\right]$.

Family 9: **When** $r_1 = r_2 = r_3 = kr_1$**,** then $\Phi(\chi) = 2\arctan\left[e^{kr_1(\chi+\omega)} - 1\right]$.

Family 10: **When** $r_1 = r_3 = kr_1$ **and** $r_2 = -kr_1$, then $\Phi(\chi) = -2\arctan\left[\frac{e^{kr_1(\chi+\omega)}}{-1+e^{kr_1(\chi+\omega)}}\right]$.

Family 11: **When** $r_3 = r_1$, then $\Phi(\chi) = -2\arctan\left[\frac{(r_+ r_2)e^{r_2(\chi+\omega)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right]$.

Family 12: **When** $r_1 = r_3$, then $\Phi(\chi) = 2\arctan\left[\frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right]$.

Family 13: **When** $r_3 = -r_1$**,** then $\Phi(\chi) = 2\arctan\left[\frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right]$.

Family 14: **When** $r_2 = -r_3$**,** then $\Phi(\chi) = -2\arctan\left[\frac{r_1 e^{r_1(\chi+\omega)}}{r_3 e^{r_1(\chi+\omega)}-1}\right]$.

Family 15: **When** $r_2 = 0$ **and** $r_1 = r_3$**,** then $\Phi(\chi) = -2\arctan\left[\frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right]$.

Family 16: **When** $r_1 = 0$ **and** $r_2 = r_3$**,** then $\Phi(\chi) = 2\arctan\left[r_3(\chi+\omega)\right]$.

Family 17: **When** $r_1 = 0$ **and** $r_2 = -r_3$**,** then $\Phi(\chi) = -2\arctan\left[\frac{1}{r_3(\chi+\omega)}\right]$,

where $A_k, B_k (k = 1, 2, ..., m), r_1, r_2$ and r_3 are constants to be determined later. The number m must be positive and we determine it by balancing all highest order derivative terms with nonlinear terms in Eq. (2.4).

Step 3. Put the guessed solution into the adjusted equation.

Step 4. Solve algebraic equations to find the solution coefficients in the assumed expression. Insert the found coefficients into the assumed solution to discover the specific equation solution.

3. The chemical kinetics system

Chemical system professionals and related fields depend on chemical equation solutions because these descriptions help them better understand chemical dynamics.

3.1. **The ITET.** We take the chemical kinetics system as follows

$$\begin{aligned} \frac{\mathrm{du_1}}{\mathrm{dt}} &= -\mathrm{k_1u_1} + \mathrm{k_2u_2u_3}, \\ \frac{\mathrm{du_2}}{\mathrm{dt}} &= \mathrm{k_1u_1} - \mathrm{k_2u_2u_3} - \mathrm{k_3u_2^2}, \\ \frac{\mathrm{du_3}}{\mathrm{dt}} &= \mathrm{k_3u_2^2}, \end{aligned} \tag{3.1}$$

utilizing the wave variable $\chi = \mathrm{kt} + \omega$ transforms to an ODE

$$\begin{aligned} \mathrm{ku_1'} &= -\mathrm{k_1u_1} + \mathrm{k_2u_2u_3}, \\ \mathrm{ku_2'} &= \mathrm{k_1u_1} - \mathrm{k_2u_2u_3} - \mathrm{k_3u_2^2}, \\ \mathrm{ku_3'} &= \mathrm{k_3u_2^2}. \end{aligned} \tag{3.2}$$

Balancing the $\mathrm{u_1'}$ and $\mathrm{u_2u_3}$, $\mathrm{u_2'}$ and $\mathrm{u_2^2}$, $\mathrm{u_3'}$ and $\mathrm{u_2^2}$ utilizing homogeneous technique, we get

$$\mathrm{M} + 1 = \mathrm{N} + \mathrm{P}, \qquad \mathrm{N} + 1 = 2\mathrm{N}, \qquad \mathrm{P} + 1 = 2\mathrm{N}, \tag{3.3}$$

then

$$\mathrm{M} = \mathrm{N} = \mathrm{P} = 1.$$

Then the trail solutions are

$$\begin{aligned} \mathrm{u_1}(\chi) &= \mathrm{A_{01}} + \mathrm{A_{11}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right] + \mathrm{B_{11}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^{-1}, \\ \mathrm{u_2}(\chi) &= \mathrm{A_{02}} + \mathrm{A_{12}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right] + \mathrm{B_{12}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^{-1}, \\ \mathrm{u_3}(\chi) &= \mathrm{A_{03}} + \mathrm{A_{13}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right] + \mathrm{B_{13}}\left[\mathrm{p} + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^{-1}. \end{aligned} \tag{3.4}$$

Appending (3.4) and (2.4) into Eq. (3.2) and by utilizing the well-known Maple software, we achieve the following sets of non-trivial solutions

Set I:

$$k = \frac{2k_3A_{02}}{(r_2 - r_3)B_{12}},\ A_{01} = -2A_{02},\ B_{11} = -2B_{12},\ A_{02} = A_{02},\ A_{12} = 0,\ B_{12} = B_{12}, \tag{3.5}$$

$$\begin{aligned} &A_{03} = -\tfrac{2(k_3A_{02}+k_1)}{k_2},\ A_{13} = 0,\ B_{13} = -\tfrac{2k_3B_{12}}{k_2},\ p = \tfrac{r_1 \pm \sqrt{(r_3^2 - r_2^2)A_{02}^2}}{(r_2 - r_3)(A_{02} - 1)}, \\ &u_1(\chi) = A_{01} + B_{11}\left[p + \tan\left(\tfrac{\Phi(\chi)}{2}\right)\right]^{-1}, \\ &u_2(\chi) = A_{02} + B_{12}\left[p + \tan\left(\tfrac{\Phi(\chi)}{2}\right)\right]^{-1}, \\ &u_3(\chi) = A_{03} + B_{13}\left[p + \tan\left(\tfrac{\Phi(\chi)}{2}\right)\right]^{-1}, \end{aligned} \tag{3.6}$$

