Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 14, No. 2, 2026, pp. 877-907
DOI:10.22034/cmde.2025.62338.2740

Inverse optimization problem for a fractional analog of the Barenblatt–Zheltov–Kochina equation

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Abstract

The generalized solvability of a nonlinear optimal control for thermal and diffusion processes in a mixed inverse problem for a Barenblatt–Zheltov–Kochina differential equation with Hilfer fractional operator is studied. The inverse problem is considered with spectral and intermediate conditions. Eigenvalues, eigenfunctions, and associated functions of the spectral problem are found and the corresponding adjoint problem is solved. Countable systems of fractional order differential equations with final value conditions are obtained. The necessary optimality conditions for nonlinear control are formulated. The determination of the optimal control function is reduced to solve a complicated nonlinear functional-integral equation, and the process of solving consists of solving separately taken two nonlinear functional-integral equations. Nonlinear functional integral equations are solved by the method of successive approximations and the unique solvability of these equations is proved by the method of contracting mapping. Approximate calculations for the optimal control function, the redefinition function, and the state function of the controlled process are obtained. The absolute and uniform convergence of the obtained Fourier series are proved.

Keywords. Barenblatt–Zheltov–Kochina differential equation, Nonlinear inverse problem, Necessary conditions for optimal control, Nonlinear control, Minimization of the functional, Hilfer fractional operator, Unique solvability.
2010 Mathematics Subject Classification. 35R30,35R11,81Q05, 47J07 .

1. Introduction

Nonlocal problems with final valued conditions are encountered in mathematical modeling of phenomena of various nature, when the initial data of the process flow domain is inaccessible for direct measurements. Some problems of the diffusion of particles in a turbulent plasma and of the processes of heat propagation are examples, where initial values are not defined. If we consider the technological process of aluminum production indicated above, it is impossible to determine the initial temporary state of the aluminum at the beginning of the technological process. First, the raw material undergoes the firing stage. We do not know in what state the raw materials enter the technological process. The technological process consists of four cycles. After each cycle, it becomes possible to determine the intermediate state of the manufactured product from sensor readings. The mathematical problem is posed as follows: knowing the intermediate state of the product, predict the state of the finished product in advance at the intermediate stage. Based on this analysis of sensor indicators, introduce control into the thermal process. If the simulation analysis needs to be repeated, the thermal process control can be adjusted up to three times.

So, we have an inverse control problem with a final valued condition and an intermediate condition to solve the thermal process equations with redefinition function at the final point.

The theory of optimal control for systems with distributed parameters is widely used in solving problems of aerodynamics, chemical reactions, diffusion, filtration, combustion, heating, etc. (see, [6, 17, 19, 29, 31, 34, 44]). Various

Received: 03 July 2024; Accepted: 05 February 2025.
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analytical and approximate methods for solving problems of optimal control systems with distributed parameters are being developed and effectively used (see, for example, [5, 18, 22, 24–26, 28, 35, 37, 41–43, 46, 48–51, 53, 60]).

The theory and applications of fractional calculus have been developed by many authors ([14, 20, 27, 33]). Let $(t_0;T) \subset \mathbb{R}^+ \equiv [0;\infty)$ be an interval on the set of positive real numbers, where $0 \leq t_0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral of a function $\eta(t)$ is defined as follows:

$$J_{t_0 t}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in (t_0;T),$$

where $\Gamma(\alpha)$ is the Gamma function.

Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows:

$$D_{t_0 t}^{\alpha}\eta(t) = \frac{d^n}{dt^n} J_{t_0 t}^{n-\alpha}\eta(t), \quad t \in (t_0;T).$$

The Caputo α-order fractional derivative of a function $\eta(t)$ is defined by

$${}_C D_{t_0 t}^{\alpha}\eta(t) = J_{t_0 t}^{n-\alpha}\eta^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{\eta^{(n)}(s)ds}{(t-s)^{\alpha-n+1}}, \quad t \in (t_0;T).$$

These derivatives are reduced to the n-th order derivatives for $\alpha = n \in \mathbb{N}$

$$D_{t_0 t}^{n}\eta(t) = {}_C D_{t_0 t}^{n}\eta(t) = \frac{d^n}{dt^n}\eta(t), \quad t \in (t_0,T).$$

The Hilfer fractional operator $D^{\alpha,\gamma}$ defined by the formula $D^{\alpha,\gamma} = J_{0t}^{\gamma-\alpha} \frac{d}{dt} J_{0t}^{1-\gamma}$, $0 < \alpha \leq \gamma \leq 1$. For the Hilfer operator $D^{\alpha,\gamma}$ for $\gamma = 0$, and $\gamma = 1$, we have $D^{\alpha,0} = {}_{RL}D_{0t}^{\alpha}$ and $D^{\alpha,1} = {}_C D_{0t}^{\alpha}$, respectively. So, the generalized integro-differentiation operator $D^{\alpha,\gamma}$ is a continuous interpolation of the well-known fractional order differentiation operators of Riemann–Liouville and Caputo, which describe diffusion processes and engineering interpretation, is given in [21, Vol. 1, P. 47–85; Vol. 4–8]. The construction of various models of theoretical physics problems using fractional calculus is described in [21, vol. 4, 5], [30, 47]. A specific physical interpretation of the generalized fractional operator $D^{\alpha,\gamma}$ is given in [45]. A detailed review devoted to the application of fractional calculus to solving applied problems is given in [21, vol. 6–8], [38, 40]. In [38], in particular, the properties of the operator $D^{\alpha,\gamma}$ were studied and an operational method for solving fractional differential equations was developed. In [36], the problem of source identification was studied for the generalized diffusion equation with the operator $D^{\alpha,\gamma}$. We also note the work [13], where inverse problems were investigated for the generalized parabolic equation of the fourth order with the operator $D^{\alpha,\gamma}$. Different boundary value and inverse problems for fractional differential and integro-differential equations were studied in the works of many authors, in particular, in [1, 2, 4, 9, 10, 12, 15, 23, 32, 39, 52, 54–59, 61].

We can see a few publications dedicated to study different problems of fractional optimal control (see [3, 7, 8, 16]). However, the application of fractional calculus in optimal control theory remains poorly investigated, despite the fact that modeling control processes using fractional integro-differentiation operators is becoming more relevant. In this paper, we consider the questions of a generalized and approximate solving of the fractional inverse problem of nonlinear optimal control for a fractional order pseudo-parabolic differential equation with a quadratic optimality criterion. The necessary optimality conditions are formulated by the maximum principle, and the control function, redefinition function and state function are calculated.

2. Statement of the problem

We consider the following fractional pseudo-parabolic equation

$$\left[D^{\alpha,\gamma} - D^{\alpha,\gamma}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2} \right] U(t,x) = f(x,p(t)), \quad (t,x) \in \Omega, \tag{2.1}$$

with final value

$$U(T,x)=\varphi(x), \quad x\in[0,1], \tag{2.2}$$

and boundary value conditions

$$U(t,0)=0, \quad U_x(t,1)=U_x(t,x_0), \quad 0\le t\le T, \quad 0<x_0<1. \tag{2.3}$$

Let $f(x,p)\in C([0,1]\times\Upsilon)$ denote the external source function, where $p(t)\in C[0,T]$ is the control function and $U(t,x)\in C(\Omega)$ represents the state function of the controlled process. The function $\varphi(x)\in L_2[0,1]$, is the redistribution distribution function. The operator $D^{\alpha,\gamma}-D^{\alpha,\gamma}\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x^2}$ is a fractional analogue of the Barenblatt–Zheltov–Kochina operator. Here, $D^{\alpha,\gamma}$ denotes the Hilfer fractional derivative, and J^{α}_{0t}, $0<\alpha$ is the Riemann–Liouville fractional integral operator. Furthermore, $\Upsilon\equiv[0,M^*]$, where $0<M^*<\infty$, $\Omega\equiv[0,T]\times[0,1]$, and $0<T<\infty$.

In finding redefinition function $\varphi(x)$, we use the following additional intermediate condition

$$U(t_1,x)=\psi(x), \quad 0<t_1<T, \quad x\in[0,1], \tag{2.4}$$

where $\psi(x)\in L_2[0,1]$.

In this paper, an optimal control problem is considered, where the final valued condition (2.2) is connected with the fact that often in practice there are situations when the object of research in the initial problem is either fundamentally inaccessible for measurement, or conducting such a measurement is expensive. The function $\varphi(x)$ in the condition (2.2) is unknown, too. There arises the necessity of using the additional condition (2.4). The necessary optimality conditions based on the maximum principle are formulated, the control function, redefinition function and the state function are calculated.

The inverse optimal control problem (2.1)–(2.4) contains a triple of unknown functions: $\{U(t,x)\in C(\Omega),\ \varphi(x)\in L_2[0,1],\ p(t)\in C[0,T]\}$.

We note that for a complete definition of this triple, it is not enough to use only the conditions (2.2)–(2.4). Therefore, in this paper, we also consider the minimization of the quadratic functional of quality. The methodology of this work can also be used to solve other problems of nonlinear optimal control associated with the heat transfer or wave processes, for example, in problems of controlling metallurgical furnaces. In solving such optimal control problems, it is necessary to study mathematical models of process control, which allow real-time prediction of the temperature distribution of heated materials depending on changes in supplied power, heating time of bodies, heating modes, etc.

So, it is important to consider the questions of generalized solvability of a mixed inverse problem in nonlinear optimal control for a fractional analog of pseudo-parabolic differential Eq. (2.1). The equation is considered with final value condition (2.2), boundary value conditions (2.3) and intermediate condition (2.4). The spectral method of variable separation based on the Fourier series is applied. Eigenvalues, eigenfunctions, and associated functions of the spectral and adjoint problems are found. Countable systems of fractional order differential equations are obtained. This paper is a further development of the works [50, 51].

3. Spectral problem

Condition A. Let x_0 be a rational number from the interval $(0,1)$ such that $x_0=\frac{p}{q}$, $p<q$, $q-p=1$, p and q be positive integers.

The state function we consider as a sum

$$U(t,x)=U_0(t,x)+U_1(t,x)+U_2(t,x)+\tilde{U}_2(t,x)$$

and the solution of mixed inverse problem (2.1)–(2.4) we search in the form of the following Fourier series

$$U(t,x)=u_0(t)\,\vartheta_0(x)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\big(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\big), \tag{3.1}$$

where

$$u_0(t) = \int_0^1 U_0(t,y)\omega_0(y)dy, \quad u_{1,n}(t) = \int_0^1 U_1(t,y)\omega_{1,n}(y)dy,$$

$$u_{2,m}(t) = \int_0^1 U_2(t,y)\tilde{\omega}_{2,m}(y)dy, \quad \tilde{u}_{2,m}(t) = \int_0^1 \tilde{U}_2(t,y)\omega_{2,m}(y)dy,$$

"*" means that the sum is taken over $n \in \mathbb{N}$, different from $k(q+p)$, $k \in \mathbb{N}$.

The functions

$$\vartheta_0(x) = x, \quad \vartheta_{1,n}(x) = \sin\sqrt{\lambda_{1,n}}x, \quad \vartheta_{2,m}(x) = \sin\sqrt{\lambda_{2,m}}x, \quad n, m \in \mathbb{N}, \tag{3.2}$$

in (3.2) are eigenfunctions of the spectral problem [11]

$$\vartheta''(x) + \lambda^2\vartheta(x) = 0, \quad \vartheta(0) = 0, \quad \vartheta'(1) = \vartheta'(x_0), \quad \lambda \geq 0, \;\; 0 < x_0 < 1, \tag{3.3}$$

with corresponding eigenvalues:

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2n\pi}{1+x_0}\right)^2, \quad \lambda_{2,m} = \left(\frac{2m\pi}{1-x_0}\right)^2, \quad n, m \in \mathbb{N}.$$

The spectral problem (3.3) for $\lambda_{2,m}$ has associated functions of the form

$$\tilde{\vartheta}_{2,m}(x) = x\cos\sqrt{\lambda_{2,m}}x. \tag{3.4}$$

For each $n, m \in \mathbb{N}$, $n \neq m(p+q)$ the functions

$$\big\{\omega_0(x);\; \omega_{1,n}(x);\; \omega_{2,m}(x)\big\}, \tag{3.5}$$

where

$$\omega_0(x) = \begin{cases} 0, & x \in [0, x_0), \\ \frac{2}{1-x_0^2}, & x \in (x_0, 1], \end{cases} \qquad \omega_{1,n}(x) = \begin{cases} \frac{4\sin\sqrt{\lambda_{1,n}}x}{1+x_0}, & x \in [0, x_0), \\ \frac{2\cos\sqrt{\lambda_{1,n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1,n}}}, & x \in (x_0, 1], \end{cases}$$

$$\omega_{2,m}(x) = \begin{cases} 0, & x \in [0, x_0), \\ \frac{4\cos\sqrt{\lambda_{2,m}}x}{1-x_0}, & x \in (x_0, 1], \end{cases}$$

are eigenfunctions of the following problem, which is an adjoint to problem (3.3)

$$\omega''(x) + \lambda\,\omega(x) = 0, \quad \lambda \geq 0, \quad x \in (0, x_0) \cup (x_0, 1), \tag{3.6}$$

$$\omega(0) = 0, \quad \omega'(1) = 0, \tag{3.7}$$

$$\omega'(x_0+0) = \omega'(x_0-0), \quad \omega(x_0+0) - \omega(x_0-0) = \omega(1). \tag{3.8}$$

The adjoint spectral problem (3.6)–(3.8) for each $\lambda_{2,m}$ has also associated functions of the form

$$\tilde{\omega}_{2,m}(x) = \begin{cases} \frac{4\sin\sqrt{\lambda_{2,m}}x}{1+x_0}, & x \in [0, x_0), \\ \frac{4(1-x)\sin\sqrt{\lambda_{2,m}}x}{1-x_0^2}, & x \in (x_0, 1]. \end{cases} \tag{3.9}$$

We note that systems of eigenfunctions (3.2), (3.4) and (3.5), (3.9) are biorthonormal in $L_2[0,1]$, that is

$$\big(\vartheta_0(x), \omega_0(x)\big) = 1, \; \big(\vartheta_0(x), \omega_{1,n}(x)\big) = \big(\vartheta_0(x), \omega_{2,m}(x)\big) = \big(\vartheta_0(x), \tilde{\omega}_{2,m}(x)\big) = 0,$$

$$\big(\vartheta_{1,n}(x), \omega_{1,k}(x)\big) = \begin{cases} 1, & n = k, \\ 0, & n \neq k, \end{cases}$$

$$\big(\vartheta_{1,n}(x), \omega_0(x)\big) = \big(\vartheta_{1,n}(x), \omega_{2,m}(x)\big) = \big(\vartheta_{1,n}(x), \tilde{\omega}_{2,m}(x)\big) = 0,$$

$$\left(\vartheta_{2,m}(x),\,\tilde{\omega}_{2,k}(x)\right)=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$
$$\left(\vartheta_{2,m}(x),\,\omega_{0}(x)\right)=\left(\vartheta_{2,m}(x),\,\omega_{1,n}(x)\right)=\left(\vartheta_{2,m}(x),\,\omega_{2,k}(x)\right)=0,$$
$$\left(\tilde{\vartheta}_{2,m}(x),\,\omega_{2,k}(x)\right)=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$
$$\left(\tilde{\vartheta}_{2,m}(x),\,\omega_{0}(x)\right)=\left(\tilde{\vartheta}_{2,m}(x),\,\tilde{\omega}_{2,k}(x)\right)=0,$$

where by $(\cdot,\,\cdot)$ is denoted the inner product in $L_2[0,1]$.

