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The behaviour of solutions boundary problem to nonlinear elliptic equations

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Abstract

We obtained estimates in generalized Morrey spaces are used to study global regularity of the solution of the Neumann boundary problem of nonlinear elliptic equations in divergence form over a bounded non-smooth domain. For these is we used of Calderon-Zygmund theory.
The investigation of nonlinear reaction, drift, diffusion processes has received many attention. This questions arise as mathematical models of different applied problems. For example, for instance diffusion processes of electrically charged species phase transition and transport in porons media.
The main goal of the article is to obtain Hölder estimates for solutions to the Neumann problem for nonlinear elliptic equations.

Keywords. Generalized Morrey spaces, Nonlinear elliptic equation, Calderon-Zygmund theory, BMO space, Reifenberg flat domain, Global regularity.

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1. Introduction

The main goal of the article is to obtain Hölder estimates for solutions to the Neumann problem for nonlinear elliptic equations. In order to study the local behavior of solutions to elliptic and parabolic partial differential equations introduced in [26] the classical Morrey spaces $L_{p,\lambda}$. There are the inclusion between the Morrey and the Hölder spaces permits to obtain regularity of the solutions to elliptic and parabolic boundary problems. In [5] Chiarenza and Frasca show boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(R^n)$ that allows them to prove continuity of fractional and classical Calderon-Zygmund operators in Morrey spaces. Calderon-Zygmund operators appear in the representation formulae of the solutions of elliptic and parabolic equations. Thus the continuity of Calderon-Zygmund integrals implies regularity of the solutions in the corresponding spaces. In [25] Mizuhara gives a generalization of Morrey spaces considering a function $\omega(x,r) : R^n \times R_+ \to R_+$ instead of r^λ. He studies continuity in this spaces of some integral operators. In [27] Nakai extends the result of Chiarenza and Frasca in these type spaces imposing certain integral and doubling conditions on ω. In work [11] Guliev studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators.

Later this result is extended on spaces with weaker condition on the pair (φ_1, φ_2) (see [19]). For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and fractional spaces and their application in the different order partial differential equations theory see [2, 7, 8, 14, 18, 21, 23, 29].

In this paper we consider Neumann boundary problem for nonlinear elliptic equation in divergence form in a bounded non-smooth domain in generalized Morrey spaces. The problem for nondivergence second order linear elliptic equations with VMO coefficients in [21], for higher order linear elliptic equations in [14] considered.

We also recall papers [6, 7, 10, 12, 13, 15, 17].

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2. Preliminaries on generalized Morrey space

Let $\varphi : \Omega \times R_+ \to R_+$ be a measurable function. A function $f \in L_p(\Omega)$, $1 < p < \infty$, belongs to the generalized Morrey space $M_{p,\varphi}(\Omega)$ if the following norm is finite

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega,\, r>0} \frac{1}{\varphi(x,r)} \Big(\frac{1}{r^n}\int_{\Omega_r} |f(y)|^p dy\Big)^{1/p} < \infty.$$

The generalized Sobolev-Morrey space $W^1_{p,\varphi}(\Omega)$ consists of all functions $u \in W^1_p(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W^1_{p,\varphi}(\Omega)} = \sum_{0\le|s|\le1} \|D^s u\|_{M_{p,\varphi}(\Omega)}. \tag{2.1}$$

The space $W^1_{p,\varphi}(\Omega) \cap W^1_p(\Omega)$ consists of all functions $u \in W^1_p(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$, $0 \le |s| \le 1$ and is endowed by the same norm.

Let S^{n-1} is a unit sphere in R^n, $B_r(x) = B(x,r) = \{y \in R^n : |x-y| < r\}$, $B^c_r(x) = R^n \setminus B_r(x)$, $2B_r(x) = B(x,2r)$, $\Omega_r(x) = \Omega \cap B_r(x)$, $x \in \Omega$, $B^+_r(x_0) \equiv B^+(x_0,r) = B(x_0,r) \cap \{x_n > 0\}$, $2B^+_r = B^+(x_0,2r)$, where $x_0 = (x',0)$.