where $\mathrm{r_1}, \mathrm{r_2}$ and c are free constants. Utilizing the (3.6) and **Family 1** and **4** respectively can be written as

$$u_{1_1}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_1 \pm \sqrt{(r_3^2 - r_2^2)A_{02}^2}}{(r_2 - r_3)(A_{02} - 1)} + \frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{2}(\chi + \omega)\right)\right]^{-1},$$

$$u_{2_1}(\chi) = A_{02} + B_{12}\left[\frac{a \pm \sqrt{(r_3^2 - r_2^2)A_{02}^2}}{(r_2 - r_3)(A_{02} - 1)} + \frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{2}(\chi + \omega)\right)\right]^{-1},$$

$$u_{3_1}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{a \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1}{r_2-r_3} - \frac{\sqrt{r_3^2-r_1^2-r_2^2}}{b-c}\tan\left(\frac{\sqrt{r_3^2-r_1^2-r_2^2}}{2}(\chi+\omega)\right)\right]^{-1}, \tag{3.7}$$

$$u_{1_2}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_1 \pm r_3A_{02}}{r_3(1-A_{02})} - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{2_2}(\chi) = A_{02} + B_{12}\left[\frac{r_1 \pm r_3A_{02}}{r_3(1-A_{02})} - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{3_2}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1 \pm r_3A_{02}}{r_3(1-A_{02})} - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right]^{-1}. \tag{3.8}$$

Utilizing the (3.6) and **Family 2, 3** and **5** in section 2 respectively get

$$u_{1_3}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_1 \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1}{r_2-r_3} + \frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{2_3}(\chi) = A_{02} + B_{12}\left[\frac{r_1 \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1}{r_2-r_3} + \frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{b-c}}}{2}(\chi+\omega)\right)\right]^{-1}, \tag{3.9}$$

$$u_{3_3}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{a \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1}{r_2-r_3} + \frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{1_4}(\chi) = -2A_{02} - 2B_{12}\left[\frac{a \pm \sqrt{-r_2^2A_{02}^2}}{r_2(A_{02}-1)} + \frac{r_1}{r_2} + \frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2}}{r_2}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{2_4}(\chi) = A_{02} + B_{12}\left[\frac{r_1 \pm \sqrt{-r_2^2A_{02}^2}}{r_2(A_{02}-1)} + \frac{r_1}{r_2} + \frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2}}{r_2}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{3_4}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1 \pm \sqrt{-r_2^2A_{02}^2}}{r_2(A_{02}-1)} + \frac{r_1}{r_2} + \frac{\sqrt{r_1^2+r_2^2}}{b}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_1^2+r_2^2}}{r_2}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{1_5}(\chi) = -2A_{02} - 2B_{12}\left[\frac{\pm\sqrt{-(r_2^2-r_3^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{2_5}(\chi) = A_{02} + B_{12}\left[\frac{\pm\sqrt{-(r_2^2-r_3^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{3_5}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{\pm\sqrt{-(r_2^2-r_3^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{\frac{\sqrt{r_2^2-r_3^2}}{r_2-r_3}}}{2}(\chi+\omega)\right)\right]^{-1}.$$

Employing the (3.6) and **Family 6** in section 2 we get

$$\mathrm{u}_{1_6}(\chi) = -2\mathrm{A}_{02} - 2\mathrm{B}_{12}\left[\frac{\pm\sqrt{-\mathrm{r}_2^2\mathrm{A}_{02}^2}}{\mathrm{r}_2(\mathrm{A}_{02}-1)} + \tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}, \frac{2\mathrm{e}^{\mathrm{b}(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right]^{-1}, \tag{3.10}$$

$$\mathrm{u}_{2_6}(\chi) = \mathrm{A}_{02} + \mathrm{B}_{12}\left[\frac{\pm\sqrt{-\mathrm{r}_2^2\mathrm{A}_{02}^2}}{\mathrm{r}_2(\mathrm{A}_{02}-1)} + \tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}, \frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right]^{-1},$$

$$u_{3_6}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{\pm\sqrt{-r_2^2A_{02}^2}}{r_2(A_{02}-1)} + \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1}, \frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]\right)\right]^{-1}.$$

Employing the (3.6) and **Family 8** in section 2 we get

$$\mathrm{u}_{1_7}(\chi) = -2\mathrm{A}_{02} - 2\mathrm{B}_{12}\left[\frac{\mathrm{r}_1 \pm \mathrm{r}_1\mathrm{A}_{02}}{(\mathrm{r}_2-\mathrm{r}_3)(\mathrm{A}_{02}-1)} + \frac{\mathrm{r}_1(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right]^{-1}, \tag{3.11}$$

$$u_{2_7}(\chi) = A_{02} + B_{12}\left[\frac{r_1 \pm r_1A_{02}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right]^{-1},$$

$$u_{3_7}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1 \pm r_1A_{02}}{(r_2-r_3)(A_{02}-1)} + \frac{r_1(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right]^{-1}.$$

Employing the (3.6) and **Family 10** in section 2 give

$$\mathrm{u}_{1_8}(\chi) = -2\mathrm{A}_{02} - 2\mathrm{B}_{12}\left[\frac{-1}{2(\mathrm{A}_{02}-1)} - \frac{\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}}{\left[\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}-1\right]}\right]^{-1}, \tag{3.12}$$

$$\mathrm{u}_{2_8}(\chi) = \mathrm{A}_{02} + \mathrm{B}_{12}\left[\frac{-1}{2(\mathrm{A}_{02}-1)} - \frac{\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}}{\left[\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}-1\right]}\right]^{-1},$$

$$\mathrm{u}_{3_8}(\chi) = -\frac{2(\mathrm{k}_3\mathrm{A}_{02}+\mathrm{k}_1)}{\mathrm{k}_2} - \frac{2\mathrm{k}_3\mathrm{B}_{12}}{\mathrm{k}_2}\left[\frac{-1}{2(\mathrm{A}_{02}-1)} - \frac{\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}}{\left[\mathrm{e}^{\mathrm{kr}_1(\chi+\omega)}-1\right]}\right]^{-1}.$$