Moreover, if the condition A is satisfying, then the systems of root functions of problems (3.3) and (3.6)–(3.8) form a Riesz basis in $L_2[0,1]$.

For the functions $f(x,p(t))=f_0(x,p_0(t))+f_1(x,p_1(t))+f_2(x,p_2(t))+\tilde{f}_2(x,\tilde{p}_2(t)),\ \varphi(x)=\varphi_0(x)+\varphi_1(x)+\varphi_2(x)+\tilde{\varphi}_2(x)$ and $\psi(x)=\psi_0(x)+\psi_1(x)+\psi_2(x)+\tilde{\psi}_2(x)$ it is assumed that

$$f(x,p(t))=f_0(p)\vartheta_0(x)+\sum_{n=1}^{\infty *}f_{1,n}(p)\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left[f_{2,m}(p)\vartheta_{2,m}(x)+\tilde{f}_{2,m}(p)\tilde{\vartheta}_{2,m}(x)\right], \tag{3.10}$$

$$\varphi(x)=\varphi_0\vartheta_0(x)+\sum_{n=1}^{\infty *}\varphi_{1,n}\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left[\varphi_{2,m}\vartheta_{2,m}(x)+\tilde{\varphi}_{2,m}\tilde{\vartheta}_{2,m}(x)\right], \tag{3.11}$$

$$\psi(x)=\psi_0\vartheta_0(x)+\sum_{n=1}^{\infty *}\psi_{1,n}\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left[\psi_{2,m}\vartheta_{2,m}(x)+\tilde{\psi}_{2,m}\tilde{\vartheta}_{2,m}(x)\right], \tag{3.12}$$

where

$$f_0(p)=\int_0^1 f_0(y,p_0(t))\omega_0(y)dy,\ \ f_{1,n}(p)=\int_0^1 f_1(y,p_1(t))\omega_{1,n}(y)dy,$$
$$f_{2,m}(p)=\int_0^1 f_2(y,p_2(t))\tilde{\omega}_{2,m}(y)dy,\ \ \tilde{f}_{2,m}(p)=\int_0^1 \tilde{f}_2(y,\tilde{p}_2(t))\omega_{2,m}(y)dy;$$
$$\varphi_0=\int_0^1 \varphi_0(y)\omega_0(y)dy,\ \ \varphi_{1,n}=\int_0^1 \varphi_1(y)\omega_{1,n}(y)dy,$$
$$\varphi_{2,m}=\int_0^1 \varphi_2(y)\tilde{\omega}_{2,m}(y)dy,\ \ \tilde{\varphi}_{2,m}=\int_0^1 \tilde{\varphi}_2(y)\omega_{2,m}(y)dy;$$
$$\psi_0=\int_0^1 \psi_0(y)\omega_0(y)dy,\ \ \psi_{1,n}=\int_0^1 \psi_1(y)\omega_{1,n}(y)dy,$$
$$\psi_{2,m}=\int_0^1 \psi_2(y)\tilde{\omega}_{2,m}(y)dy,\ \ \tilde{\psi}_{2,m}=\int_0^1 \tilde{\psi}_2(y)\omega_{2,m}(y)dy.$$

4. Reducing the mixed inverse problem to countable systems of fractional equations

Problem. Find control function $p(t) \in \{p : |\, p(t)\,| \le M^*,\, t \in [0,T]\}$, redefinition function $\varphi(x)$ and corresponding state function $U(t,x)$, which deliver a minimum to functionality

$$J[p] = \int_0^1 \left[\varphi(y) - \xi(y)\right]^2 dy + \alpha \int_0^T p^2(t)\, dt, \tag{4.1}$$

where $0 < \alpha = \text{const}$, and $\xi(x) = \xi_0(x) + \xi_1(x) + \xi_2(x) + \tilde{\xi}_2(x)$ is given continuous function such that

$$\xi(x) = \xi_0 \vartheta_0(x) + \sum_{n=1}^{\infty *} \xi_{1,n} \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} \left[\xi_{2,m}\vartheta_{2,m}(x) + \tilde{\xi}_{2,m}\tilde{\vartheta}_{2,m}(x)\right], \tag{4.2}$$

$$\xi_0 = \int_0^1 \xi_0(y)\omega_0(y)dy, \;\; \xi_{1,n} = \int_0^1 \xi_1(y)\omega_{1,n}(y)dy,$$

$$\xi_{2,m} = \int_0^1 \xi_2(y)\tilde{\omega}_{2,m}(y)dy, \;\; \tilde{\xi}_{2,m} = \int_0^1 \tilde{\xi}_2(y)\omega_{2,m}(y)dy,$$

$$|\,\xi_0\,| + \sum_{n=1}^{\infty *} |\,\xi_{1,n}\,| + \sum_{m=1}^{\infty} \left[|\,\xi_{2,m}\,| + \left|\,\tilde{\xi}_{2,m}\,\right|\right] < \infty. \tag{4.3}$$

We use the following well-known spaces

$$\bar{C}^{1,2}_U(\Omega) = \left\{U : U(t,x) \in C^{1,2}(\Omega),\, U(t,0) = 0,\, U_x(t,1) = U_x(t,x_0),\, 0 \le t \le T,\, 0 < x_0 < 1\right\},$$
$$\bar{C}^{1,2}_\Phi(\Omega) = \left\{\Phi : \Phi(t,x) \in C^{1,2}(\Omega),\, \Phi(0,x) = 0\right\}.$$

The closure of these spaces with the norm

$$\|\, U\,\|_{\bar{H}(\Omega)} = \sqrt{\int_0^T \int_0^1 |\, U(t,y)\,|^2 \, dy\, dt} < \infty,$$

denoted respectively by $\bar{H}_U(\Omega)$, $\bar{H}_\Phi(\Omega)$.

Definition 4.1. The function $U(t,x) \in \bar{H}_U(\Omega)$ is called a generalized solution to the nonlocal problem (2.1)–(2.3), if this function satisfies the differential equation (2.1) with conditions (2.2) and (2.3) almost everywhere.

Using definition and Fourier series (3.1) and (3.10), taking into account that the properties of eigenfunctions (3.2), (3.4), (3.5), and (3.9), from Eq. (2.1) we come to the following scalar and three countable systems (CS) of ordinary fractional order differential equations

$$D^{\alpha,\gamma} u_0(t) = f_0(p_0(t)), \tag{4.4}$$

$$D^{\alpha,\gamma} u_{1,n}(t) = -\mu_{1,n} u_{1,n}(t) + g_{1,n}(t), \tag{4.5}$$

$$D^{\alpha,\gamma} u_{2,m}(t) = -\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\left(D^{\alpha,\gamma}\tilde{u}_{2,m}(t) + \tilde{u}_{2,m}(t)\right) - \mu_{2,m}u_{2,m}(t) + g_{2,m}(t), \tag{4.6}$$

$$D^{\alpha,\gamma} \tilde{u}_{2,m}(t) = -\mu_{2,m} \tilde{u}_{2,m}(t) + \tilde{g}_{2,m}(t), \tag{4.7}$$

where

$$g_{i,n}(t) = \frac{1}{1+\lambda_{i,n}} f_{i,n}(p_i(t)), \quad \mu_{1,n} = \frac{\lambda_{1,n}}{1+\lambda_{1,n}}, \quad \mu_{2,m} = \frac{\lambda_{2,m}}{1+\lambda_{2,m}},$$

$$\lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2, \quad \lambda_{2,m} = (2qm\pi)^2, \quad n, m \in \mathbb{N}, \quad n \neq m(p+q).$$

We solve the differential Equations (4.4)–(4.7). Using the series (3.11) and (3.12) from given conditions (2.2) and (2.4), we determine the final and intermediate conditions for the unknown Fourier coefficients

$$u_0(T) = \int_0^1 U_0(T,y)\omega_0(y)dy = \int_0^1 \varphi_0(y)\omega_0(y)dy = \varphi_0, \tag{4.8}$$

$$u_{1,n}(T) = \int_0^1 U_1(T,y)\omega_{1,n}(y)dy = \int_0^1 \varphi_1(y)\omega_{1,n}(y)dy = \varphi_{1,n}, \tag{4.9}$$

$$u_{2,m}(T) = \int_0^1 U_2(T,y)\tilde{\omega}_{2,m}(y)dy = \int_0^1 \varphi_2(y)\tilde{\omega}_{2,m}(y)dy = \varphi_{2,m}, \tag{4.10}$$

$$\tilde{u}_{2,m}(T) = \int_0^1 \tilde{U}_2(T,y)\omega_{2,m}(y)dy = \int_0^1 \tilde{\varphi}_2(y)\omega_{2,m}(y)dy = \tilde{\varphi}_{2,m}; \tag{4.11}$$

$$u_0(t_1) = \int_0^1 U_0(t_1,y)\omega_0(y)dy = \int_0^0 \psi_0(y)\omega_0(y)dy = \psi_0, \tag{4.12}$$

$$u_{1,n}(t_1) = \int_0^1 U_1(t_1,y)\omega_{1,n}(y)dy = \int_0^1 \psi_1(y)\omega_n(y)dy = \psi_{1,n}, \tag{4.13}$$

$$u_{2,n}(t_1) = \int_0^1 U_2(t_1,y)\tilde{\omega}_{2,n}(y)dy = \int_0^1 \psi_2(y)\tilde{\omega}_{2,n}(y)dy = \psi_{2,n}, \tag{4.14}$$

$$\tilde{u}_{2,n}(t_1) = \int_0^1 \tilde{U}_2(t_1,y)\omega_{2,n}(y)dy = \int_0^1 \tilde{\psi}_2(y)\omega_{2,n}(y)dy = \tilde{\psi}_{2,n}. \tag{4.15}$$

5. Scalar fractional differential equation

5.1. **Direct problem.** Our purpose is to find redefinition function $\varphi_0(x)$, and using the functional (4.1), determine optimal control function $p_0(t)$. However, first we solve the fractional differential Eq. (4.4) with final condition (4.8). In this order, we apply the operator J_{0t}^{α} to both sides of the Eq. (4.4) and obtain the presentation

$$u_0(t) = \frac{C_0}{\Gamma(\gamma)}\, t^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} f_0(p_0(s))\, ds, \tag{5.1}$$

where C_0 is arbitrary constant. We define by the aid of condition (4.8). So, using final value condition (4.8), from represent (5.1) we have

$$C_0 = \Gamma(\gamma)\, T^{1-\gamma}\varphi_0 + T^{1-\gamma} \int_0^T (T-s)^{\alpha-1} f_0(p_0(s))\, ds. \tag{5.2}$$

Substituting the constant (5.2) into the Eq. (5.1), we derive a new representation

$$t^{1-\gamma}u_0(t) = T^{1-\gamma}\varphi_0 + \int_0^T K_0(t,s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds, \tag{5.3}$$

where

$$K_0(t,s) = \frac{1}{\Gamma(\gamma)}\begin{cases} -T^{1-\gamma}(T-s)^{\alpha-1}, & t \le s \le T, \\ -T^{1-\gamma}(T-s)^{\alpha-1} + t^{1-\gamma}(t-s)^{\alpha-1}, & 0 \le s < t. \end{cases}$$

From the presentation (5.3), we have the solution of the problem (2.1)–(2.3), corresponding to the eigenvalues $\lambda_0 = 0$ and the eigenfunctions $\vartheta_0(x) = x$:

$$t^{1-\gamma}U_0(t,x) = T^{1-\gamma}\varphi_0(x) + \vartheta_0(x)\int_0^T K_0(t,s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds \tag{5.4}$$

for fixed values of $\varphi_0(x)$ and $p_0(t)$.