The investigation of nonlinear reaction, drift, diffusion processes has received many attention. This questions arise as mathematical models of different applied problems. For example, for instance diffusion processes of electrically charged species phase transition and transport in porons media.

Also we note for the semiconductor theory weight functions have derivative. Another example comes from phase separation problems. In this case usually used the Fermi integral and Fermi functions. This problems is connected with of nonlinear process which arise as mathematical models of different applied problems. The qualitative property of solution is investigated. The theorems obtained in this article have important applied significance in various issues of mechanics, physics, biology and other applied fields.

We give assumption about domain $\Omega \subset R^n$, $n \ge 2$. The measure the deviation of $\partial\Omega$ from being an $(n-1)$ -dimensional affine space at each scale $\rho > 0$, use the following so-called "Reifenberg flatness".

Definition 2.1. A bounded domain Ω is said to be (δ, R) -Reifenberg flat if for every $x \in \partial\Omega$ and every $\rho \in (0,R]$, there exists a coordinate system $\{y_1,\dots,y_n\}$, which can depend on ρ and x such that $x = 0$ in this coordinate system and that

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}, \tag{2.2}$$

where $B_\rho(y) = \{x \in R^n : |x-y| < \rho\}$ denotes the open ball on R^n centered $y \in R^n$ and radius $\rho > 0$. Later $|E|$ denotes the n-dimensional Lebesgue measure of a set $E \subset R^n$.

The above definition warrants a few comments. Because our the main problem has a scaling invariance property, the constant R can be taken as 1 or any other constant greater than 1. However, the constant δ is a small positive constant still invariant under such scaling. In fact, the Reifenberg flatness (2.2) is meaningful when $0 < \delta < \frac{1}{8}$, see [**?**], and with such small δ, these flatness conditions mean that the deviation of $\partial\Omega$ from being an $(n-1)$-dimensional affine space is small enough at each scale $\rho > 0$. By (2.2), the following measure density condition is obtained for all $y \in \Omega$ and $\rho \in (0,R)$

$$|\Omega \cap B_\rho(y)| \ge \left(\frac{1-\delta}{2}\right)^n |B_\rho(y)| \ge \left(\frac{3}{10}\right)^n |B_\rho(y)|.$$

We give the definitions of the functional spaces to which the coefficients and the data of the problem belong.

Let the nonlinearity $a = a(x,\xi) : R^n \to R^n \times R^n$ is measurable in x for all $\xi \in R^n$ and continuous in ξ for almost all $x \in R^n$. We give the regularity assumption on the nonlinearity $a(x,\xi)$. First set

$$\theta(a; B_\rho(y))(x) = \sup_{\xi\in R^n\ \{0\}} \frac{|a(x,\xi) - \bar a_{B_\rho(y)}(\xi)|}{|\xi|},$$

where

$$\bar{a}_{B_\rho(y)}(\xi) = \int\limits_{B_\rho(y)} a(x,\xi)dx = \frac{1}{|B_\rho(y)|}\int\limits_{B_\rho(y)} a(x,\xi)dx,$$

is the integral average of $a(x,\xi)$ in the variable x over $B_\rho(y)$ for fixed $\xi \in R^n$. The function $\theta(a; B_\rho(y))$ provides the measurement of the oscillation of $\frac{a(x,\xi)}{|\xi|}$ in the variable x over $B_\rho(y)$, uniformly in ξ.

Definition 2.2. A vector field a is said to be (δ, R)-vahishing if

$$\sup_{0<\rho\leq R}\sup_{y\in R^n} \fint\limits_{B_\rho(y)} \theta\left(a;\, B_\rho\left(y\right)\right)(x)\, dx \leq \delta.$$

3. Statement of the problem

Let Ω be a bounded domain in R^n, $n \geq 2$, with its non-smooth boundary $\partial\Omega$. We suppose domain Ω to be (δ, R)-Reifenberg flat. Let $f = f(x) : \Omega \to R^n$ be a given vector-valued function at least in $L^2(\Omega, R^n)$ and consider the following nonlinear elliptic equation

$$\begin{cases} div\, a(x, Du) = div\, f, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where the nonlinearity $a(x,\xi)$ as section 2. Here assume monotonicity and growth conditions on $a = a(x,\xi)$ as follows

$$\begin{cases} c_0|\xi-\eta|^p \leq [a(x,\xi) - a(x,\eta)], & \leq (\xi-\eta)^p \\ |a(x,\xi)| + |D_\xi a(x,\xi)||\xi|, & \leq c_1|\xi|^{p-2}, \end{cases} \tag{3.2}$$

for all $\xi, \eta \in R^n$ and for almost every $x \in R^n$ and c_0, c_1 some positive constants.