Utilizing the (3.6) and **Family 11** in section 2 give

$$\mathrm{u}_{1_9}(\chi) = -2\mathrm{A}_{02} - 2\mathrm{B}_{12}\left[\frac{\mathrm{a} \pm \sqrt{(\mathrm{r}_1^2-\mathrm{r}_2^2)\mathrm{A}_{02}^2}}{(\mathrm{r}_2-\mathrm{r}_1)(\mathrm{A}_{02}-1)} - \frac{(\mathrm{r}_1+\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}{(\mathrm{r}_1-\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right]^{-1}, \tag{3.13}$$

$$\mathrm{u}_{2_9}(\chi) = \mathrm{A}_{02} + \mathrm{B}_{12}\left[\frac{\mathrm{r}_1 \pm \sqrt{(\mathrm{r}_1^2-\mathrm{r}_2^2)\mathrm{A}_{02}^2}}{(\mathrm{r}_2-\mathrm{r}_1)(\mathrm{A}_{02}-1)} - \frac{(\mathrm{r}_+\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}{(\mathrm{r}_1-\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right]^{-1},$$

$$u_{3_9}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1 \pm \sqrt{(r_1^2-r_2^2)A_{02}^2}}{(r_2-r_1)(A_{02}-1)} - \frac{(r_1+r_2)e^{r_2(\chi+\omega)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right]^{-1}.$$

Employing the (3.6) and **Family 12** in section 2 give

$$u_{1_{10}}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_3 \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right]^{-1}, \tag{3.14}$$

$$u_{2_{10}}(\chi) = A_{02} + B_{12}\left[\frac{r_3 \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right]^{-1},$$

$$u_{3_{10}}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_3 \pm \sqrt{(r_3^2-r_2^2)A_{02}^2}}{(r_2-r_3)(A_{02}-1)} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right]^{-1}.$$

Employing the (3.6) and **Family 13** in section 2 give

$$u_{1_{11}}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_1 \pm \sqrt{(r_1^2-r_2^2)A_{02}^2}}{(r_2+r_1)(A_{02}-1)} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right]^{-1}, \tag{3.15}$$

$$u_{2_{11}}(\chi) = A_{02} + B_{12}\left[\frac{a \pm \sqrt{(r_1^2-r_2^2)A_{02}^2}}{(r_2+r_1)(A_{02}-1)} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right]^{-1},$$

$$u_{3_{11}}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1 \pm \sqrt{(r_1^2-r_2^2)A_{02}^2}}{(r_2+r_1)(A_{02}-1)} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right]^{-1}.$$

Utilizing the (3.6) and **Family 14** in section 2 give

$$u_{1_{12}}(\chi) = -2A_{02} - 2B_{12}\left[\frac{r_1}{2r_3(A_{02}-1)} - \frac{r_1e^{r_1(\chi+\omega)}}{r_3e^{r_1(\chi+\omega)}-1}\right]^{-1}, \tag{3.16}$$

$$u_{2_{12}}(\chi) = A_{02} + B_{12}\left[\frac{r_1}{2r_3(A_{02}-1)} - \frac{r_1e^{r_1(\chi+\omega)}}{r_3e^{r_1(\chi+\omega)}-1}\right]^{-1},$$

$$u_{3_{12}}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{r_1}{2r_3(A_{02}-1)} - \frac{r_1e^{r_1(\chi+\omega)}}{r_3e^{r_1(\chi+\omega)}-1}\right]^{-1}.$$

Employing the (3.6) and **Family 15** in section 2 give

$$u_{1_{13}}(\chi) = -2A_{02} - 2B_{12}\left[\frac{1 \pm A_{02}}{(1-A_{02})} - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right]^{-1}, \tag{3.17}$$

$$u_{2_{13}}(\chi) = A_{02} + B_{12}\left[\frac{1 \pm A_{02}}{(1-A_{02})} - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right]^{-1},$$

$$u_{3_{13}}(\chi) = -\frac{2(k_3A_{02}+k_1)}{k_2} - \frac{2k_3B_{12}}{k_2}\left[\frac{1 \pm A_{02}}{(1-A_{02})} - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right]^{-1},$$

where $\chi = \frac{2k_3A_{02}}{(r_2-r_3)B_{12}}t + \omega$.

Set II:

$$k = k,\ A_{01} = \frac{k(r_3-r_2)+2k_3A_{12}}{-k(r_3-r_2)}\sqrt{\frac{r_3+r_2}{r_2-r_3}}A_{12},\ B_{11}=0,\ A_{02} = \sqrt{\frac{r_3+r_2}{r_2-r_3}}A_{12}, \tag{3.18}$$

$$A_{12} = A_{12},\quad B_{12} = 0,\quad p = 0,$$

$$A_{03} = -\frac{1}{2kk_2\sqrt{r_3^2 - r_2^2}}[k^2(r_2^2 - r_3^2) - 2kk_3A_{12}(r_2 + r_3) + 2kk_1(r_2 - r_3) - 4k_1k_3A_{12}], \tag{3.19}$$

$$A_{13} = \frac{k(r_2 - r_3) - 2k_3A_{12}}{2k_2}, \quad A_{11} = \frac{-k(r_2 - r_3) + 2k_3A_{12}}{k(r_2 - r_3)}A_{12},$$

$$u_1(\chi) = A_{01} + A_{11}\tan\left(\frac{\Phi(\chi)}{2}\right),$$
$$u_2(\chi) = A_{02} + A_{12}\tan\left(\frac{\Phi(\chi)}{2}\right),$$
$$u_3(\chi) = A_{03} + A_{13}\tan\left(\frac{\Phi(\chi)}{2}\right),$$