Theorem 5.1. *Assume that condition A and the inequalities* $|\,\varphi_0(x)\,| < \infty$ *and* $\max\limits_{(t,x)\in\Omega}|\,f_0(x, p_0(t))\,| < \infty$ *are satisfied. Then, for fixed values of the redistribution function* $\varphi_0(x)$ *and of the control function* $p_0(t)$*, it follows that* $U_0(t,x) \in \bar{H}(\Omega)$*, where* $U_0(t,x)$ *is defined by representation (5.4).*

Proof. For fixed values of the redefinition function $\varphi_0(x)$ and of the control function $p_0(t)$, we substitute formula (5.4) into the integral $\Im_0 = \int\limits_0^T\int\limits_0^1 t^{2(1-\gamma)}U_0^2(t,x)\,dx\,dt$ and we square it

$$\Im_0 = \int_0^T\int_0^1 t^{2(1-\gamma)}\left\{T^{2(1-\gamma)}x^2\varphi_0^2 + 2x^2T^{1-\gamma}\left|\varphi_0\right|\int_0^T\left|K_0(t,s)\right|\int_0^1\left|f_0\left(y, p_0(s)\right)\right|\left|\omega_0(y)\right|dyds\right.$$
$$\left.+\left[x\int_0^T K_0(t,s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds\right]^2\right\}dxdt$$
$$\le T^{3-2\gamma}\left\{T^{2(1-\gamma)}\varphi_0^2 + 2T^{1-\gamma}C_{0,1}M_{0,1}\left|\varphi_0\right|\int_0^1\left|\omega_0(y)\right|dy + \left[C_{0,1}M_{0,1}\int_0^1\omega_0(y)dy\right]^2\right\}$$
$$\le T^{3-2\gamma}\left\{T^{2(1-\gamma)}\varphi_0^2 + 2T^{1-\gamma}\frac{2C_{0,1}M_{0,1}}{1+x_0}\,|\,\varphi_0\,| + \left[\frac{2C_{0,1}M_{0,1}}{1+x_0}\right]^2\right\} < \infty,$$

where

$$\max_t\int_0^T|\,K_0(t,s)\,|\,ds \le C_{0,1} = \text{const}, \quad \max_{(t,x)}|\,f_0\left(x, p_0(t)\right)|\le M_{0,1} = \text{const}.$$

The Theorem 5.1 is proved. □

5.2. **Inverse problem.** In the presentation (5.4) functions $\varphi_0(x)$ and $p_0(t)$ are unknown. To find $\varphi_0(x)$, we apply the condition (4.12) into Equation (5.3):

$$\varphi_0 = \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0 - T^{\gamma-1}\int_0^T K_0(t_1,s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds.$$

Hence, we obtain

$$\varphi_0(x) = \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0(x) - T^{\gamma-1}\vartheta_0(x)\int_0^T K_0(t_1,s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\, ds. \tag{5.5}$$

For the function (5.5), we have estimate

$$|\,\varphi_0(x)\,| \le |\,\psi_0\,| + \frac{2C_{0,1}M_{0,1}}{1+x_0}T^{\gamma-1} < \infty. \tag{5.6}$$

Substituting (5.5) into presentation (5.4), we obtain

$$t^{1-\gamma}U_0(t,x) = t_1^{1-\gamma}\psi_0(x) + x\int_0^T [K_0(t,s) - K_0(t_1,s)]\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\, ds. \tag{5.7}$$

(5.7) is the solution of the problem (2.1)–(2.3) for fixed values of control function $p_0(t)$.

5.3. **Optimal control function.** Now we will start to find the control function $p_0(t)$. Let $p_0^*(t)$ is optimal control function

$$\Delta\, J\,[p_0^*(t)\,] = J\,[p_0^*(t) + \Delta\, p_0^*(t)\,] - J\,[p_0^*(t)\,] \ge 0,$$

where $p_0^*(t) + \Delta\, p_0^*(t) \in \bar{H}[0,T]$.

We consider the following function

$$t^{1-\gamma}Q_0(t,x)\left[x t_1^{1-\gamma}\psi_0 + x\int_0^T [K_0(t,s) - K_0(t_1,s)]\int_0^1 f_0\left(y, p_0^*(s)\right)\omega_0(y)dy\, ds\right] = \alpha[p_0^*(t)]^2, \tag{5.8}$$

where $Q_0(t,x)$ defines by solving the following mixed problem

$$D^{\alpha,\gamma}Q_0(t,x) + D^{\alpha,\gamma}Q_{0x\,x}(t,x) + Q_{0x\,x}(t,x) = 0, \quad (t,x)\in\Omega, \tag{5.9}$$

$$Q_0(T,x) = -2\left[\varphi(x) - \xi(x)\right], \tag{5.10}$$

$$Q_0(t,0) = 0, \quad Q_{0x}(t,1) = U_{0x}(t,x_0), \quad 0\le t\le T, \quad 0<x_0<1, \tag{5.11}$$

which is conjugated to problem (2.1)–(2.3). The Equation (5.8) we rewrite in convenient for us form

$$t^{1-\gamma}Q_0(t,x)\left[\Phi_0(t,x) + \int_0^T \bar{K}_0(t,s,x) * f_0\left(p_0^*(s)\right)ds\right] = \alpha[p_0^*(t)]^2, \tag{5.12}$$

where

$$\Phi_0(t,x) = x t_1^{1-\gamma}\psi_0,\;\; \bar{K}_0(t,s,x) * f_0\left(p_0^*(s)\right) = \Big[K_0(t,s) - K_0(t_1,s)\Big]\int_0^1 f_0\left(y, p_0^*(s)\right)\omega_0(y)dyds.$$

According to the maximum principle, we calculate in (5.12) derivative with respect to the control function and come to the following necessary condition for optimality

$$t^{1-\gamma}Q_0(t,x)\int_0^T \bar{K}(t,s,x) * f_p\left(p_0^*(s)\right)ds - 2\alpha\, p_0^*(t) = 0. \tag{5.13}$$

Calculating derivative in (5.13) with respect to the control function $p^*(t)$, we obtain another necessary condition for optimality

$$t^{1-\gamma}Q_0(t,x)\int_0^T \bar{K}(t,s,x) * f_{pp}\left(p_0^*(s)\right)ds - 2\alpha < 0. \tag{5.14}$$

We solve the conjugated differential equation (5.9) by the same way as we solved the Eq. (2.1). According to the conditions of (5.11), the nonzero solution of the Eq. (5.9) we find from the fractional differential equations

$$D^{\alpha,\gamma} q_0(t) = 0, \tag{5.15}$$

where

$$q_0(t) = \int_0^1 Q_0(t, y)\omega_0(y)dy.$$

To solve the differential equation (5.15), we use the condition of (5.10) in the following form

$$q_0(T) = -2\int_0^1 [\varphi_0(y) - \xi_0(y)]\,\vartheta_0(y)dy = -2\varphi_0 + 2\xi_0. \tag{5.16}$$

Substituting presentation (5.5) into the formula (5.16) and by virtue of (4.2), we obtain

$$q_0(T) = 2\xi_0 - 2\frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0 - 2T^{\gamma-1}\int_0^T K_0(t_1, s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds. \tag{5.17}$$

The general solution of the homogeneous equation (5.15) has a form

$$q_0(t) = \frac{B_0}{\Gamma(\gamma)}t^{\gamma-1}, \tag{5.18}$$

where we determine the arbitrary coefficient of integration B_0 from the condition (5.17)

$$B_0 = 2\xi_0\Gamma(\gamma)T^{1-\gamma} - 2t_1^{1-\gamma}\Gamma(\gamma)\psi_0 - 2\Gamma(\gamma)\int_0^T K_0(t_1, s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dyds. \tag{5.19}$$

Substituting (5.19) into general solution (5.18) of homogeneous fractional equation (5.15), we obtain

$$t^{1-\gamma}q_0(t) = 2T^{1-\gamma}\xi_0 - 2t_1^{1-\gamma}\psi_0 - 2\int_0^T K_0(t_1, s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dyds.$$

Hence, we obtain a desire function

$$t^{1-\gamma}Q_0(t, x) = 2xT^{1-\gamma}\xi_0 - 2xt_1^{1-\gamma}\psi_0 - 2x\int_0^T K_0(t_1, s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds.$$

The last equation we rewrite it in the compact form

$$t^{1-\gamma}Q_0(t, x) = \Psi_0(t, x) + \int_0^T \bar{\bar{K}}_0(t, s, x) * f_0\left(p_0(s)\right)ds, \tag{5.20}$$

where

$$\Psi_0(t, x) = 2xT^{1-\gamma}\xi_0 - 2xt_1^{1-\gamma}\psi_0,$$

$$\bar{\bar{K}}_0(t, s, x) * f_0\left(p_0(s)\right) = -2xK_0(t_1, s)\int_0^1 f_0\left(y, p_0(s)\right)\omega_0(y)dy\,ds.$$

Taking into account (5.20), the optimality condition (5.13) we rewrite as

$$\int_0^T \bar{K}_0(t,s,x) * f_p\left(p_0(s)\right) ds = \frac{2\alpha\, p_0(t)}{\Psi_0(t,x) + \int_0^T \bar{\bar{K}}_0(t,s,x) * f_0\left(p_0(s)\right) ds}. \tag{5.21}$$

Substituting (5.20) into condition (5.14), we obtain

$$\int_0^T \bar{K}_0(t,s,x)\, f_{pp}\left(p_0^*(s)\right) ds < \frac{2\alpha}{\Psi_0(t,x) + \int_0^T \bar{\bar{K}}_0(t,s,x)\, f_0\left(p_0^*(s)\right) ds}. \tag{5.22}$$

By virtue of (5.22), we solve the Eq. (5.21) with respect to the control function $p_0(t)$. However, it is difficult to solve the Eq. (5.21) by simple way. So, we use the following techniques. If the nonlinear functional-integral equation (5.21) is solvable, then it is true that we have the following two functional-integral equations for solving:

$$\frac{2\alpha\, p_0(t)}{\Psi_0(t,x) + \int_0^T \bar{\bar{K}}_0(t,s,x) * f_0\left(p_0(s)\right) ds} = t^{1-\gamma} g_0(t), \tag{5.23}$$

$$\int_0^T \bar{K}_0(t,s,x) * f_p\left(p_0(s)\right) ds = t^{1-\gamma} g_0(t), \tag{5.24}$$

where $g_0(t) \in C[0,T]$ is yet unknown function. First, we solve the Eq. (5.23) by the method of successive approximations. However, we assume that the function $g_0(t)$ in (5.23) is known. Therefore, the nonlinear Fredholm functional-integral equation (5.23) we rewrite as follows

$$p_0(t) = \frac{g_0(t)}{2\alpha}\left[t^{1-\gamma}\Psi_0(t,x) + \int_0^T t^{1-\gamma}\bar{\bar{K}}_0(t,s,x) * f_0\left(p_0(s)\right) ds\right]. \tag{5.25}$$

For an arbitrary function $p(t) \in C[0,T]$, we consider the following continuous norm

$$\|\, p(t)\,\|_C = \max_{t\in[0,T]} |\, p(t)\,| \,.$$

Theorem 5.2. *Let the following conditions are fulfilled:*

(1) $\xi_0(x),\ \psi_0(x) \in L_2[0,1]$,
(2) $0 < \max\limits_{(t,x)} |\, f_0\left(x, p_0(t)\right) | \le M_{0,1},\ 0 < M_{0,1} = \text{const}$,
(3) $\left|\, f_0\left(x, p_0^1(t)\right) - f_0\left(x, p_0^2(t)\right) \right| \le N_{0,1}\, \left|\, p_0^1(t) - p_0^2(t)\,\right|,\quad 0 < N_{0,1} = \text{const}$,
(4) $\rho_{0,1} = \frac{\|\, g_0(t)\,\|_C}{\alpha} T^{1-\gamma}\frac{C_{0,1}N_{0,1}}{1+x_0} < 1$.

Then the nonlinear Fredholm functional integral equation (5.25) has a unique solution in the space of continuous functions $C[0,T]$, which is found from the following iterative process:

$$\begin{cases} p_0^{k+1}(t) = \frac{g_0(t)}{2\alpha}\left[t^{1-\gamma}\Psi_0(t,x) + \int_0^T t^{1-\gamma}\bar{\bar{K}}_0(t,s,x) \,*\, f_0\left(p_0^k(s)\right) ds\right], \\ p_0^1(t) = \frac{g_0(t)}{2\alpha} t^{1-\gamma}\Psi_0(t,x), \qquad k = 0,1,2,\cdots. \end{cases} \tag{5.26}$$

Proof. According of the conditions of the Theorem 5.2 and estimate (4.3), from successive approximations (5.26), we obtain that for the first approximation there holds the following estimate

$$\left\|\, p_0^1(t)\,\right\|_C \le \frac{\|\, g_0(t)\,\|_C}{2\alpha} T^{1-\gamma}\,\|\, \Psi_0(t,x)\,\|_C$$

$$\leq \frac{\| g_0(t) \|_C}{2\alpha} T^{1-\gamma} \left| 2xT^{1-\gamma}\xi_0 - 2xt_1^{1-\gamma}\psi_0 \right| \tag{5.27}$$
$$\leq \frac{\| g_0(t) \|_C}{\alpha} T^{2(1-\gamma)} [| \xi_0 | + | \psi_0 |] < \infty.$$

For the first difference, we derive that there holds the following estimate

$$\| p_0^2(t) - p_0^1(t) \|_C \leq \frac{\| g_0(t) \|_C}{2\alpha} T^{1-\gamma} \int_0^T \left| \bar{\bar{K}}_0(t,s,x) \right| \, | f_0 \left(p_0^1(s) \right) | \, ds$$
$$\leq \frac{\| g_0(t) \|_C}{\alpha} T^{1-\gamma} \max_{(t,x)} | f_0 \left(x, p_0^1(t) \right) | \int_0^T | K_0(t_1,s) | \, ds \int_0^1 \omega_0(y) dy$$
$$\leq \frac{\| g_0(t) \|_C}{\alpha} T^{1-\gamma} \frac{2C_{0,1}M_{0,1}}{1+x_0} < \infty. \tag{5.28}$$