We consider a weak solution $W_p^1(\Omega)$, which means that the for any $\varphi \in W_p^1(\Omega)$ integral formula hold

$$\int\limits_\Omega a(x, Du)D\varphi dx = \int\limits_\Omega f D\varphi dx.$$

The existence and uniqueness of a weak solution to problem (3.1) can be obtained by the Minty-Browder method for monotone operators, see [20], under the assumption $f \in L^2(\Omega, R^n)$, with the estimate

$$\||Du|^2\|_{L^1(\Omega,R^n)} \leq C\||f|^2\|_{L^1(\Omega,R^n)},$$

the constant C is independent of u and f.

We are now ready to state the main result.

Theorem 3.1. *Let $p \in (1,\infty)$, $q_1 \in (p-1, n(p-1))$, $\frac{n}{q_1} - \frac{n}{q} = \frac{1}{p-1}$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q}}\, ess\sup_{s<\sigma<\infty} \varphi_1\left(\Omega_\sigma\left(y\right)\right)\sigma^{\frac{n}{q_1}} \leq C\varphi_2\left(\Omega_r\left(y\right)\right), \tag{3.3}$$

where C does not depend on y and r. Assume that (3.2) hold and u is a weak solution to problem (3.1). There is a small $\delta(n, \Lambda_1, \gamma) > 0$ such that if Ω is (δ, R_0) - Reifenberg flat and the nonlinearity $\mathbf{a}$ satisfies the small $(\delta, R_0) - BMO$ condition for some $R_0 > 0$ and $|f|^{\frac{1}{p-1}} \in M_{q_1,\varphi_1}(\Omega)$, then $Du \in M_{q,\varphi_2}(\Omega)$ with the estimate

$$\|Du\|_{M_{q,\varphi_2}(\Omega)} \leq C_1\, \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}, \tag{3.4}$$

the constant C_1 depending on n, q, Λ_1,γ, R_0, $|\Omega|$.

The boundedness results of maximal operators in generalized Morrey spaces is obtained in [1].

Lemma 3.2. *Assume that* $1 < q_1 < \infty$ *and the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q_1}} \operatorname{ess\,sup}_{s<\sigma<\infty} \varphi\left(\Omega_\sigma\left(x\right)\right) \sigma^{\frac{n}{q_1}} \leq C\varphi\left(\Omega_r\left(x\right)\right), \tag{3.5}$$

holds, where C *does not depend on* x *and* r. *Then there is a constant* $C_{q_1} > 0$ *such that*

$$\|f\|_{M_{q_1,\varphi}(R^n)} \leq \|Mf\|_{M_{q_1,\varphi}(R^n)} \leq C_q\|f\|_{M_{q_1,\varphi}(R^n)}, \; f \in M_{q_1,\varphi}(R^n).$$

4. Auxiliary results

We give the following invariance property under normalization and scaling, later some result witch is needed from measure theory in the generalized Morrey space and one version of the Calderon-Zygmund type covering lemma. This results we is used to prove the main theorem.

Lemma 4.1. *Let* $u \in W_2^1(\Omega)$ *be a weak solution to the problem* (3.1) *and nonlinearity* $a(x,\xi)$ *satisfies* (3.2) *and is* (δ, R) *-vanishing. For each* $\lambda > 1$ *and* $0 < r < 1$*, define the rescaled maps*

$$\tilde{a}(x,\xi) = \frac{a(rx,\lambda\xi)}{\lambda},$$
$$\widetilde{\Omega} = \left\{\frac{1}{r}x : x \in \Omega\right\}, \tilde{u}(x) = \frac{u(rx)}{\lambda r},$$
$$\tilde{f}(x) = \frac{f(rx)}{\lambda_0}.$$