where a, b, and c are free constants. Utilizing of the (3.19) and **Family 1** and **4** in section 2 respectively can be written as

$$u_{1_{14}}(\chi) = \frac{-k(r_2 - r_3) + 2k_3A_{12}}{k(r_2 - r_3)}A_{12}\left[\sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{2}(\chi + \omega)\right)\right], \tag{3.20}$$

$$u_{2_{14}}(\chi) = A_{12}\left[\sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{2}(\chi + \omega)\right)\right],$$

$$u_{3_{14}}(\chi) = -\frac{1}{2kk_2\sqrt{r_3^2 - r_2^2}}[k^2(r_2^2 - r_3^2) - 2kk_3A_{12}(r_2 + r_3) + 2kk_1(r_2 - r_3) - 4k_1k_3A_{12}]$$
$$+\frac{k(r_2 - r_3) - 2k_3A_{12}}{2k_2}\left[\frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2 - r_2^2}}{2}(\chi + \omega)\right)\right],$$

$$u_{1_{15}}(\chi) = -\frac{kc + 2k_3A_{12}}{k}A_{12}\left[1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2}}{2}(\chi + \omega)\right)\right],$$

$$u_{2_{15}}(\chi) = A_{12}\left[1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2}}{2}(\chi + \omega)\right)\right]$$

$$u_{3_{15}}(\chi) = \frac{1}{2kk_2r_3}[k^2r_3^2 + 2kk_3A_{12}r_3 + 2kk_1r_3 + 4k_1k_3A_{12}]$$
$$-\frac{kr_3 + 2k_3A_{12}}{2k_2}\left[1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2}}{2}(\chi + \omega)\right)\right].$$

Employing the (3.19) and **Family 2, 3**, and **5** in section 2 respectively get

$$u_{1_{16}}(\chi) = \frac{-k(r_2 - r_3) + 2k_3A_{12}}{k(r_2 - r_3)}A_{12}\left[\sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3} + \frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{r_2 - r_3}\tanh\left(\frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{2}(\chi + \omega)\right)\right], \tag{3.21}$$

$$u_{2_{16}}(\chi) = A_{12}\left[\sqrt{\frac{c + b}{c - b}} + \frac{r_1}{r_2 - r_3} + \frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{r_2 - r_3}\tanh\left(\frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{2}(\chi + \omega)\right)\right],$$

$$u_{3_{16}}(\chi) = -\frac{1}{2kk_2\sqrt{r_3^2 - r_2^2}}[k^2(r_2^2 - r_3^2) - 2kk_3A_{12}(b + c) + 2kk_1(b - c) - 4k_1k_3A_{12}]$$

$$+\frac{\mathrm{k}(\mathrm{r}_2-\mathrm{r}_3)-2\mathrm{k}_3\mathrm{A}_{12}}{2\mathrm{k}_2}\left[\frac{\mathrm{r}_1}{\mathrm{r}_2-\mathrm{r}_3}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{b}-\mathrm{c}}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{1_{17}}(\chi)=\frac{-\mathrm{r}_2\mathrm{k}+2\mathrm{k}_3\mathrm{A}_{12}}{\mathrm{k}\mathrm{r}_2}\mathrm{A}_{12}\left[\mathrm{i}+\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{2_{17}}(\chi)=\mathrm{A}_{12}\left[\mathrm{i}+\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{3_{17}}(\chi)=-\frac{1}{2\mathrm{k}\mathrm{k}_2\sqrt{-\mathrm{r}_2^2}}[\mathrm{k}^2\mathrm{b}^2-2\mathrm{k}\mathrm{k}_3\mathrm{A}_{12}\mathrm{b}+2\mathrm{k}\mathrm{k}_1\mathrm{b}-4\mathrm{k}_1\mathrm{k}_3\mathrm{A}_{12}]$$
$$+\frac{\mathrm{k}\mathrm{r}_2-2\mathrm{k}_3\mathrm{A}_{12}}{2\mathrm{k}_2}\left[\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{1_{18}}(\chi)=\frac{\mathrm{k}(\mathrm{r}_3-\mathrm{r}_2)+2\mathrm{k}_3\mathrm{A}_{12}}{-\mathrm{k}(\mathrm{r}_3-\mathrm{r}_2)}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}_2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{2_{18}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}_2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{3_{18}}(\xi)=-\frac{1}{2\mathrm{k}\mathrm{k}_2\sqrt{\mathrm{r}_3^2-\mathrm{r}_2^2}}[\mathrm{k}^2(\mathrm{r}_2^2-\mathrm{r}_3^2)-2\mathrm{k}\mathrm{k}_3\mathrm{A}_{12}(\mathrm{r}_2+\mathrm{r}_3)+2\mathrm{k}\mathrm{k}_1(\mathrm{r}_2-\mathrm{r}_3)-4\mathrm{k}_1\mathrm{k}_3\mathrm{A}_{12}]$$
$$+\frac{\mathrm{k}(\mathrm{b}-\mathrm{c})-2\mathrm{k}_3\mathrm{A}_{12}}{2\mathrm{k}_2}\left[\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}_2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right].$$

Utilizing the (3.19) and **Family 6** in section 2 we get

$$\mathrm{u}_{1_{19}}(\chi)=\frac{-\mathrm{k}\mathrm{r}_2+2\mathrm{k}_3\mathrm{A}_{12}}{\mathrm{k}\mathrm{r}_2}\mathrm{A}_{12}\left[\mathrm{i}+\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right], \tag{3.22}$$

$$\mathrm{u}_{2_{19}}(\chi)=\mathrm{A}_{12}\left[\mathrm{i}+\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right],$$

$$\mathrm{u}_{3_{19}}(\chi)=-\frac{1}{2kk_2\sqrt{-r_2^2}}[k^2r_2^2-2kk_3A_{12}r_2+2kk_1r_2-4k_1k_3A_{12}]$$
$$+\frac{\mathrm{k}\mathrm{r}_2-2\mathrm{k}_3\mathrm{A}_{12}}{2\mathrm{k}_2}\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right).$$