Analogously, for the arbitrary successive difference, we have the estimate

$$\| p_0^{k+1}(t) - p_0^k(t) \|_C \leq \frac{\| g_0(t) \|_C}{2\alpha} T^{1-\gamma} \int_0^T \left| \bar{\bar{K}}_0(t,s,x) \right| \cdot \left| f_0 \left(p_0^k(s) \right) - f_0 \left(p_0^{k-1}(s) \right) \right| ds$$
$$\leq \frac{\| g_0(t) \|_C}{2\alpha} T^{1-\gamma} \int_0^T \left| \bar{\bar{K}}_0(t,s,x) \right| \int_0^1 \left| f_0 \left(y, p_0^k(s) \right) - f_0 \left(y, p_0^{k-1}(s) \right) \right| dyds$$
$$\leq \frac{\| g_0(t) \|_C}{2\alpha} T^{1-\gamma} N_{0,1} \int_0^T \left| \bar{\bar{K}}_0(t,s,x) \right| \, \left| p_0^k(s) - p_0^{k-1}(s) \right| ds \int_0^1 \omega_0(y) dy$$
$$\leq \frac{\| g_0(t) \|_C}{\alpha} T^{1-\gamma} \frac{C_{0,1}N_{0,1}}{1+x_0} \left\| p_0^k(t) - p_0^{k-1}(t) \right\|_C = \rho_{0,1} \left\| p_0^k(t) - p_0^{k-1}(t) \right\|_C, \tag{5.29}$$

where

$$\rho_{0,1} = \frac{\| g_0(t) \|_C}{\alpha} T^{1-\gamma} \frac{C_{0,1}N_{0,1}}{1+x_0}.$$

According to the last condition of the Theorem 5.2, $\rho_{0,1} < 1$. From the validity of the estimates (5.27)–(5.29), it follows that the operator on the right-hand side of (5.25) is contracting and for this operator there exists a unique fixed point in the space of continuous functions $C[0,T]$. Therefore, the nonlinear functional integral equation (5.25) has a unique solution in the space $C[0,T]$. The Theorem 5.2 is proved. □

We denote the solution of the nonlinear integral functional equation (5.25) as

$$p_0(t) = h_0(t, g_0(t)). \tag{5.30}$$

Substituting (5.30) into (5.24), we obtain the following nonlinear Fredholm functional integral equation of the second kind with respect to function $g_0(t)$

$$t^{1-\gamma} g_0(t) = \int_0^T \bar{K}_0(t,s,x) \, * \, f_p \left(h_0(s, g_0(s)) \right) ds. \tag{5.31}$$

Theorem 5.3. *Let the following conditions be satisfied:*

1) $\xi_0(x)$, $\psi_0(x) \in L_2[0,1]$,
2) $0 < \max_{(t,x)} | f_p \left(x, h_0(t) \right) | \leq M_{0,2}$, $0 < M_{0,2} = \text{const}$,
3) $\left| f_p \left(x, h_0^1(t) \right) - f_p \left(x, h_0^2(t) \right) \right| \leq N_{0,2} \left| h_0^1(t) - h_0^2(t) \right|$, $0 < N_{0,2} = \text{const}$,

4) $\left|\, h_0\left(t, g_0^1(t)\right) - h_0\left(t, g_0^2(t)\right) \right| \leq N_{0,3} \left|\, g_0^1(t) - g_0^2(t) \right|, \quad 0 < N_{0,3} = \text{const},$
5) $\rho_{0,2} = \frac{4C_{0,1}N_{0,2}N_{0,3}}{1+x_0} < 1.$

Then the nonlinear Fredholm integral equation (5.31) has a unique solution in the class of continuous functions $g_0(t) \in C[0,T]$, which can be found from the following iterative process:

$$g_0^0(t) = 0, \quad t^{1-\gamma} g_0^{k+1}(t) = \int_0^T \bar{K}_0(t,s,x) * f_p\left(h_0(s, g_0^k(s))\right) ds. \tag{5.32}$$

Proof. From successive approximations (5.32), we obtain the following estimate for the first difference

$$\begin{aligned}\left\| g_0^1(t) - g_0^0(t) \right\|_C &\leq \max_t \int_0^T \left|\, \bar{K}_0(t,s,x) * f_{0,p}\left(h_0(s,0)\right) \right| ds \\ &\leq \max_t \int_0^T \left| \left[K_0(t,s) - K_0(t_1,s)\right] \int_0^1 f_{0,p}\left(y, h_0(0)\right) \omega_0(y) dy \right| ds \\ &\leq 2C_{0,1} \max_x \left|\, f_{0,p}\left(x, h_0(0)\right) \right| \int_0^1 \omega_0(y) dy \leq \frac{4C_{0,1}M_{0,2}}{1+x_0} < \infty,\end{aligned} \tag{5.33}$$

where

$$\max_t \int_0^T \left|\, K_0(t,s) - K_0(t_1,s) \right| ds \leq 2C_{0,1} = \text{const}.$$

Now, we obtain the estimate for arbitrary successive difference

$$\begin{aligned}\left\| g_0^{k+1}(t) - g_0^k(t) \right\|_C &\leq \max_{(t,x)} \int_0^T \left| \bar{K}_0(t,s,x) * \left[f_p\left(h_0(s, g_0^k(s))\right) - f_p\left(h(s, g_0^{k-1}(s))\right) \right] \right| ds \\ &\leq N_{0,2} \max_t \int_0^T \left|\, K_0(t,s) - K_0(t_1,s) \right| ds \int_0^1 \left|\, h_0\left(t, g_0^k(t)\right) - h_0\left(t, g_0^{k-1}(t)\right) \right| \omega_0(y) dy \\ &\leq 2C_{0,1}N_{0,2}N_{0,3} \left\| g_0^k(t) - g_0^{k-1}(t) \right\|_C \int_0^1 \omega_0(y) dy \leq \rho_{0,2} \left\| g_0^k(t) - g_0^{k-1}(t) \right\|_C,\end{aligned} \tag{5.34}$$

where

$$\rho_{0,2} = \frac{4C_{0,1}N_{0,2}N_{0,3}}{1+x_0} < 1.$$

It follows from the validity of these estimates (5.33) and (5.34) that the operator (5.31) is contracting and there exists a unique fixed point in the space of continuous functions $C[0,T]$. Therefore, the nonlinear integral Equation (5.31) has a unique solution in the space of continuous functions $g(t) \in C[0,T]$. The Theorem 5.3 is proved. □

We finished solving process for the Eq. (5.21). Substituting the solution of Eq. (5.31) into (5.30), we determine the control function $\bar{p}(t)$. Then the values of control function we substitute into Eq. (5.5) and obtain redefinition function. The values of control function we substitute into Eq. (5.7) and obtain the state function (see, [24, 25, 50, 51]). Thus, the process of solving of the fractional equation (4.4) is finished for the case of eigenvalues $\lambda_0 = 0$ and the eigenfunctions $\vartheta_0(x) = x$.

6. CS of fractional differential equation (4.5)

6.1. **Direct problem.** Now, we consider the case of $\lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2$ and the eigenfunctions $\vartheta_{1,n}(x) = \sin\sqrt{\lambda_{1,n}}x$. So, we consider CS of fractional differential equation (4.5):

$$D^{\alpha,\gamma} u_{1,n}(t) = -\mu_{1,n} u_{1,n}(t) + g_{1,n}(t)$$

with final condition (4.9): $u_{1,n}(T) = \varphi_{1,n}$, where

$$g_{1,n}(t) = \frac{1}{1+\lambda_{1,n}} f_{1,n}(p_1(t)), \quad \mu_{1,n} = \frac{\lambda_{1,n}}{1+\lambda_{1,n}}, \quad \lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2, \quad n \in \mathbb{N}.$$

Applying the operator J_{0t}^{α} to both sides of the equation and taking into account the formula [38], we obtain

$$J_{0t}^{\gamma} D_{0t}^{\gamma} u_{1,n}(t) = u_{1,n}(t) - \frac{1}{\Gamma(\gamma)} J_{0t}^{1-\gamma} A_{1,n} t^{\gamma-1},$$

we have

$$u_{1,n}(t) = \frac{A_{1,n}}{\Gamma(\gamma)} t^{\gamma-1} + J_{0+}^{\alpha} g_{1,n}(t) - \mu_{1,n} J_{0+}^{\alpha} u_{1,n}(t), \quad A_{1,n} = \text{const}.$$

We represent the solution of the countable system (4.5) in the form

$$u_{1,n}(t) = \frac{A_{1,n}}{\Gamma(\gamma)} t^{\gamma-1} + J_{0+}^{\alpha} g_{1,n}(t) - \mu_{1,n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) \left[\frac{A_{1,n}}{\Gamma(\gamma)} s^{\gamma-1} + J_{0+}^{\alpha} g_{1,n}(s)\right] ds.$$

We will rewrite this representation in the following form

$$u_{1,n}(t) = A_{1,n} \left[\frac{t^{\gamma-1}}{\Gamma(\gamma)} - \frac{\mu_{1,n}}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) s^{\gamma-1} ds\right]$$
$$+ J_{0+}^{\alpha} g_{1,n}(t) - \mu_{1,n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) J_{0+}^{\alpha} g_{1,n}(s)\, ds. \tag{6.1}$$

We will do some obvious calculations in representation (6.1)

$$\frac{t^{\gamma-1}}{\Gamma(\gamma)} - \frac{\mu_{1,n}}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) s^{\gamma-1} ds = t^{\gamma-1} E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)$$

and

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) J_{0+}^{\alpha} g_{1,n}(s)\, ds = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) ds \int_0^s (s-\theta)^{\alpha-1} g_{1,n}(\theta)\, d\theta$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^t g_{1,n}(s)\, ds \int_s^t (t-s)^{\alpha-1}(s-\theta)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-\theta)^{\alpha}\right) d\theta$$
$$= \int_0^t g_{1,n}(s)\, (t-s)^{2\alpha-1} E_{\alpha,2\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) ds.$$

By virtue of these relations, the presentation (6.1), we write as

$$u_{1,n}(t) = A_{1,n} t^{\gamma-1} E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) g_{1,n}(s) ds, \tag{6.2}$$

where

$$E_{\alpha,\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha\, m+\gamma)}, \quad z, \alpha, \gamma \in \mathbb{C}, \quad \mathrm{Re}\,(\alpha) > 0,$$

is Mittag–Leffler function [21, Vol. 1, P. 269–295].

In obtaining the Eq. (6.2) we took into account the following representations [21, Vol. 1, P. 269–295]:

$$E_{\alpha,\gamma}(z) = \frac{1}{\Gamma(\gamma)} + z\, E_{\alpha,\gamma+\alpha}(z), \quad \alpha > 0, \quad \gamma > 0,$$

$$\frac{1}{\Gamma(\alpha)} \int\limits_0^z (z-t)^{\alpha-1} E_{\alpha,\gamma}\left(k\, t^{\alpha}\right) t^{\gamma-1} dt = z^{\gamma+\alpha-1} E_{\alpha,\gamma+\alpha}\left(k\, z^{\alpha}\right), \quad \alpha > 0, \quad \gamma > 0.$$

Using the condition (4.9), we will find from (6.2) the unknown constant

$$A_{1,n} = \frac{T^{1-\gamma}}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)} \left[\varphi_{1,n} - \int\limits_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right) g_{1,n}(s) ds\right]. \tag{6.3}$$

Substituting (6.3) into Eq. (6.2), we obtain the new representation

$$u_{1,n}(t) = \varphi_{1,n}\sigma_{1,n}(t) - \sigma_{1,n}(t) \int\limits_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right) g_{1,n}(s) ds$$

$$+ \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right) g_{1,n}(s) ds, \tag{6.4}$$

where

$$\sigma_{1,n}(t) = \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)} \left[\frac{t}{T}\right]^{\gamma-1}.$$

The Equation (6.4) we represent in the convenient form

$$t^{1-\gamma} u_{1,n}(t) = \varphi_{1,n}\sigma_{1,0,n}(t) + \frac{1}{1+\lambda_{1,n}} \int\limits_0^T t^{1-\gamma} K_{1,n}(t,s)\, f_{1,n}\left(p_1(s)\right) ds, \tag{6.5}$$

where

$$K_{1,n}(t,s) = \begin{cases} -\sigma_{1,n}(t)(T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right), & t \le s \le T, \\ -\sigma_{1,n}(t)(T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right) + (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right), & s < t, \end{cases}$$

$$\sigma_{1,0,n}(t) = \sigma_{1,n}(t) t^{1-\gamma} = \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)} \left[\frac{t}{T}\right]^{\gamma-1} t^{1-\gamma} = \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)} T^{1-\gamma}.$$

We note that the function $\sigma_{1,n}(t)$ in (6.4) has singularity at the point $t = 0$. However, the function $\sigma_{1,0,n}(t)$ in (6.5) has no singularity at this point $t = 0$. For all $\alpha \in (0,1)$, $\alpha < \gamma \le 1$, $0 < \mu_{1,n} < 1$, we have the estimate $0 < \left| E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right) \right| \le M_{1,0} = \text{const}$. Moreover, in our further calculations we take into account that

$$\frac{1}{1+\lambda_{1,n}} < \frac{1}{\lambda_{1,n}} = \left[\frac{p+q}{2q\pi}\right]^2 \frac{1}{n^2},$$

$$\left| t^{1-\gamma} K_{1,n}(t,s) \right| \leq M_{1,1} \cdot (T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right) + T^{1-\gamma}(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right),$$

$$\int_0^T \left| t^{1-\gamma} K_{1,n}(t,s) \right| ds \leq 2(M_{1,1}+T^{1-\gamma}) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mu_{1,n}(T-s)^{\alpha}\right) ds$$
$$\leq 2\Gamma(\alpha)(M_{1,1}+T^{1-\gamma}) T^{1+\alpha-\gamma} M_{1,0} = M_{1,2} < \infty,$$

where

$$M_{1,1} \geq \max_t \sigma_{1,0,n}(t) = \max_t \left| \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)} T^{1-\gamma} \right|.$$

Substituting CS (6.5) into following Fourier series (see (3.1))

$$U_1(t,x) = \sum_{n=1}^{\infty *} u_{1,n}(t)\, \vartheta_{1,n}(x),$$

we obtain

$$t^{1-\gamma} U_1(t,x) = \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) \times \left\{ \varphi_{1,n} \sigma_{1,0,n}(t) + \frac{1}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} K_{1,n}(t,s) \int_0^1 f_1\left(y, p_1(s)\right) \omega_{1,n}(y)\, dy\, ds \right\}. \tag{6.6}$$

We consider the space $L_2[0,1]$ of summable squared functions $\vartheta(x)$ with the norm

$$\| \vartheta(x) \|_{L_2[0,1]} = \sqrt{\int_0^1 |\vartheta(y)|^2\, dy} < \infty.$$

Theorem 6.1. *Let the following conditions be satisfied:* $\varphi_1(x) \in L_2[0,1]$, $\max_t \| f_1(x, p_1(t)) \|_{L_2[0,1]} < \infty$. *Then for function (6.6) there holds* $U_1(t,x) \in \bar{H}(\Omega)$.