Then

(1) $\tilde{u} \in W_2^1(\tilde{\Omega})$ *is the weak solution of*

$$div\, \tilde{a}(x, D\tilde{u}) = div\, \tilde{f} \;\text{ in }\; \tilde{\Omega},$$

(2) $\tilde{a}(x.\xi)$ *satisfies the conditions* (3.2) *with the same constants* c_0 *and* c_1,

(3) $\tilde{a}$ *is* $\left(\delta, \frac{R}{r}\right)$*-vanishing and* $\tilde{\Omega}$ *is* $\left(\delta, \frac{R}{r}\right)$*- Reifenberg flat.*

Proof. The proof follows by direct computations. Also for example see [4].

Now we give the Hardy-Littlewood maximal function and its basic properties. Let g be a locally integrable function on R^n. Then the Hardy-Littlewood maximal function is given by

$$(My)(x) = \sup_{\rho>0} \int_{B_\rho(x)} |g(y)|dy = \sup_{\rho>0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |g(y)|dy.$$

If g is defined only on a bounded subset of R^n, then we define as $Mg = M\bar{g}$, where $\bar{g}$ is the zero extension of g in R^n. This maximal function holds the so-called weak inequality. More specifically, there exists a positive constant $c = c(n)$ such that, for any $\lambda > 0$

$$|\{x \in R^n : (Mg)(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{R^n} |g(x)|dx. \tag{4.1}$$

□

We next state the Hardy-Littlewood maximal operators is bounded from generalized Morrey space $M_{p,\varphi}(R^n)$ to itself.

Lemma 4.2. *(see [16]) Assume that there is a positive constant* C *such that for any fixed* $x \in R^n$, $r > 0$ *it holds*

$$\sup_{r<s<\infty} \frac{\operatorname{ess\,inf}_{s<\sigma<\infty} \; \varphi\left(B_\sigma\left(x\right)\right) \sigma^{\frac{n}{p}}}{s^{\frac{n}{p}}} < C\varphi\left(B_r\left(x\right)\right). \tag{4.2}$$

Then there is a constant $C_p > 0$ *such that*

$$\|f\|_{M_{p,\varphi}(R^n)} \leq \|Mf\|_{M_{p,\varphi}(R^n)} \leq C_p\|f\|_{M_{p,\varphi}(R^n)},$$

$\forall f \in M_{p,\varphi}(R^n)$ with compact support in R^n.

We will use the following version of the Vitali covering lemma.

Lemma 4.3. *(see [9]) Assume that Ω is $(\delta,1)$-Reifenberg flat domain for some small $\delta > 0$. Let E and D be measurable sets with $E \subset D \subset \Omega$. Suppose that there exists small $\varepsilon > 0$ such that*

(1) *for any $y \in \Omega$ $\quad |E \cap B_1(y)| < \varepsilon |B_1(y)|$,*
(2) *for each $y \in \Omega$ and $r \in (0,1)$, if $|E \cap B_r(y)| \geq \varepsilon |B_r(y)|$, then $B_r(y) \cap \Omega \subset D$.*

Then

$$|E| \leq C \cdot \varepsilon |D|,$$

the constant C depending only on n, Φ, and the constant $\dfrac{1}{1-\delta}$.

Proof. The proof of this lemma also can be found in [3, Lemma 5.4] or [28, Lemma 3.4] with slight modification. ☐

We also use the following standard arguments of measure theory.

Lemma 4.4. *(see [16]) Let f is a nonnegative an measurable function defined on a bounded domain $\Omega \subset R^n$ and φ be a weight satisfying (4.2) and in addition a kind of monotonicity condition*

$$\varphi(B_r(y))r^n \leq \varphi(B_s(z))s^n \quad \text{for all} \quad B_r(y) \subset B_s(z), \tag{4.3}$$

and let $\theta > 0$, $\lambda > 1$ be constants. Then $f \in M_{p,\varphi}(\Omega)$ if and only if

$$S = \sup_{y \in \Omega, r > 0} \sum_{k \geq 1} \frac{\lambda^k |\{x \in \Omega : f(x) > \theta \lambda^k\}|}{\varphi(B_r(y))r^n} < \infty,$$

and

$$\frac{1}{C} S \leq \|f\|_{M_{p,\varphi}(\Omega)} \leq C(1+\delta),$$

where $C > 0$ is a constant depending only on θ, λ, φ.