Employing the (3.19) and **Family 8** in section 2 we get

$$\mathrm{u}_{1_{20}}(\chi)=\frac{-\mathrm{k}(\mathrm{r}_2-\mathrm{r}_3)+2\mathrm{k}_3\mathrm{A}_{12}}{\mathrm{k}(\mathrm{r}_2-\mathrm{r}_3)}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_2+\mathrm{r}_3}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right], \tag{3.23}$$

$$\mathrm{u}_{2_{20}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_2+\mathrm{r}_3}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right],$$

$$\mathrm{u}_{3_{20}}(\chi)=-\frac{1}{2kk_2\sqrt{r_3^2-r_2^2}}[k^2(r_2^2-r_3^2)-2kk_3A_{12}(r_2+r_3)+2kk_1(r_2-r_3)-4k_1k_3A_{12}]$$
$$+\frac{(\mathrm{k}(\mathrm{r}_2-\mathrm{r}_3)-2\mathrm{k}_3\mathrm{A}_{12})\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{2\mathrm{k}_2(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}.$$

Employing the (3.19) and **Family 11** in section 2 give

$$u_{1_{21}}(\chi) = \frac{k(r_1 - r_2) + 2k_3A_{12}}{k(r_2 - r_1)} A_{12}\left[\sqrt{\frac{a+b}{a-b}} - \frac{(a+b)e^{r_2(\chi+\omega)} - 1}{(r_1 - r_2)e^{r_2(\chi+\omega)} - 1}\right], \tag{3.24}$$

$$u_{2_{21}}(\chi) = A_{12}\left[\sqrt{\frac{r_1+r_2}{r_1-r_2}} - \frac{(r_1+r_2)e^{r_2(\chi+\omega)} - 1}{(r_1 - r_2)e^{r_2(\chi+\omega)} - 1}\right],$$

$$u_{3_{21}}(\chi) = -\frac{1}{2kk_2\sqrt{r_1^2 - r_2^2}}[k^2(r_2^2 - r_1^2) - 2kk_3A_{12}(r_1+r_2) + 2kk_1(r_2 - r_1) - 4k_1k_3A_{12}]$$
$$- \frac{k(r_2 - r_1) - 2k_3A_{12}}{2k_2}\frac{(a+b)e^{r_2(\chi+\omega)} - 1}{(r_1 - r_2)e^{r_2(\chi+\omega)} - 1}.$$

Utilizing the (3.19) and **Family 12** in section 2 give

Set III:

$$k = -\frac{2k_3A_{12}}{r_2 - r_3},\ A_{01} = \frac{k_2A_{12}B_{13}}{k_1},\ B_{11} = B_{12} = A_{02} = 0,\ A_{12} = A_{12},\ p = \sqrt{\frac{r_2+r_3}{r_3-r_2}}, \tag{3.25}$$

$$A_{03} = -\frac{2k_1}{k_2},\quad A_{13} = -\frac{2k_3}{k_2}A_{12},\quad A_{11} = -2A_{12},\quad u_1(\chi) = A_{01} + A_{11}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right], \tag{3.26}$$

$$u_2(\chi) = A_{12}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right],$$

$$u_3(\chi) = A_{03} + A_{13}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right] + B_{13}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right]^{-1},$$

where a, b and c are free constants. Employing the (3.26) and **Family 1** and **4** in section 2 respectively can be written as

$$u_{1_{25}}(\chi) = \frac{k_2A_{12}B_{13}}{k_1} - 2A_{12}\left[\sqrt{\frac{r_3+r_2}{r_3-r_2}} + \frac{r_1}{r_2-r_3} - \frac{\sqrt{r_3^2-r_1^2-r_2^2}}{r_2-r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2-r_2^2}}{2}(\chi+\omega)\right)\right], \tag{3.27}$$

$$u_{2_{25}}(\chi) = A_{12}\left[\sqrt{\frac{r_3+r_2}{r_3-r_2}} + \frac{r_1}{r_2-r_3} - \frac{\sqrt{r_3^2-r_1^2-r_2^2}}{r_2-r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2-r_2^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{25}}(\chi) = \frac{-2k_1}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\sqrt{\frac{r_3+r_2}{r_3-r_2}} + \frac{r_1}{r_2-r_3} - \frac{\sqrt{r_3^2-r_1^2-r_2^2}}{r_2-r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2-r_2^2}}{2}(\chi+\omega)\right)\right]$$
$$+B_{13}\left[\sqrt{\frac{c+b}{c-b}} + \frac{r_1}{r_2-r_3} - \frac{\sqrt{r_3^2-r_1^2-r_2^2}}{r_2-r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2-r_2^2}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$u_{1_{26}}(\chi) = \frac{k_2A_{12}B_{13}}{k_1} - 2A_{12}\left[1 - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{2_{26}}(\chi) = A_{12}\left[1 - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{26}}(\chi) = \frac{-2k_1}{k_2} - \frac{2k_3}{k_2}A_{12}\left[1 - \frac{r_1}{r_3} + \frac{\sqrt{r_3^2-r_1^2}}{r_3}\tan\left(\frac{\sqrt{r_3^2-r_1^2}}{2}(\chi+\omega)\right)\right]$$

$$+\mathrm{B}_{13}\left[1-\frac{\mathrm{r}_1}{\mathrm{r}_3}+\frac{\sqrt{\mathrm{r}_3^2-\mathrm{r}_1^2}}{\mathrm{r}_3}\tan\left(\frac{\sqrt{\mathrm{r}_3^2-\mathrm{r}_1^2}}{2}(\chi+\omega)\right)\right]^{-1}.$$