Proof. For fixed values of the redefinition function $\varphi_1(x)$ and of the control function $p_1(t)$, we substitute formula (6.6) into the integral $\Im_1 = \int_0^T \int_0^1 t^{2(1-\gamma)} U_1^2(t,y)\, dydt$ and we square it

$$\Im = \int_0^T \int_0^1 \left\{ \sum_{n=1}^{\infty *} \vartheta_{1,n}(y) \Big[\varphi_{1,n} \sigma_{1,0,n}(t) + \; + \frac{1}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} K_{1,n}(t,s) \int_0^1 f_1\left(z, p_1(s)\right) \omega_{1,n}(z)\, dz\, ds \Big] \right\}^2 dydt$$
$$= \int_0^T \int_0^1 \left\{ \sum_{n=1}^{\infty *} \varphi_{1,n} \sigma_{1,0,n}(t)\, \omega_{1,n}(y) \right\}^2 dydt + 2 \int_0^T \int_0^1 \left\{ \sum_{n=1}^{\infty *} \varphi_{1,n} \sigma_{1,0,n}(t)\, \omega_{1,n}(y) \right\}$$
$$\times \left\{ \sum_{n=1}^{\infty *} \frac{1}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} K_{1,n}(t,s) \int_0^1 f_1\left(z, p_1(s)\right) \omega_{1,n}(z)\, dz\, ds \right\} \omega_{1,n}(y)\, dy\, dt$$
$$+ \int_0^T \int_0^1 \left\{ \sum_{n=1}^{\infty *} \frac{1}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} K_{1,n}(t,s) \int_0^1 f_1\left(z, p_1(s)\right) \omega_{1,n}(z) dz ds \right\}^2 \omega_{1,n}(y) dydt. \tag{6.7}$$

We take into account that $\vartheta_{1,n}(x) = \sin\sqrt{\lambda_{1,n}}x$, $\lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2$ and apply the Cauchy–Schwarz and Bessel inequalities. Then from (6.7), we obtain the following estimate

$$\Im_1 \le [M_{1,1}]^2 T\left[\sum_{n=1}^{\infty *}|\varphi_{1,n}|\right]^2 + 2M_{1,2}M_{1,1}T\sum_{n=1}^{\infty *}|\varphi_{1,n}|\sqrt{\sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}^2}}\max_t\sqrt{\sum_{n=1}^{\infty *}\left|\int_0^1 f_1(y,p_1(t))\,\omega_{1,n}(y)dy\right|^2}$$

$$+ M_{1,2}T\sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}^2}\max_t\sum_{n=1}^{\infty *}\left|\int_0^1 f_1(y,p_1(t))\,\omega_{1,n}(y)dy\right|^2, \tag{6.8}$$

where $M_{1,2} = 2\Gamma(\alpha)(M_{1,1} + T^{1-\gamma})T^{1+\alpha-\gamma}M_{1,0}$.

Since $\max_t\sqrt{\sum_{n=1}^{\infty *}\left|\int_0^1 f_1(y,p_1(t))\,\omega_{1,n}(y)dy\right|^2} \le \max_t\| f_1(x,p_1(t))\|_{L_2[0,1]} < \infty$, from (6.8) implies the assertion of Theorem 6.1. □

6.2. **Inverse problem.** Now, we determinate redefinition function $\varphi_1(x)$ from the condition (4.13). According to the series (3.12), we apply the condition (4.13) into presentation (4.1)

$$t_1^{1-\gamma}\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\psi_{1,n} = \sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left[\varphi_{1,n}\sigma_{1,0,n}(t_1) + \frac{1}{1+\lambda_{1,n}}\int_0^T t_1^{1-\gamma}K_{1,n}(t_1,s)\int_0^1 f_1(y,p_1(s))\,\omega_{1,n}(y)\,dy\,ds\right]. \tag{6.9}$$

Taking into account that the functions $\vartheta_{1,n}(x)$, and $\omega_{1,n}(x)$ constitute a complete biorthonormal system in $L_2[0,1]$, we obtain

$$(\vartheta_{1,n}(x),\,\omega_{1,k}(x)) = \begin{cases} 0, & n \ne k, \\ 1, & n = k, \end{cases}$$

from (6.9), we obtain

$$t_1^{1-\gamma}\psi_n = \varphi_{1,n}\sigma_{1,0,n}(t_1) + \frac{1}{1+\lambda_{1,n}}\int_0^T t_1^{1-\gamma}K_{1,n}(t_1,s)\int_0^1 f_1(y,p_1(s))\,\omega_{1,n}(y)\,dy\,ds. \tag{6.10}$$

From the representation (6.10), we unique define Fourier coefficients $\varphi_{1,n}$ for redefinition function $\varphi_1(x)$:

$$\varphi_{1,n} = \frac{t_1^{1-\gamma}\psi_{1,n}}{\sigma_{1,0,n}(t_1)} - \frac{1}{1+\lambda_{1,n}}\int_0^T \frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}K_{1,n}(t_1,s)\int_0^1 f_1(y,p_1(s))\,\omega_{1,n}(y)\,dy\,ds, \tag{6.11}$$

where $t_1 > 0$ and

$$\sigma_{1,0,n}(t_1) = \frac{E_{\alpha,\gamma}(-\mu_n t_1^\alpha)}{E_{\alpha,\gamma}(-\mu_n T^\alpha)}T^{1-\gamma} > 0.$$

Substituting the Fourier coefficients (6.11) into Fourier series (3.11), we have

$$\varphi_1(x) = \sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)} \times \left\{\psi_{1,n} - \frac{1}{1+\lambda_{1,n}}\int_0^T K_{1,n}(t_1,s)\int_0^1 f_1(y,p_1(s))\,\omega_{1,n}(y)dyds\right\}, \tag{6.12}$$

for fixed values of control function $p_1(t)$. Then it is not difficult to see that for fixed values of control function $p_1(t)$ the series (6.12) is convergence

$$|\varphi_1(x)| \le \sum_{n=1}^{\infty *}\frac{t_1^{1-\gamma}}{|\sigma_{1,0,n}(t_1)|}|\psi_{1,n}| + \frac{2\Gamma(\alpha)(M_{1,1}+T^{1-\gamma})T^{1+\alpha-\gamma}M_{1,0}}{|\sigma_{1,0,n}(t_1)|}\sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}}\left|\int_0^1 f_1(y,p_1(s))\,\omega_{1,n}(y)\,dy\right|$$

$$\leq M_{1,3} \sum_{n=1}^{\infty *} |\, \psi_{1,n} \,| + M_{1,4} \sqrt{\sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}^2} \, \| \, f_1(x, p_1(t)) \, \|_{L_2[0,1]}} < \infty, \tag{6.13}$$

where

$$M_{1,3} = \frac{t_1^{1-\gamma}}{|\, \sigma_{1,0,n}(t_1) \,|}, \quad M_{1,4} = \frac{M_{1,2}}{|\, \sigma_{1,0,n}(t_1) \,|}.$$

6.3. **Optimal control function.** Now we will start to find the control function $p_1(t)$ for the case of $\lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2$ and the eigenfunctions $\vartheta_{1,n}(x) = \sin \sqrt{\lambda_{1,n}} x$. First, substituting the presentation (6.11) into series (6.6), we obtain

$$t^{1-\gamma} U_1(t,x) = \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) t_1^{1-\gamma} \sigma_{1,1,n}(t) \psi_{1,n} + \sum_{n=1}^{\infty *} \frac{\vartheta_{1,n}(x)}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} \bar{K}_{1,n}(t,s) \int_0^1 f_1\left(y, p_1(s)\right) \omega_{1,n}(y) \, dy \, ds, \tag{6.14}$$

where

$$\bar{K}_{1,n}(t,s) = t^{1-\gamma} K_{1,n}(t,s) - t_1^{1-\gamma} K_{1,n}(t_1,s) \sigma_{1,1,n}(t),$$

$$\sigma_{1,1,n}(t) = \frac{\sigma_{1,0,n}(t)}{\sigma_{1,0,n}(t_1)} = \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} t_1^{\alpha}\right)}.$$

Let $p_1^*(t)$ is optimal control function:

$$\Delta \, J\left[p_1^*(t)\right] = J\left[p_1^*(t) + \Delta \, p_1^*(t)\right] - J\left[p_1^*(t)\right] \geq 0,$$

where $p_1^*(t) + \Delta \, p_1^*(t) \in \bar{H}[0,T]$. We consider the following function

$$\begin{aligned} \alpha \left[p_1^*(t)\right]^2 = \; & t^{1-\gamma} Q_1(t,x) \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) t_1^{1-\gamma} \sigma_{1,1,n}(t) \psi_{1,n} \\ & + t^{1-\gamma} Q_1(t,x) \sum_{n=1}^{\infty *} \frac{\vartheta_{1,n}(x)}{1+\lambda_{1,n}} \int_0^T t^{1-\gamma} \bar{K}_{1,n}(t,s) \int_0^1 f_1\left(y, p_1^*(s)\right) \omega_{1,n}(y) \, dy \, ds, \end{aligned} \tag{6.15}$$

where $Q(t,x)$ is defined as the solution to the following mixed boundary value problem:

$$D^{\alpha,\gamma} Q_1(t,x) + D^{\alpha,\gamma} Q_{1xx}(t,x) + Q_{1xx}(t,x) = 0, \quad (t,x) \in \Omega, \tag{6.16}$$

$$Q_1(T,x) = -2\left[\varphi_1(x) - \xi_1(x)\right], \;\; Q_1(t,0) = 0, \;\; Q_{1x}(t,1) = Q_{1x}(t,x_0), \tag{6.17}$$

which is conjugated to problem (2.1)–(2.3).

The Eq. (6.15) we rewrite in convenient form:

$$t^{1-\gamma} Q_1(t,x) \left[\Phi_1(t,x) + \int_0^T \bar{K}_1(t,s,x) * f_1\left(p_1^*(s)\right) ds\right] = \alpha \left[p_1^*(t)\right]^2, \tag{6.18}$$

where

$$\Phi_1(t,x) = \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) t_1^{1-\gamma} \sigma_{1,1,n}(t) \psi_{1,n},$$

$$\bar{K}_1(t,s,x) * f_1\left(p_1^*(s)\right) = \sum_{n=1}^{\infty *} \frac{\vartheta_{1,n}(x)}{1+\lambda_{1,n}} t^{1-\gamma} \bar{K}_{1,n}(t,s) \int_0^1 f_1\left(y, p_1^*(s)\right) \omega_{1,n}(y) dy.$$

According to the maximum principle, we calculate derivative of the control function and come to the following necessary condition for optimality

$$t^{1-\gamma}Q_1(t,x)\int\limits_0^T \bar{K}_1(t,s,x)\,*\,f_p\left(p_1^*(s)\right)ds - 2\alpha\, p_1^*(t) = 0. \tag{6.19}$$

Calculating derivative in (6.19) with respect to the control function $p_1^*(t)$, we obtain another necessary condition for optimality

$$t^{1-\gamma}Q_1(t,x)\int\limits_0^T \bar{K}_1(t,s,x) * f_{1pp}\left(p_1^*(s)\right)ds - 2\alpha < 0. \tag{6.20}$$

We solve the conjugated differential equation (6.16) by the same as we solved the fractional differential equation (2.1). According to the second condition of (6.17), the nonzero solution of the Eq. (6.16) we find from the CS of fractional differential equations

$$D^{\alpha,\gamma}q_{1,n}(t) = \mu_{1,n}q_{1,n}(t), \tag{6.21}$$

where

$$q_{1,n}(t) = \int\limits_0^1 Q_1(t,y)\omega_{1,n}(y)dy.$$

To solve the CS of differential equations (6.21) we use the first condition of (6.17) in the following form

$$q_{1,n}(T) = -2\int\limits_0^1 \left[\varphi_1(y) - \xi_1(y)\right]\omega_{1,n}(y)dy = -2\varphi_{1,n} + 2\xi_{1,n}. \tag{6.22}$$

Substituting presentation (6.11) into the formula (6.22), we obtain

$$q_{1,n}(T) = 2\xi_{1,n} - \frac{2\psi_{1,n}}{\sigma_{1,0,n}(t_1)} + \frac{2}{1+\lambda_{1,n}}\int\limits_0^T \frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}K_{1,n}(t_1,s)\int\limits_0^1 f_{1,n}\left(y,p_1(s)\right)\omega_{1,n}(y)\,dy\,ds. \tag{6.23}$$

The general solution of the CS of homogeneous equation (6.21) has a form

$$q_{1,n}(t) = B_{1,n}t^{\gamma-1}E_{\alpha,\gamma}\left(\mu_{1,n}t^{\alpha}\right), \tag{6.24}$$

where we determine $B_{1,n}$ from the condition (6.23):

$$B_{1,n} = \frac{2T^{1-\gamma}\xi_{1,n}}{E_{\alpha,\gamma}\left(\mu_{1,n}T^{\alpha}\right)} - \sigma_{1,2,n}\psi_{1,n} + \frac{\sigma_{1,2,n}t_1^{1-\gamma}}{1+\lambda_{1,n}}\int\limits_0^T K_{1,n}(t_1,s)\int\limits_0^1 f_1\left(y,p_1(s)\right)\omega_{1,n}(y)dy\,ds, \tag{6.25}$$

$$\sigma_{1,2,n} = \frac{2E_{\alpha,\gamma}\left(-\mu_{1,n}T^{\alpha}\right)}{E_{\alpha,\gamma}\left(\mu_{1,n}T^{\alpha}\right)E_{\alpha,\gamma}\left(-\mu_{1,n}t_1^{\alpha}\right)}.$$