Lemma 4.5. *Assume $u \in W_2^1(\Omega)$ be the weak solution of (3.1). Then there exists a constant $N = N(c_0, c_1, n) > 1$ such that for each $\varepsilon \in (0,1)$ fixed, one can select small $\delta = \delta(\varepsilon, c_0, c_1, n, \varphi) \in \left(0, \frac{1}{8}\right)$ such that if $a(x,\xi)$ is $(\delta,1)$-vanishing, Ω is $(\delta,1)$-Reifenberg flat, and if for $0 < r < 1$ and $y \in \Omega$, $B_r(y)$ satisfies*

$$|\{x \in \Omega : M(|Du|^2) > N^2\} \cap B_r(y)| \geq C|B_r(y)|.$$

Then we have

$$B_r(y) \cap \Omega \subset \{x \in \Omega : M(|Du|^2) > 1\} \cup \{x \in \Omega : M(|f|^2) > \delta^2\}.$$

Proof. Proof this lemma is based on the same method as in the proof in [24, Theorem 4.10]. ☐

5. Proof the main theorem

Now, we are ready to prove the main theorem.

Proof of Theorem 3.1. By to Lemma 4.1, it suffices to prove that

$$\||Du|^2\|_{M_{p,\varphi}(\Omega)} \leq C, \tag{5.1}$$

under the assumption $\||f|^2\|_{M_{p,\varphi}(\Omega)} \leq \delta^2$.

We take

$$u_1 = \frac{\delta u}{\sqrt{\||f|^2\|_{M_{p,\varphi}(\Omega)} + \sigma}}, \quad f_1 = \frac{\delta f}{\sqrt{\||f|^2\|_{M_{p,\varphi}(\Omega)} + \sigma}},$$

in place of u and f, respectively, in the problem (3.1) estimate

$$\frac{1}{c} \min\{\|g\|^{q_0}_{M_{p,\varphi}(\Omega)}, \|g\|^{q_1}_{M_{p,\varphi}(\Omega)}\} \leq \int_\Omega |g(x)| dx \leq c \max\{\|g\|^{q_0}_{M_{p,\varphi}(\Omega)}, \|g\|^{q_1}_{M_{p,\varphi}(\Omega)}\}. \tag{5.2}$$

This estimate follows from [4].

It follows from Lemma 4.2 and estimate (5.1) that

$$\||f_1|^2\|_{M_{p,\varphi}(\Omega)} \leq \delta^2, \int_\Omega |f_1|^2 dx \leq C\delta^{2\tau_2}, \tag{5.3}$$

where $\tau_2 = \frac{q_0}{q_1}$. Therefore if (5.1) is obtained with Du_1 instead of Du, then the proof is completed after letting $\sigma \to 0$. However, in view of [4] and Lemma 4.2,

$$\||Du|^2\|^\alpha_{M_{p,\varphi}(\Omega)} \leq C \int_\Omega |Du|^2 dx \leq C \int_\Omega (M(|Du|^2))dx,$$

for some $\alpha > 0$. Consequently, it suffices to show that, by Lemma 4.4,

$$S = \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(N^{2k})|\{x \in \Omega : M(|Du|^2) > N^{2k}\}|}{\varphi(B_r(y))r^n} < \infty.$$

Now we to derive the decay estimates of the measure of the upper-level set $\{x \in \Omega : M(|Du|^2) > N^{2k}\}$ for $k = 1, 2, \ldots$. To apply Lemma 4.3, first fix ε and take δ and N as given in Lemma 4.5. Then define the sets

$$E = \{x \in \Omega : M(|Du|^2) > N^2\},$$
$$D = \{x \in \Omega : M(|Du|^2) > 1\} \cup \{x \in \Omega : M(|f|^2) > \delta^2\}.$$