Utilizing the (3.26) and **Family 2, 3,** and **5** in section 2 respectively get

$$u_{1_{27}}(\chi)=\frac{k_2A_{12}B_{13}}{k_1}-2A_{12}\left[\sqrt{\frac{r_2-r_3}{r_3-r_2}}+\frac{r_1}{r_2-r_3}+\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r-2-r_3}\tanh\left(\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{2}(\chi+\omega)\right)\right], \quad (3.28)$$

$$u_{2_{27}}(\chi)=A_{12}\left[\sqrt{\frac{r_2-r_3}{r_3-r_2}}+\frac{r_1}{r_2-r_3}+\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r-2-r_3}\tanh\left(\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{27}}(\chi)=\frac{-2k_1}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\sqrt{\frac{r_2-r_3}{r_3-r_2}}+\frac{r_1}{r_2-r_3}+\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r-2-r_3}\tanh\left(\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{2}(\chi+\omega)\right)\right]$$

$$+\mathrm{B}_{13}\left[\sqrt{\frac{\mathrm{r}_2+\mathrm{r}_3}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\mathrm{r}_1}{\mathrm{r}_2-\mathrm{r}_3}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}_2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$\mathrm{u}_{1_{28}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\mathrm{i}+\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{2_{28}}(\chi)=\mathrm{A}_{12}\left[\mathrm{i}+\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{3_{28}}(\chi)=\frac{-2k_1}{k_2}-\frac{2k_3}{k_2}A_{12}\left[i+\frac{r_1}{r_2}+\frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{r_1^2+r_2^2}}{2}(\chi+\omega)\right)\right]$$

$$+\mathrm{B}_{13}\left[\mathrm{i}+\frac{\mathrm{r}_1}{\mathrm{r}_2}+\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{\mathrm{r}_2}\tanh\left(\frac{\sqrt{\mathrm{r}_1^2+\mathrm{r}_2^2}}{2}(\chi+\omega)\right)\right]^{-1},$$

$$\mathrm{u}_{1_{29}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}-2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{2_{29}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{b}-\mathrm{c}}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right],$$

$$\mathrm{u}_{3_{29}}(\chi)=\frac{-2\mathrm{k}_1}{\mathrm{k}_2}-\frac{2\mathrm{k}_3}{\mathrm{k}_2}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}-2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right]$$

$$+\mathrm{B}_{13}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{\mathrm{r}_2-\mathrm{r}_3}\tanh\left(\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}}{2}(\chi+\omega)\right)\right]^{-1}.$$

Employing the (3.26) and **Family 6** in section 2 we get

$$\mathrm{u}_{1_{30}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\mathrm{i}+\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right], \quad (3.29)$$

$$\mathrm{u}_{2_{30}}(\chi)=\mathrm{A}_{12}\left[\mathrm{i}+\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right],$$

$$\mathrm{u}_{3_{30}}(\chi)=\frac{-2k_1}{k_2}-\frac{2k_3}{k_2}A_{12}\left[i+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1},\frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]\right)\right]$$

$$+\mathrm{B}_{13}\left[\mathrm{i}+\tan\left(\frac{1}{2}\arctan\left[\frac{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}-1}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1},\frac{2\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}}{\mathrm{e}^{2\mathrm{r}_2(\chi+\omega)}+1}\right]\right)\right]^{-1}.$$

Utilizing the (3.26) and **Family 8** in section 2 we obtain

$$\mathrm{u}_{1_{31}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right], \tag{3.30}$$

$$\mathrm{u}_{2_{31}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right],$$

$$\mathrm{u}_{3_{31}}(\chi)=\frac{-2\mathrm{k}_1}{\mathrm{k}_2}-\frac{2\mathrm{k}_3}{\mathrm{k}_2}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{\sqrt{\mathrm{r}_2^2-\mathrm{r}_3^2}(\chi+\omega)+2}{(\mathrm{r}_2-\mathrm{r}_3)(\chi+\omega)}\right]+B_{13}\left[\sqrt{\frac{r_3+r_2}{r_3-r_2}}+\frac{\sqrt{r_2^2-r_3^2}(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right]^{-1}.$$

Employing the (3.26) and **Family 10** in section 2 give

$$\mathrm{u}_{1_{32}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}+2\mathrm{A}_{12}\frac{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}}{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}-1},\qquad \mathrm{u}_{2_{32}}(\xi)=-\mathrm{A}_{12}\frac{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}}{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}-1}, \tag{3.31}$$

$$\mathrm{u}_{3_{32}}(\chi)=\frac{-2\mathrm{k}_1}{\mathrm{k}_2}+\frac{2\mathrm{k}_3}{\mathrm{k}_2}\mathrm{A}_{12}\frac{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}}{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}-1}-\mathrm{B}_{13}\frac{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}-1}{\mathrm{e}^{\mathrm{r}_1\mathrm{k}'(\chi+\omega)}}.$$

Employing the (3.26) and **Family 11** in section 2 give

$$\mathrm{u}_{1_{33}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_1+\mathrm{r}_2}{\mathrm{r}_1-\mathrm{r}_2}}-\frac{(\mathrm{r}_1+\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}{(\mathrm{r}_1-\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right], \tag{3.32}$$

$$\mathrm{u}_{2_{33}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_1+\mathrm{r}_2}{\mathrm{r}_1-\mathrm{r}_2}}-\frac{(\mathrm{r}_1+\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}{(\mathrm{r}_1-\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right],$$

$$\mathrm{u}_{3_{33}}(\xi)=\frac{-2\mathrm{k}_1}{\mathrm{k}_2}-\frac{2\mathrm{k}_3}{\mathrm{k}_2}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_1+\mathrm{r}_2}{\mathrm{r}_1-\mathrm{r}_2}}-\frac{(\mathrm{r}_1+\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}{(\mathrm{r}_1-\mathrm{r}_2)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right]+B_{13}\left[\sqrt{\frac{r_1+r_2}{r_1-r_2}}-\frac{(r_1+r_2)e^{r_2(\chi+\omega)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right]^{-1}.$$