Substituting (6.25) into general solution (6.24) of homogeneous fractional equation (6.21), we obtain

$$t^{1-\gamma}Q_1(t,x) = \Psi_1(t,x) + \int\limits_0^T \bar{\bar{K}}_1(t,s,x) * f_1\left(p_1(s)\right)ds, \tag{6.26}$$

where

$$\Psi_1(t,x) = \sum_{n=1}^{\infty*}\vartheta_{1,n}(x)\Big[2T^{1-\gamma}\xi_{1,n}\sigma_{1,3,n}(t) - \sigma_{1,4,n}(t)\psi_{1,n}\Big],$$

$$\bar{\bar{K}}_1(t,s,x) \, * \, f_1\left(p_1(s)\right) = \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) \frac{\sigma_{1,4,n} t_1^{1-\gamma}}{1+\lambda_{1,n}} K_{1,n}(t_1,s) \int\limits_0^1 f_1\left(y, p_1(s)\right) \omega_{1,n}(y) dy,$$

$$\sigma_{1,3,n}(t) = \frac{E_{\alpha,\gamma}\left(\mu_{1,n} t^{\alpha}\right)}{E_{\alpha,\gamma}\left(\mu_{1,n} T^{\alpha}\right)}, \quad \sigma_{1,4,n}(t) = \sigma_{1,2,n} E_{\alpha,\gamma}\left(\mu_{1,n} t^{\alpha}\right) = 2\sigma_{1,3,n}(t) \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} T^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} t_1^{\alpha}\right)}.$$

Taking into account (6.26), the optimality condition (6.19) we rewrite as

$$\int\limits_0^T \bar{K}_1(t,s,x) \, * \, f_{1p}\left(p_1(s)\right) ds = \frac{2\alpha\, p_1(t)}{\Psi_1(t,x) + \int\limits_0^T \bar{\bar{K}}_1(t,s,x) \, * \, f_1\left(p_1(s)\right) ds}. \tag{6.27}$$

Substituting (6.26) into condition (6.20), we obtain

$$\int\limits_0^T \bar{K}_1(t,s,x)\, f_{1pp}\left(p_1^*(s)\right) ds \left[\Psi_1(t,x) + \int\limits_0^T \bar{\bar{K}}_1(t,s,x)\, f_1\left(p_1^*(s)\right) ds\right] < 2\alpha. \tag{6.28}$$

By virtue of (6.28), we solve the Eq. (6.27) with respect to the control function $p_1(t)$. If the nonlinear functional integral equation (6.27) is solvable, then it is true that the following relations hold

$$\frac{2\alpha p_1(t)}{\Psi_1(t,x) + \int\limits_0^T \bar{\bar{K}}_1(t,s,x) * f_1\left(p_1(s)\right) ds} = t^{1-\gamma} g_1(t), \tag{6.29}$$

$$\int\limits_0^T \bar{K}_1(t,s,x) * f_{1p}\left(p_1(s)\right) ds = t^{1-\gamma} g_1(t), \tag{6.30}$$

where $g_1(t) \in C[0,T]$ is yet unknown function. So, from the nonlinear functional integral equation (6.27) we came two different nonlinear equations (6.29) and (6.30). First, we solve the Eq. (6.29) by the method of successive approximations. However, we assume that the function $g_1(t)$ in (6.29) is known. Therefore, the nonlinear Fredholm functional integral equation (6.29) we rewrite as follows

$$p_1(t) = \frac{g_1(t)}{2\alpha}\left[t^{1-\gamma}\Psi_1(t,x) + \int\limits_0^T t^{1-\gamma} \bar{\bar{K}}_1(t,s,x) \, * \, f_1\left(p_1(s)\right) ds\right]. \tag{6.31}$$

For an arbitrary function $p(t) \in C[0,T]$, we consider the following continuous norm

$$\| \, p(t) \, \|_C = \max_{t \in \Omega_T} | \, p(t) \, | \, .$$

Theorem 6.2. *Let the following conditions are fulfilled:*

1) $\xi_1(x),\, \psi_1(x) \in L_2[0,1]$,
2) $0 < \max\limits_t \| \, f_1\left(x, p_1(t)\right) \|_{L_2[0,1]} \le N_{1,1},\ 0 < N_{1,1} = \text{const}$,
3) $\left| \, f_1\left(x, p_1^1(t)\right) - f_1\left(x, p_1^2(t)\right) \right| \le N_{1,2}(x) \left| \, p_1^1(t) - p_1^2(t) \, \right|, \quad 0 < \| \, N_{1,2}(x) \, \|_{L_2[0,1]}$,
4) $\rho_{1,1} = M_{1,5} M_{1,6} \, \| \, g_1(t) \, \|_C \, \| \, N_{1,2}(x) \, \|_{L_2[0,1]} < 1$,

where $M_{1,5} = M_{1,0}\alpha^{-1}\Gamma(\alpha)(M_{1,1} + T^{1-\gamma})T^{2+\alpha-2\gamma}, \quad M_{1,6} = \sqrt{\sum\limits_{n=1}^{\infty *} \lambda_{1,n}^{-2}}.$

Then the nonlinear Fredholm functional integral equation (6.31) has a unique solution in the space of continuous functions $C[0,T]$, which is found from the following iterative process

$$\begin{cases} p_1^{k+1}(t) = \frac{g_1(t)}{2\alpha}\left[t^{1-\gamma}\Psi_1(t,x) + \int_0^T t^{1-\gamma}\bar{\bar{K}}_1(t,s,x)\, f_1\left(p_1^k(s)\right) ds\right], \\ p_1^1(t) = \frac{g_1(t)}{2\alpha}t^{1-\gamma}\Psi_1(t,x), \qquad\qquad k = 0,1,2,\cdots. \end{cases} \tag{6.32}$$

Proof. By virtue of (4.3), from successive approximations (6.32), we obtain that for the first approximation there holds the following estimate

$$\begin{aligned} \| p_1^1(t) \|_C &\le \frac{\| g_1(t) \|_C}{2\alpha} T^{1-\gamma} \| \Psi_1(t,x) \|_C \le \frac{\| g_1(t) \|_C}{2\alpha} T^{1-\gamma} \times \max_t \left| \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) \left[2T^{1-\gamma}\xi_{1,n}\sigma_{1,3,n}(t) - \sigma_{1,4,n}(t)\psi_{1,n} \right] \right| \\ &\le M_{1,0}C_{1,0}\frac{\| g_1(t) \|_C}{\alpha} \left[\sum_{n=1}^{\infty *} |\, \xi_{1,n} \,| + \sum_{n=1}^{\infty *} |\, \psi_{1,n} \,| \right] < \infty, \quad M_{1,0},\ C_{1,0} = \text{const}. \end{aligned} \tag{6.33}$$

Taking into account the estimate (6.33) and approximations (6.32), for the first difference, we derive the following estimate

$$\begin{aligned} \| p_1^2(t) - p_1^1(t) \|_C &\le M_{1,0}\frac{\| g_1(t) \|_C}{2\alpha} T^{1-\gamma} \times \int_0^T \left| \sum_{n=1}^{\infty *} \vartheta_{1,n}(x) \frac{\sigma_{1,4,n}(t) t_1^{1-\gamma}}{\lambda_{1,n}} K_{1,n}(t_1,s) \int_0^1 f_{1,n}\left(y, p_1^1(s)\right) \omega_{1,n}(y) dy \right| \\ &\le M_{1,0}\frac{\|g_1(t)\|_C}{\alpha}\Gamma(\alpha)(M_{1,1}+T^{1-\gamma})T^{2+\alpha-2\gamma}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{\lambda_n^2}}\sqrt{\sum_{n=1}^{\infty *}\left[\max_t \left|f_{1,n}\left(p_1^1(t)\right)\right|\right]^2} \\ &\le M_{1,5} \| g_1(t) \|_C\, M_{1,6} \max_t \left\| f_1\left(x, p_1^1(t)\right) \right\|_{L_2[0,1]} \le M_{1,5} \| g_1(t) \|_C\, M_{1,6} N_{1,1} < \infty, \end{aligned} \tag{6.34}$$

where

$$M_{1,5} = M_{1,0}\alpha^{-1}\Gamma(\alpha)(M_{1,1}+T^{1-\gamma})T^{2+\alpha-2\gamma}, \quad M_{1,6} = \sqrt{\sum_{n=1}^{\infty *}\lambda_{1,n}^{-2}}.$$

Analogously, for the arbitrary successive difference, we have estimate

$$\begin{aligned} \left\|p_1^{k+1}(t) - p_1^k(t)\right\|_C &\le M_{1,5}\, \|g_1(t)\|_C \sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}} \max_t \left| f_{1,n}\left(p_1^k(t)\right) - f_n\left(p_1^{k-1}(t)\right)\right| \\ &\le M_{1,5}\, \|g_1(t)\|_C \sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}} \max_t \int_0^1 N_{1,2}(y) \left|p_1^k(t) - p_1^{k-1}(t)\right| dy \\ &\le M_{1,5}M_{1,6} \| g_1(t) \|_C \left\| p_1^k(t) - p_1^{k-1}(t) \right\|_C \sqrt{\sum_{n=1}^{\infty *}\left[\int_0^1 N_{1,2}(y)dy\right]^2} \\ &\le \rho_{1,1} \left\| p_1^k(t) - p_1^{k-1}(t) \right\|_C, \end{aligned} \tag{6.35}$$

where

$$\rho_{1,1} = M_{1,5}M_{1,6} \| g_1(t) \|_C \| N_{1,2}(x) \|_{L_2[0,1]}.$$

According to the last condition of the Theorem 6.1, $\rho_{1,1} < 1$. From the validity of the estimates (6.33)–(6.35), it follows that the operator on the right-hand side of (6.31) is contracting and for this operator there exists a unique fixed point in the space of continuous functions $C[0,T]$. Therefore, the nonlinear functional integral equation (6.31) has a unique solution in the space $C[0,T]$. The Theorem 6.2 is proved. □

We denote this solution of the nonlinear functional integral equation (6.31) as

$$p_1(t) = h_1(t, g_1(t)). \tag{6.36}$$

Substituting (6.36) into (6.30), we obtain the following nonlinear Fredholm integral equation of the second kind with respect to $g_1(t)$

$$t^{1-\gamma} g_1(t) = \int_0^T \bar{K}_1(t, s, x) \, * \, f_{1p}\left(h_1(s, g_1(s))\right) ds. \tag{6.37}$$

Theorem 6.3. *Let the following conditions be satisfied:*

1) $\xi_1(x),\ \psi_1(x) \in L_2[0, 1]$,
2) $0 < \max_t \| f_{1p}\,(x, h_1(g_1)) \|_{L_2[0,1]} \le N_{1,3}, \ \ 0 < N_{1,3} = \text{const}$,
3) $\left| f_{1p}\left(x, h_1^1(t)\right) - f_p\left(x, h_1^2(t)\right) \right| \le N_{1,4}(x) \left| h_1^1(t) - h_1^2(t) \right|, \ \ 0 < \| N_{1,4}(x) \|_{L_2[0,1]}$,
4) $\left| h_1\left(t, g_1^1(t)\right) - h_1\left(t, g_1^2(t)\right) \right| \le N_{1,5} \left| g_1^1(t) - g_1^2(t) \right|, \ \ 0 < N_{1,5} = \text{const}$,
5) $\rho_{1,2} = 2 M_{1,2} N_{1,5} M_{1,6} \| N_{1,4}(x) \|_{L_2[0,1]} < 1$.

Then the nonlinear Fredholm integral equation (6.37) has a unique solution in the class of continuous functions $g_1(t) \in C[0, T]$, which can be found from the following iterative process:

$$g_1^0(t) = 0, \quad t^{1-\gamma} g_1^{k+1}(t) = \int_0^T \bar{K}_1(t, s, x) \, * \, f_{1p}\left(h_1(s, g_1^k(s))\right) ds. \tag{6.38}$$

Proof. Taking into account that $g_1^0(t) = 0$, from successive approximations (6.38), we obtain the following estimate for the first difference

$$\begin{aligned}
\left\| g_1^1(t) - g_1^0(t) \right\|_C &\le \max_t \int_0^T \left| \bar{K}_1(t, s, x) \, * \, f_{1p}\big(h_1(s, 0)\big) \right| ds \\
&\le \max_t \int_0^T \left| \sum_{n=1}^{\infty *} \frac{\vartheta_{1,n}(x)}{1 + \lambda_{1,n}} t^{1-\gamma} \bar{K}_{1,n}(t, s) \int_0^1 f_{1p}\left(y, h_1(s, 0)\right) \omega_{1,n}(y) \, dy \right| ds \\
&\le 2 M_{1,2} \max_t \left| \sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}} \int_0^1 f_{1p}\left(y, h_1(t, 0)\right) \omega_{1,n}(y) \, dy \right| \\
&\le 2 M_{1,2} \sqrt{\sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}^2}} \sqrt{\sum_{n=1}^{\infty *} \left[\max_t \left| f_{1,n\,p}\left(h_1(t, 0)\right) \right| \right]^2} \le 2 M_{1,2} N_{1,3} M_{1,6} < \infty,
\end{aligned} \tag{6.39}$$

where

$$M_{1,2} = 2\Gamma(\alpha)(M_{1,1} + T^{1-\gamma}) T^{1+\alpha-\gamma} M_{1,0}, \ \ M_{1,1} \ge \max_t \sigma_{1,0,n}(t) = \max_t \left| \frac{E_{\alpha,\gamma}\left(-\mu_{1,n} t^\alpha\right)}{E_{\alpha,\gamma}\left(-\mu_{1,n} T^\alpha\right)} T^{1-\gamma} \right|.$$

Now, we obtain the estimate for arbitrary successive difference

$$\begin{aligned}
\left\| g_1^{k+1}(t) - g_1^k(t) \right\|_C &\le \max_t \int_0^T \left| \sum_{n=1}^{\infty *} \frac{\vartheta_{1,n}(x)}{1 + \lambda_n} t^{1-\gamma} \bar{K}_{1,n}(t, s) \left[f_{1,n\,p}\left(h_1\big(s, g_1^k(s)\big)\right) - f_{1,n\,p}\left(h_1\big(s, g_1^{k-1}(s)\big)\right) \right] \right| ds \\
&\le 2 M_{1,2} \max_t \left| h_1\left(t, g_1^{k-1}(t)\right) - h_1\left(t, g_1^{k-1}(t)\right) \right| \sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}} \left| \int_0^1 N_{1,4}(y) \, \omega_{1,n}(y) \, dy \right|
\end{aligned}$$

$$\leq 2M_{1,2}N_{1,5}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}^{2}}}\sqrt{\sum_{n=1}^{\infty *}\left[\int_{0}^{1}N_{1,4}(y)\,dy\right]^{2}}\,\left\|\,g_1^k(t)-g_1^{k-1}(t)\,\right\|_C$$

$$\leq \rho_{1,2}\left\|\,g_1^k(t)-g_1^{k-1}(t)\,\right\|_C, \tag{6.40}$$

where

$$\rho_{1,2}=2M_{1,2}N_{1,5}M_{1,6}\,\|\,N_{1,4}(x)\,\|_{L_2[0,1]}\,.$$

It follows from the validity of the estimates (6.39) and (6.40) that the operator on the right-hand side of (6.37) is contracting and there exists a unique fixed point in the space of continuous functions $C[0,T]$. Therefore, the nonlinear integral equation (6.37) has a unique solution in the space of continuous functions $g_1(t)\in C[0,T]$. The Theorem 6.3 is proved. □

Thus, we finished solving process for the Eq. (6.27). Substituting the solution of Eq. (6.37) into (6.36), we determine the control function $\bar{p}_1(t)$. Then we define redefinition function (6.12) and state function (6.14) (see, [24, 25, 50, 51]).