Check its hypotheses. It is clearly, $E \subset D \subset \Omega$, and for each $y \in \Omega$

$$\frac{|E \cap B_1(y)|}{|B_1(y)|} \leq C \left(\frac{|E \cap B_1(y)|}{|B_1(y)|}\right)^{\tau_1} \stackrel{def}{=} A_1,$$

for some constants $C > 1$ and $\tau_1 \in (0, 1)$ in case $E \subset B_1$. Here constants C and τ_1 depend only on n, p and thus not on E and B_1. Then

$$A_1 \leq C|E|^{\tau_1} \leq C \left(\int_\Omega |Du|^2 dx\right)^{\tau_1} \leq C \left(\int_\Omega |f|^2 dx\right)^{\tau_1} \leq C\delta^{2\tau_1\tau_2} < \varepsilon,$$

for δ small enough. Because the second condition of Lemma 4.3 is already checked in Lemma 4.5,

$$|\{x \in \Omega : M(|Du|^2) > N^{2k}\}| \leq C\varepsilon|\{x \in \Omega : M(|Du|^2) > 1\}| + C\varepsilon|\{x \in \Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|. \tag{5.4}$$

On the other hand the main problem (3.1) has the invariance property from normalization Lemma 4.1 an therefore the same result (5.4) may be obtained for $\left(\frac{u}{N}, \frac{f}{N}\right), \left(\frac{u}{N^2}, \frac{f}{N^2}\right), \ldots$, inductively.

After this iteration the following power decay estimates are obtained for $k = 1, 2, \ldots$

$$|\{x \in \Omega : M(|Du|^2) > N^{2k}\}| \leq \varepsilon_1^k|\{x \in \Omega : M(|Du|^2) > 1\}| + \sum_{i=1}^k \varepsilon_1^i|\{x \in \Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|,$$

where $\varepsilon_1 = C\varepsilon$. Then a direct computation yields

$$\begin{aligned} S &= \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(N^{2k})|\{x \in \Omega : M(|Du|^2) > N^{2k}\}|}{\varphi(B_r(y))r^n} \\ &\leq \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(N^{2k})\varepsilon_1^k|\{x \in \Omega : M(|Du|^2) > 1\}|}{\varphi(B_r(y))r^n} \\ &+ \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(N^{2k}) \sum_{i=1}^k \varepsilon_1^i|\{x \in \Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} = S_1 + S_2. \end{aligned}$$

There exists a constant ν_1, depending only on φ and N such that $N^2 \leq \nu_1$, and therefore for $k = 1, 2, \dots$

$$N^{2k} \leq \nu_1^k.$$

We estimate S_1 and S_2 as follows:

$$S_1 \leq \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{\nu_1^k \varepsilon_1^k |\Omega|)}{\varphi(B_r(y))r^n} \leq C \sum_{k\geq 1} (\nu_1\varepsilon_1)^k,$$

$$\begin{aligned}
S_2 &= \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(N^{2(k-i)}N^{2i}) \sum_{i=1}^{k} \varepsilon_1^i |\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\
&\leq \sup_{y\in\Omega, r>0} \sum_{i\geq 1}\sum_{k\geq i} \frac{(N^{2(k-i)})\nu_1^i\varepsilon_1^i |\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\
&\leq C \sum_{i\geq 1} (\nu_1\varepsilon_1)^i \sum_{k\geq i} \frac{(N^{2(k-i)})|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\
&\leq C \sum_{i\geq 1} (\nu_1\varepsilon_1)^i \sum_{j\geq 0} \frac{(N^{2j})|\{x\in\Omega : M(\left|\frac{f}{\delta}\right|^2) > N^{2j}\}|}{\varphi(B_r(y))r^n} \\
&\leq C \sum_{i\geq 1} (\nu_1\varepsilon_1)^i \int_\Omega \left(\left|\frac{f}{\delta}\right|^2\right) dx \\
&\leq C \sum_{i\geq 1} (\nu_1\varepsilon_1)^i \left\|\frac{|f|^2}{\delta^2}\right\|_{M_{p,\varphi}(\Omega)}^{q_0} \leq C \sum_{i\geq 1} (\nu_1\varepsilon_1)^i.
\end{aligned}$$

Therefore

$$S \leq C \sum_{k\geq 1} (\nu_1\varepsilon_1)^k$$

where $\varepsilon_1 = C\varepsilon$ as in Lemma 4.3. First take sufficiently small $\varepsilon > 0$ to get

$$\nu_1\varepsilon_1 < 1.$$

Then one can select correspondingly small $\delta = \delta(c_0, c_1, n, \varphi) > 0$ from Lemma 4.5. This completes the proof.