Utilizing the (3.26) and **Family 12** in section 2 give

$$\mathrm{u}_{1_{34}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{(\mathrm{b}+\mathrm{c})\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}+1}{(\mathrm{r}_2-\mathrm{r}_3)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right], \tag{3.33}$$

$$\mathrm{u}_{2_{34}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{c}+\mathrm{b}}{\mathrm{c}-\mathrm{b}}}+\frac{(\mathrm{b}+\mathrm{c})\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}+1}{(\mathrm{b}-\mathrm{c})\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right],$$

$$\mathrm{u}_{3_{34}}(\chi)=\frac{-2\mathrm{k}_1}{\mathrm{k}_2}-\frac{2\mathrm{k}_3}{\mathrm{k}_2}\mathrm{A}_{12}\left[\sqrt{\frac{\mathrm{r}_3+\mathrm{r}_2}{\mathrm{r}_3-\mathrm{r}_2}}+\frac{(\mathrm{r}_2+\mathrm{r}_3)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}+1}{(\mathrm{r}_2-\mathrm{r}_3)\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-1}\right]+B_{13}\left[\sqrt{\frac{r_3+r_2}{r_3-r_2}}+\frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right]^{-1}.$$

Employing the (3.26) and **Family 13** in section 2 give

$$\mathrm{u}_{1_{35}}(\chi)=\frac{\mathrm{k}_2\mathrm{A}_{12}\mathrm{B}_{13}}{\mathrm{k}_1}-2\mathrm{A}_{12}\left[\sqrt{\frac{-\mathrm{r}_1+\mathrm{r}_2}{-\mathrm{r}_1-\mathrm{r}_2}}+\frac{\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}+\mathrm{r}_2-\mathrm{r}_1}{\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-\mathrm{r}_2-\mathrm{r}_1}\right], \tag{3.34}$$

$$\mathrm{u}_{2_{35}}(\chi)=\mathrm{A}_{12}\left[\sqrt{\frac{-\mathrm{r}_1+\mathrm{r}_2}{-\mathrm{r}_1-\mathrm{r}_2}}+\frac{\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}+\mathrm{r}_2-\mathrm{r}_1}{\mathrm{e}^{\mathrm{r}_2(\chi+\omega)}-\mathrm{r}_2-\mathrm{r}_1}\right],$$

$$u_{3_{35}}(\chi)=\frac{-2k_1}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\sqrt{\frac{-a+b}{-a-b}}+\frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-b-a}\right]+B_{13}\left[\sqrt{\frac{-a+b}{-a-b}}+\frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right]^{-1}.$$

Employing the (3.26) and **Family 15** in section 2 give

$$u_{1_{36}}(\chi) = \frac{k_2 A_{12} B_{13}}{k_1} - 2A_{12}\left[1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right], \; u_{2_{36}}(\chi) = A_{12}\left[1 - \frac{r_3(\chi+\omega)2}{r_3(\chi+\omega)}\right], \tag{3.35}$$

$$u_{3_{36}}(\chi) = \frac{-2k_1}{k_2} - \frac{2k_3}{k_2}A_{12}\left[1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right] + B_{13}\left[1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right]^{-1},$$

where $\chi = -\frac{2k_3 A_{12}}{r_2 - r_3}t + \omega$.

Set IV:

$$k = -\frac{2k_3 A_{12}}{r_2 - r_3}, \;\; A_{01} = -2A_{02}, \;\; B_{11} = B_{12} = 0, \;\;\; A_{12} = A_{12}, \; p = \frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}}, \tag{3.36}$$

$$A_{03} = -\frac{2(k_1 + A_{02}k_3)}{k_2}, \;\; B_{13} = 0, \;\; A_{13} = -\frac{2k_3}{k_2}A_{12}, \;\; A_{11} = -2A_{12}, \tag{3.37}$$

$$u_1(\chi) = A_{01} + A_{11}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right],$$

$$u_2(\chi) = A_{02} + A_{12}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right], \;\; u_3(\chi) = A_{03} + A_{13}\left[p + \tan\left(\frac{\Phi(\chi)}{2}\right)\right],$$

where a, b and c are free constants. Employing the (3.37) and **Family 1** and **4** in section 2 respectively can be written as

$$u_{1_{37}}(\xi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3} - \frac{\sqrt{r_3^2 - r_2^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_2^2 - r_1^2}}{2}(\chi+\omega)\right)\right], \tag{3.38}$$

$$u_{2_{37}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{a}{b - c} - \frac{\sqrt{r_3^2 - r_2^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_2^2 - r_1^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{37}}(\chi) = -\frac{2(k_1 + A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{a}{b - c} - \frac{\sqrt{r_3^2 - r_2^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{1_{38}}(\chi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{2_{38}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{38}}(\chi) = -\frac{2(k_1 + A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_1}{r_3} - \frac{\sqrt{r_3^2 - r_1^2}}{r_2 - r_3}\tan\left(\frac{\sqrt{r_3^2 - r_1^2}}{2}(\chi+\omega)\right)\right].$$

Utilizing the (3.37) and **Family 2, 3,** and **5** in section 2 respectively get

$$u_{1_{39}}(\chi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3} + \frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{r_2 - r_3}\tanh\left(\frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{2}(\chi+\omega)\right)\right], \tag{3.39}$$

$$u_{2_{39}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{c + b}{c - b}} + \frac{r_1}{r_2 - r_3} + \frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{r_2 - r_3}\tanh\left(\frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{39}}(\chi) = -\frac{2(k_1 + A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2 + r_3}{r_3 - r_2}} + \frac{r_1}{r_2 - r_3}\right.$$

$$+\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{r_2-r_3}\tanh\left(\frac{\sqrt{r_1^2+r_2^2-r_3^2}}{2}(\chi+\omega)\right)\Bigg],$$

$$u_{1_{40}}(\chi)=-2A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\frac{r_1}{r_2}+\frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{r_1^2+r_2^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{2_{40}}(\chi)=A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\frac{r_1}{r_2}+\frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{r_1^2+r_2^2}}{2}(\chi+\omega)\right)\right],$$

$$u_{3_{40}}(\chi)=-\frac{2(k_1+A_{02}k_3)}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\frac{r_1}{r_2}+\frac{\sqrt{r_1^2+r_2^2}}{r_2}\tanh\left(\frac{\sqrt{r_1^2+r_2^2}}{2}(\xi+C)\right)\right].$$