7. CS of fractional differential equation (4.7)

7.1. **Direct problem.** Now for the case of $\lambda_{2,m}=(2qm\pi)^2$ and for the associated functions $\tilde{\vartheta}_{2,m}(x)=x\cos\sqrt{\lambda_{2,m}}x$ we consider CS of fractional differential equation (4.7) with final condition (4.11). Applying the operator J_{0+}^{α} to both sides of this equation, taking into account the formula:

$$J_{0t}^{\gamma}D_{0t}^{\gamma}\tilde{u}_{2,m}(t)=\tilde{u}_{2,m}(t)-\frac{1}{\Gamma(\gamma)}\,J_{0t}^{1-\gamma}A_{2,m}t^{\gamma-1},$$

we have

$$t^{1-\gamma}\tilde{u}_{2,m}(t)=\tilde{\varphi}_{2,m}\tilde{\sigma}_{2,0,m}(t)+\frac{1}{1+\lambda_{2,m}}\int_{0}^{T}t^{1-\gamma}\tilde{K}_{2,m}(t,s)\,\tilde{f}_{2,m}(\tilde{p}_2(s))ds, \tag{7.1}$$

where

$$\tilde{K}_{2,m}(t,s)=\begin{cases}-\sigma_{2,m}(t)(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(T-s)^{\alpha}\right), & t\leq s\leq T,\\ -\sigma_{2,m}(t)(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(T-s)^{\alpha}\right)+(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(t-s)^{\alpha}\right), & s<t,\end{cases}$$

$$\sigma_{2,m}(t)=\frac{E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}\left[\frac{t}{T}\right]^{\gamma-1},\quad \sigma_{2,0,m}(t)=\frac{E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}T^{1-\gamma}.$$

Substituting CS (7.1) into following Fourier series

$$\tilde{U}_2(t,x)=\sum_{m=1}^{\infty}\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x),$$

we obtain

$$t^{1-\gamma}\tilde{U}_2(t,x)=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\times\left\{\tilde{\varphi}_{2,m}\sigma_{2,0,m}(t)+\frac{1}{1+\lambda_{2,m}}\int_{0}^{T}t^{1-\gamma}\tilde{K}_{2,m}(t,s)\int_{0}^{1}\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)\,dy\,ds\right\}. \tag{7.2}$$

Theorem 7.1. *Let the following conditions be satisfied:* $\tilde{\varphi}_2(x)\in L_2[0,1]$, $\max\limits_{0\leq t\leq T}\left\|\,\tilde{f}_2\left(x,\tilde{p}_2(t)\right)\right\|_{L_2[0,1]}<\infty$. *Then for function (7.2) there holds* $\tilde{U}_2(t,x)\in\bar{H}(\Omega)$.

7.2. **Inverse problem.** Now we determinate redefinition function $\tilde{\varphi}_2(x)$ from the condition (4.15). Appling the condition (4.15) into presentation (7.2):

$$t_1^{1-\gamma}\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\tilde{\psi}_{2,m}=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\left[\tilde{\varphi}_{2,m}\sigma_{2,0,m}(t_1)+\frac{1}{1+\lambda_{2,m}}\int_0^T t_1^{1-\gamma}\tilde{K}_{2,m}(t_1,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)\,dy\,ds\right]. \tag{7.3}$$

Taking into account in that the functions $\tilde{\vartheta}_{2,m}(x)$, $\omega_{2,m}(x)$ form a complete system of biorthonormal functions in $L_2[0,1]$:

$$\Big(\tilde{\vartheta}_{2,m}(x),\,\omega_{2,k}(x)\Big)=\begin{cases}0, & m\neq k,\\ 1, & m=k,\end{cases}$$

from (7.3) we obtain

$$\tilde{\varphi}_{2,m}=\frac{t_1^{1-\gamma}\tilde{\psi}_{2,m}}{\sigma_{2,0,m}(t_1)}-\frac{1}{1+\lambda_{2,m}}\int_0^T\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\tilde{K}_{2,m}(t_1,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)dyds, \tag{7.4}$$

and redefinition function $\tilde{\varphi}_2(x)$

$$\tilde{\varphi}_2(x)=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\left\{\frac{t_1^{1-\gamma}\tilde{\psi}_{2,m}}{\sigma_{2,0,m}(t_1)}-\frac{1}{1+\lambda_{2,m}}\int_0^T\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\tilde{K}_{2,m}(t_1,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)\,dy\,ds\right\}, \tag{7.5}$$

for fixed values of control function $\tilde{p}_2(t)$. Then it is not difficult to see for fixed values of control function $\tilde{p}_2(t)$ that the series (7.5) is convergence.

7.3. **Optimal control function.** Now we will start to find the control function $\tilde{p}_2(t)$. First, substituting the presentation (7.4) into series (7.3), we obtain

$$t^{1-\gamma}\tilde{U}_2(t,x)=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\,t_1^{1-\gamma}\sigma_{2,2,m}(t)\tilde{\psi}_{2,m}+\sum_{m=1}^{\infty}\frac{\tilde{\vartheta}_{2,m}(x)}{1+\lambda_{2,m}}\int_0^T t^{1-\gamma}\bar{K}_{2,m}(t,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)\,dy\,ds, \tag{7.6}$$

where

$$\bar{\tilde{K}}_{2,m}(t,s)=t^{1-\gamma}\tilde{K}_{2,m}(t,s)-t_1^{1-\gamma}\tilde{K}_{2,m}(t_1,s)\sigma_{2,2,m}(t),\quad \sigma_{2,2,m}(t)=\frac{E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}t_1^{\alpha}\right)}.$$

In order to find the control function $\tilde{p}_2(t)$ from the function (7.6) and minimization of functional (4.1) we come to the equation

$$\int_0^T\bar{\tilde{K}}_2(t,s,x)*\tilde{f}_{2p}\left(\tilde{p}_2(s)\right)ds=\frac{2\alpha\,\tilde{p}_2(t)}{\tilde{\Psi}_2(t,x)+\int_0^T\bar{\bar{\tilde{K}}}_1(t,s,x)\,*\,\tilde{f}_2\left(\tilde{p}_2(s)\right)ds}, \tag{7.7}$$

where

$$\bar{\tilde{K}}_2(t,s,x)*\tilde{f}_2\left(\tilde{p}_2(s)\right)=\sum_{m=1}^{\infty}\frac{\tilde{\vartheta}_{2,m}(x)}{1+\lambda_{2,m}}t^{1-\gamma}\bar{\tilde{K}}_{2,m}(t,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)\,dy,$$

$$\tilde{\Psi}_2(t,x)=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\left[2T^{1-\gamma}\tilde{\xi}_{2,m}\sigma_{2,3,m}(t)-\sigma_{2,4,m}(t)\tilde{\psi}_{2,m}\right],$$

$$\bar{\bar{\tilde{K}}}_2(t,s,x)*\tilde{f}_2\left(\tilde{p}_2(s)\right)=\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\frac{\sigma_{2,4,m}t_1^{1-\gamma}}{1+\lambda_{2,m}}\tilde{K}_{2,m}(t_1,s)\int_0^1\tilde{f}_2\left(y,\tilde{p}_2(s)\right)\omega_{2,m}(y)dy,$$

$$\sigma_{2,2,m}=\frac{2E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}{E_{\alpha,\gamma}\left(\mu_{2,m}T^{\alpha}\right)E_{\alpha,\gamma}\left(-\mu_{2,m}t_1^{\alpha}\right)},\quad \sigma_{2,3,m}(t)=\frac{E_{\alpha,\gamma}\left(\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(\mu_{2,m}T^{\alpha}\right)},$$

$$\sigma_{2,4,m}(t)=\sigma_{2,2,m}E_{\alpha,\gamma}\left(\mu_{2,m}t^{\alpha}\right)=2\sigma_{2,3,m}(t)\frac{E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}t_1^{\alpha}\right)}.$$

We solve the Eq. (7.7) with respect to the control function $\tilde{p}_2(t)$. We consider

$$\frac{2\alpha\tilde{p}_2(t)}{\tilde{\Psi}_2(t,x)+\int\limits_0^T\bar{\bar{\tilde{K}}}_2(t,s,x)\,*\,\tilde{f}_2\left(\tilde{p}_2(s)\right)ds}=t^{1-\gamma}\tilde{g}_2(t), \tag{7.8}$$

$$\int\limits_0^T\bar{\bar{\tilde{K}}}_2(t,s,x)\,*\,\tilde{f}_{2p}\left(\tilde{p}_2(s)\right)ds=t^{1-\gamma}\tilde{g}_2(t), \tag{7.9}$$

where $\tilde{g}_2(t)\in C[0,T]$ is yet unknown function. First, we solve the Eq. (7.8) by the method of successive approximations. However, we assume that the function $\tilde{g}_2(t)$ in (7.9) is known. Therefore, the nonlinear Fredholm functional integral equation (7.8) we rewrite as follows

$$\tilde{p}_2(t)=\frac{\tilde{g}_2(t)}{2\alpha}\left[t^{1-\gamma}\tilde{\Psi}_2(t,x)+\int\limits_0^T t^{1-\gamma}\bar{\bar{\tilde{K}}}_2(t,s,x)*\tilde{f}_2\left(\tilde{p}_2(s)\right)ds\right]. \tag{7.10}$$

Theorem 7.2. *Let the following conditions are fulfilled:*

1) $\tilde{\xi}_2(x),\ \tilde{\psi}_2(x)\in L_2[0,1]$,
2) $0<\max\limits_t\left\|\tilde{f}_2\left(x,\tilde{p}_2(t)\right)\right\|_{L_2[0,1]}\le\tilde{N}_{2,1},\ 0<\tilde{N}_{2,1}=\text{const}$,
3) $\left|\tilde{f}_2\left(x,\tilde{p}_2^1(t)\right)-\tilde{f}_2\left(x,\tilde{p}_2^2(t)\right)\right|\le\tilde{N}_{2,2}(x)\left|\tilde{p}_2^1(t)-\tilde{p}_2^2(t)\right|,\ \ 0<\left\|\tilde{N}_{2,2}(x)\right\|_{L_2[0,1]}$,
4) $\tilde{\rho}_{2,1}=\tilde{C}_{2,1}\left\|\tilde{g}_2(t)\right\|_C\left\|\tilde{N}_{2,2}(x)\right\|_{L_2[0,1]}<1,\ \ \tilde{C}_{2,1}=\text{const}$.

Then the nonlinear Fredholm functional integral equation (7.10) has a unique solution in the space of continuous functions $C[0,T]$.

We denote this solution of the nonlinear integral functional equation (7.10) as

$$\tilde{p}_2(t)=\tilde{h}_2(t,\tilde{g}_2(t)). \tag{7.11}$$

Substituting (7.11) into (7.9), we obtain the following nonlinear Fredholm integral equation of the second kind with respect to $\tilde{g}_2(t)$

$$t^{1-\gamma}\tilde{g}_2(t)=\int\limits_0^T\bar{\bar{\tilde{K}}}_1(t,s,x)\,*\,\tilde{f}_{2p}\left(\tilde{h}_2(s,\tilde{g}_2(s))\right)ds. \tag{7.12}$$

Theorem 7.3. *Let the following conditions be satisfied:*

1) $\tilde{\xi}_2(x),\ \tilde{\psi}_2(x)\in L_2[0,1]$,
2) $0<\max\limits_t\left\|\tilde{f}_{2p}\left(x,\tilde{h}_2(\tilde{g}_2)\right)\right\|_{L_2[0,1]}\le\tilde{N}_{2,3},\ 0<\tilde{N}_{2,3}=\text{const}$,
3) $\left|\tilde{f}_{2,p}\left(x,\tilde{h}_2^1(t)\right)-\tilde{f}_{2,p}\left(x,\tilde{h}_2^2(t)\right)\right|\le\tilde{N}_{2,4}(x)\left|\tilde{h}_2^1(t)-\tilde{h}_2^2(t)\right|,\ \ 0<\left\|\tilde{N}_{2,4}(x)\right\|_{L_2[0,1]}$,
4) $\left|\tilde{h}_2\left(t,\tilde{g}_2^1(t)\right)-\tilde{h}_2\left(t,\tilde{g}_2^2(t)\right)\right|\le\tilde{N}_{2,5}\left|\tilde{g}_2^1(t)-g_1^2(t)\right|,\ \ 0<\tilde{N}_{2,5}=\text{const}$,
5) $\tilde{\rho}_{2,2}=\tilde{C}_{2,2}\tilde{N}_{2,5}\left\|\tilde{N}_{2,4}(x)\right\|_{L_2[0,1]}<1,\ \ \tilde{C}_{2,2}=\text{const}$.

Then the nonlinear Fredholm integral equation (7.12) has a unique solution in the class of continuous functions $\tilde{g}_2(t)\in C[0,T]$.

Proof. of the Theorems 7.2 and 7.3 are similar to the proof of the Theorems 6.2 and 6.3.

Substituting the solution of Eq. (7.12) into (7.11), we determine the control function $\bar{\tilde{p}}_2(t)$. Substituting the control function $\bar{\tilde{p}}_2(t)$ into and we obtain redefinition function (7.5) and state function (7.6), respectively (see, [24, 25, 50, 51]).

By similar way, can be study the CS of fractional differential equation (4.6) with final condition (4.10). The solution of the CS of fractional differential equation (4.7) with final condition (4.11) we denote by $\tilde{F}_{2,m}(t)$. Then for the solution of (4.6) we have the presentation

$$t^{1-\gamma}u_{2,m}(t) = \varphi_{2,m}\sigma_{2,0,m}(t) - \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int\limits_0^t t^{1-\gamma}K_{2,m}(t,s)\left(D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\right)ds$$
$$+\frac{1}{1+\lambda_{2,m}}\int\limits_0^T t^{1-\gamma}K_{2,m}(t,s)\,f_{2,m}(p_2(s))ds, \tag{7.13}$$

where

$$K_{2,m}(t,s) = \begin{cases} -\sigma_{2,m}(t)(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(T-s)^{\alpha}\right), & t \le s \le T, \\ -\sigma_{2,m}(t)(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(T-s)^{\alpha}\right)+(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(t-s)^{\alpha}\right), & s < t, \end{cases}$$

$$\sigma_{2,m}(t) = \frac{E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}\left[\frac{t}{T}\right]^{\gamma-1}, \quad \sigma_{2,0,m}(t) = \frac{E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_{2,m}T^{\alpha}\right)}T^{1-\gamma},$$

$$\mu_{2,m} = \frac{\lambda_{2,m}}{1+\lambda_{2,m}}, \quad \lambda_{2,m} = (2qm\pi)^2, \quad m = 1,\, 2,\, \ldots$$

□

8. Building the optimal process and calculating the minimal values of functional

According to formulas (4.1), (4.2), (5.21), (6.27), and (7.7), the minimum value of the functional is found from the following formula

$$J[\bar{p}] = \int\limits_0^1 \left\{ \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0(x) - T^{\gamma-1}\vartheta_0(x)\int\limits_0^T K_0(t_1,s)\int\limits_0^1 f_0\left(y,\bar{p}_0(s)\right)\omega_0(y)dy\,ds - \xi_0(x) \right.$$
$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left\{\frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}\left[\psi_{1,n} - \frac{1}{1+\lambda_{1,n}}\int\limits_0^T K_{1,n}(t_1,s)\int\limits_0^1 f_1\left(y,\bar{p}_1(s)\right)\omega_{1,n}(y)dyds\right] - \xi_{1,n}\right\}$$
$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\left\{\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\left[\psi_{2,m} + \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int\limits_0^{t_1} K_{2,m}(t_1,s)\left[D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\right]ds\right.\right.$$
$$\left.\left.-\frac{1}{1+\lambda_{2,m}}\int\limits_0^T K_{2,m}(t_1,s)\int\limits_0^1 f_2\left(y,\bar{p}_2(s)\right)\tilde{\omega}_{2,m}(y)\,dy\,ds\right] - \xi_{2,m}\right\}$$
$$\left.+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)dx\left\{\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\left[\tilde{\psi}_{2,m} - \frac{1}{1+\lambda_{2,m}}\int\limits_0^T \tilde{K}_{2,m}(t_1,s)\int\limits_0^1 \tilde{f}_2\left(y,\bar{\tilde{p}}_2(s)\right)\omega_{2,m}(y)\,dy\,ds\right] - \tilde{\xi}_{2,m}\right\}\right\}^2$$
$$+\alpha\int\limits_0^T\left[\bar{p}_0(t)+\bar{p}_1(t)+\bar{p}_2(t)+\bar{\tilde{p}}_2(t)\right]^2dt. \tag{8.1}$$

Theorem 8.1. *Let the conditions of Theorems 5.1–7.3 be satisfied. Then functional (8.1) takes a finite value.*

Proof. The *proof* of the Theorem 8.1 is similar to the proof of estimates (5.6) and (6.13). □

According to (8.1), the approximate value of the functional is calculated from the following iterative process

$$J[\bar{p}^k] = \int_0^1 \Bigg\{ \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0(x) - T^{\gamma-1}\vartheta_0(x)\int_0^T K_0(t_1,s)\int_0^1 f_0\left(y,\bar{p}_0^k(s)\right)\omega_0(y)dy\,ds - \xi_0(x) +$$
$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left\{\frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}\left[\psi_{1,n} - \frac{1}{1+\lambda_{1,n}}\int_0^T K_{1,n}(t_1,s)\int_0^1 f_1\left(y,\bar{p}_1^k(s)\right)\omega_{1,n}(y)dyds\right] - \xi_{1,n}\right\}$$
$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\Bigg\{\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\Bigg[\psi_{2,m} + \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int_0^{t_1} K_{2,m}(t_1,s)\left[D^{\alpha,\gamma}\tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s)\right]ds$$
$$-\frac{1}{1+\lambda_{2,m}}\int_0^T K_{2,m}(t_1,s)\int_0^1 f_2\left(y,\bar{p}_2^k(s)\right)\tilde{\omega}_{2,m}(y)\,dy\,ds\Bigg] - \xi_{2,m}\Bigg\}$$
$$+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)dx\left\{\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\left[\tilde{\psi}_{2,m} - \frac{1}{1+\lambda_{2,m}}\int_0^T \tilde{K}_{2,m}(t_1,s)\int_0^1 \tilde{f}_2\left(y,\bar{\tilde{p}}_2^k(s)\right)\omega_{2,m}(y)dyds\right] - \tilde{\xi}_{2,m}\right\}\Bigg\}^2$$
$$+\alpha\int_0^T\left[\bar{p}_0^k(t) + \bar{p}_1^k(t) + \bar{p}_2^k(t) + \bar{\tilde{p}}_2^k(t)\right]^2 dt. \tag{8.2}$$

According to (3.11), (5.5), (6.12), (7.5) the redefinition function is determined as follows

$$\bar{\varphi}(x) = \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0(x) - T^{\gamma-1}\vartheta_0(x)\int_0^T K_0(t_1,s)\int_0^1 f_0\left(y,\bar{p}_0(s)\right)\omega_0(y)dy\,ds$$
$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}\left\{\psi_{1,n} - \frac{1}{1+\lambda_{1,n}}\int_0^T K_{1,n}(t_1,s)\int_0^1 f_1\left(y,\bar{p}_1(s)\right)\omega_{1,n}(y)\,dy\,ds\right\}$$
$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\Bigg\{\psi_{2,m} + \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int_0^{t_1} K_{2,m}(t_1,s)\left[D^{\alpha,\gamma}\tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s)\right]ds$$
$$-\frac{1}{1+\lambda_{2,m}}\int_0^T K_{2,m}(t_1,s)\int_0^1 f_2\left(y,\bar{p}_2(s)\right)\tilde{\omega}_{2,m}(y)\,dy\,ds\Bigg\}$$
$$+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\left\{\tilde{\psi}_{2,m} - \frac{1}{1+\lambda_{2,m}}\int_0^T \tilde{K}_{2,m}(t_1,s)\int_0^1 \tilde{f}_2\left(y,\bar{\tilde{p}}_2(s)\right)\omega_{2,m}(y)dyds\right\}. \tag{8.3}$$

The redefinition function (8.3) can be approximately found using the iterative process

$$\bar{\varphi}^k(x) = \frac{t_1^{1-\gamma}}{T^{1-\gamma}}\psi_0(x) - T^{\gamma-1}\vartheta_0(x)\int_0^T K_0(t_1,s)\int_0^1 f_0\left(y,\bar{p}_0^k(s)\right)\omega_0(y)dyds$$
$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\frac{t_1^{1-\gamma}}{\sigma_{1,0,n}(t_1)}\left\{\psi_{1,n} - \frac{1}{1+\lambda_{1,n}}\int_0^T K_{1,n}(t_1,s)\int_0^1 f_1\left(y,\bar{p}_1^k(s)\right)\omega_{1,n}(y)dyds\right\}$$
$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\Bigg\{\psi_{2,m} + \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int_0^{t_1} K_{2,m}(t_1,s)\left[D^{\alpha,\gamma}\tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s)\right]ds$$

$$
-\frac{1}{1+\lambda_{2,m}}\int\limits_0^T K_{2,m}(t_1,s)\int\limits_0^1 f_2\left(y,\bar{p}_2^k(s)\right)\tilde{\omega}_{2,m}(y)dyds\Bigg\}
$$

$$
+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\frac{t_1^{1-\gamma}}{\sigma_{2,0,m}(t_1)}\left\{\tilde{\psi}_{2,m}-\frac{1}{1+\lambda_{2,m}}\int\limits_0^T \tilde{K}_{2,m}(t_1,s)\int\limits_0^1 \tilde{f}_2\left(y,\bar{\bar{p}}_2^k(s)\right)\omega_{2,m}(y)dyds\right\}. \tag{8.4}
$$

According to relations (3.1), (5.4), (6.9), (7.2), and (7.13) the optimal process is determined by the following formula

$$
\begin{aligned}
t^{1-\gamma}\bar{U}(t,x) &= T^{1-\gamma}\bar{\varphi}_0(x)+\vartheta_0(x)\int\limits_0^T K_0(t,s)\int\limits_0^1 f_0\left(y,\bar{p}_0(s)\right)\omega_0(y)dy\,ds \\
&+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left[\bar{\varphi}_{1,n}\sigma_{1,0,n}(t)+\frac{1}{1+\lambda_{1,n}}\int\limits_0^T t^{1-\gamma}K_{1,n}(t,s)\,f_{1,n}\left(y,\bar{p}_1(s)\right)ds\right] \\
&+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\left[\bar{\varphi}_{2,m}\sigma_{2,0,m}(t)-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int\limits_0^t t^{1-\gamma}K_{2,m}(t,s)\left(D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\right)ds\right. \\
&\left.+\frac{1}{1+\lambda_{2,m}}\int\limits_0^T t^{1-\gamma}K_{2,m}(t,s)f_{2,m}(\bar{p}_2(s))ds\right] \\
&+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\left[\bar{\tilde{\varphi}}_{2,m}\tilde{\sigma}_{2,0,m}(t)+\frac{1}{1+\lambda_{2,m}}\int\limits_0^T t^{1-\gamma}\tilde{K}_{2,m}(t,s)\tilde{f}_{2,m}(\bar{\bar{p}}_2(s))ds\right].
\end{aligned} \tag{8.5}
$$

The optimal process (8.5) can be approximately found using the iterative process

$$
\begin{aligned}
t^{1-\gamma}\bar{U}^k(t,x) &= T^{1-\gamma}\bar{\varphi}_0(x)+\vartheta_0(x)\int\limits_0^T K_0(t,s)\int\limits_0^1 f_0\left(y,\bar{p}_0^k(s)\right)\omega_0(y)dy\,ds \\
&+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\left[\bar{\varphi}_{1,n}\sigma_{1,0,n}(t)+\frac{1}{1+\lambda_{1,n}}\int\limits_0^T t^{1-\gamma}K_{1,n}(t,s)\,f_{1,n}\left(y,\bar{p}_1^k(s)\right)ds\right] \\
&+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\left[\bar{\varphi}_{2,m}\sigma_{2,0,m}(t)-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int\limits_0^t t^{1-\gamma}K_{2,m}(t,s)\left(D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\right)ds\right. \\
&\left.+\frac{1}{1+\lambda_{2,m}}\int\limits_0^T t^{1-\gamma}K_{2,m}(t,s)f_{2,m}(\bar{p}_2^k(s))ds\right] \\
&+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\left[\bar{\tilde{\varphi}}_{2,m}\tilde{\sigma}_{2,0,m}(t)+\frac{1}{1+\lambda_{2,m}}\int\limits_0^T t^{1-\gamma}\tilde{K}_{2,m}(t,s)\tilde{f}_{2,m}(\bar{\bar{p}}_2^k(s))ds\right].
\end{aligned} \tag{8.6}
$$

9. Conclusion

In this paper, a methodology is developed for solving a nonlinear optimal control problem associated with a final-time inverse problem for the Barenblatt–Zheltov–Kochina equation involving the Hilfer fractional operator under boundary value conditions.

The spectral method combined with separation of variables is employed. The eigenvalues, eigenfunctions, and associated functions of both the spectral and adjoint problems are determined, leading to countable systems of fractional-order differential equations.

Using the maximum principle, necessary optimality conditions for the control function are formulated under a quadratic performance criterion. The optimal control function is uniquely determined from a system of nonlinear integral equations via the method of successive approximations. Explicit representations are derived for the redistribution function, the optimal control function, and the corresponding state function.

Iterative schemes are proposed for the approximate computation of the optimal process, the redistribution function, and the minimum value of the cost functional. In particular, the iterative procedures (5.26), (5.32), (6.32), (6.38), (8.2), (8.4), and (8.6) are established.

The results obtained contribute to the further development of the mathematical theory of nonlinear optimal control for systems governed by partial differential equations.

Funding

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, 2024-2026).

Conflicts of interest

The authors of this work declare that they have no conflicts of interest.

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