Employing the (3.37) and **Family 6** in section 2 we get

$$u_{1_{41}}(\chi)=-2A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1},\ \frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]\right)\right], \tag{3.40}$$

$$u_{2_{41}}(\chi)=A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1},\ \frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]\right)\right],$$

$$u_{3_{41}}(\chi)=-\frac{2(k_1+A_{02}k_3)}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}}\pm i+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2r_2(\chi+\omega)}-1}{e^{2r_2(\chi+\omega)}+1},\ \frac{2e^{r_2(\chi+\omega)}}{e^{2r_2(\chi+\omega)}+1}\right]\right)\right].$$

Employing the (3.37) and **Family 8** in section 2 we get

$$u_{1_{42}}(\chi)=-2A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_2+r_3}{r_3-r_2}}+\frac{\sqrt{r_2^2-r_3^2}(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right], \tag{3.41}$$

$$u_{2_{42}}(\chi)=A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_2+r_3}{r_3-r_2}}+\frac{\sqrt{r_2^2-r_3^2}(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right],$$

$$u_{3_{42}}(\chi)=-\frac{2(k_1+A_{02}k_3)}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_2+r_3}{r_3-r_2}}+\frac{\sqrt{r_2^2-r_3^2}(\chi+\omega)+2}{(r_2-r_3)(\chi+\omega)}\right].$$

Employing the (3.37) and **Family 10** in section 2 give

$$u_{1_{43}}(\chi)=-2A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}-\frac{e^{r_1k'(\chi+\omega)}}{e^{r_1k'(\chi+\omega)}-1}\right], \tag{3.42}$$

$$u_{2_{43}}(\chi)=A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}-\frac{e^{r_1k'(\chi+\omega)}}{e^{r_1k'(\chi+\omega)}-1}\right],$$

$$u_{3_{43}}(\chi)=-\frac{2(k_1+A_{02}k_3)}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}}-\frac{e^{r_1k'(\chi+\omega)}}{e^{r_1k'(\chi+\omega)}-1}\right].$$

Utilizing the (3.37) and **Family 11** in section 2 give

$$u_{1_{44}}(\chi)=-2A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_1+r_2}{r_1-r_2}}-\frac{(r_1+r_2)e^{r_2(\chi)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right], \tag{3.43}$$

$$u_{2_{44}}(\chi)=A_{02}+A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_1+r_2}{r_1-r_2}}-\frac{(r_1+r_2)e^{r_2(\chi+\omega)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right],$$

$$u_{3_{44}}(\chi)=-\frac{2(k_1+A_{02}k_3)}{k_2}-\frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}}\pm\sqrt{\frac{r_1+r_2}{r_1-r_2}}-\frac{(r_1+r_2)e^{r_2(\chi+\omega)}-1}{(r_1-r_2)e^{r_2(\chi+\omega)}-1}\right].$$

Employing the (3.37) and **Family 12** in section 2 give

$$u_{1_{45}}(\chi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2+r_3}{r_3-r_2}} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right], \tag{3.44}$$

$$u_{2_{45}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2+r_3}{r_3-r_2}} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right],$$

$$u_{3_{45}}(\chi) = -\frac{2(k_1+A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{r_2+r_3}{r_3-r_2}} + \frac{(r_2+r_3)e^{r_2(\chi+\omega)}+1}{(r_2-r_3)e^{r_2(\chi+\omega)}-1}\right].$$

Employing the (3.37) and **Family 13** in section 2 give

$$u_{1_{46}}(\chi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{-r_1+r_2}{-r_1-r_2}} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right], \tag{3.45}$$

$$u_{2_{46}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{-r_1+r_2}{-r_1-r_2}} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right],$$

$$u_{3_{46}}(\chi) = -\frac{2(k_1+A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm \sqrt{\frac{-r_1+r_2}{-r_1-r_2}} + \frac{e^{r_2(\chi+\omega)}+r_2-r_1}{e^{r_2(\chi+\omega)}-r_2-r_1}\right].$$

Utilizing the (3.37) and **Family 15** in section 2 give

$$u_{1_{47}}(\chi) = -2A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right], \tag{3.46}$$

$$u_{2_{47}}(\chi) = A_{02} + A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right],$$

$$u_{3_{47}}(\chi) = -\frac{2(k_1+A_{02}k_3)}{k_2} - \frac{2k_3}{k_2}A_{12}\left[\frac{A_{02}}{A_{12}} \pm 1 - \frac{r_3(\chi+\omega)+2}{r_3(\chi+\omega)}\right],$$

where $\chi = -\frac{2k_3A_{12}}{b-c}t + \omega$.

4. Conclusion

We analyzed the chemical equation with the improved $\tan(\Phi(\xi)/2)$-expansion technique to study its details and present them in a visual form. By applying a combination of the ITET. A variety of new wave profiles were obtained, closely aligned with soliton theory. The resulting wave profiles hold great potential for application in chemistry and various other fields. The current project primarily focused on discovering new wave profiles for a wide range of NLPs utilizing the ITET. NLPs have been highlighted across numerous fields, including engineering, physics, and applied mathematics; therefore, it is reasonable to anticipate that this project will have widespread applicability. Future research will develop this model by adding linked neuron systems to better mimic natural brain activity. Our research team should explore how neurons influence and work with fractional soliton propagation in these systems. Our flexible NLP model allows researchers to study its applications in engineering physics and applied mathematics, plus other scientific fields, to expand this research's effects beyond neuroscience.

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