6. Conclusion

Thus we obtained estimates in generalized Morrey spaces are used to study global regularity of the solution of the Neumann boundary problem of nonlinear elliptic equations in divergence form over a bounded non-smooth domain.

This estimated are need for investigation of nonlinear reaction, drift, diffusion processes has received many attention. This questions arise as mathematical models of different applied problems.

Also is to obtain Hölder estimates for solutions to the Neumann problem for nonlinear elliptic equations.

References

[1] A. Akbulut, V. S. Guliyev, and R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem., *137*(1) (2012), 27-43.

[2] G. Bonanno, G. Molica Bisci, V. D. Radulescu, *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces*, Nonlinear Anal., *75*(12) (2012), 4441-4456.

[3] S. Byun and D. K. Palagachev, *Weighted Lp-estimates for elliptic equations with measurable coefficients in non-smooth domains*, Potential Anal., *41*(1) (2014), 51–79.

[4] S. Byun and L. Wang, *Elliptic equations with BMO nonlinearity in Reifenberg domains*, Adv. Math., *219*(6) (2008), 1937–1971.

[5] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat., *7* (1987), 273–279.

[6] F. Chiarenza, M. Frasca, and P. Longo, *$W^{2,p}$-solvability of Dirichlet problem for nondivergence ellipic equations with VMO coefficients*, Trans. Amer. Math. Soc., *336* (1993), 841-853.

[7] G. Di Fazio, D. K. Palagachev, and M. A. Ragusa, *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal., *166*(2) (1999), 179-196.

[8] D. Fan, S. Lu, and D. Yang, *Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients*, Georgian Math. J., *5* (1998), 425-440.

[9] H. Dong and D. Kim, *L_p solvability of divergence type parabolic and elliptic systems with partially BMO coefficients*, Calc. Var. Partial Differential Equations, *40*(3-4) (2011), 357–389.

[10] T. S. Gadjiev, *On behavior of solutions of mixed problems for quasilinear elliptic equations*, Differentsyal'nye Uravneniya, 1031-1036.

[11] T. S. Gadjiev, *On behavior of solutions of mixed problems for quasilinear elliptic equations*. Diff. Equ., (1991), 1031-1036.

[12] T. S. Gadjiev, *On solutions of mixed boundary-value problems for second order quasilinear elliptic type equations in arbitrary bounded domains*, Mat. Fizika i nelineynaya mekhanika, (1986).

[13] T. S. Gadjiev, S. Y. Aliyev, and S. Galandarova, *Apriori estimates for solutions to Dirichlet boundary value problems for poluharmonic equations in generalized Morrey spaces*, TWMS J. Pure Appl. Math, *3*(2) (2018), 232-243 .

[14] T. Gadjiev, S. Galandarova, and V. Guliyev, *Regularity in generalized Morrey spaces of solutions to higher order nondivergence elliptic equations with VMO coefficients*, Electron. J. Qual. Theory Differ. Equ., (55) (2019), 1-17.

[15] T. S. Gadjiev, V. S. Guliyev, and K. G. Suleymanova, *The Dirichlet problem for the uniformly elliptic equation in generalized weighted Morrey spaces*, Studia Scientiarum Mathematicarum Hungarica, *6* (1), 68-90.

[16] T. Gadjiev, M. Kerimova, and G. Gasanova, *Solvability of a boundary-value problem for degenerate equations*, Ukr. Math. J., *72*(4) (2020), 435-451.

[17] T. S. Gadjiev and K. Mamedova *The behavior of solutions of nonlinear degenerate parabolic equations in nonregular domains and removability of singularity on boundary*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. Mathematics, *40*(4) (2020), 66-74.

[18] T. Gadjiev and S. N. Mammadova, *Behaviour of solutions of parabolic equations*, Advances in Applied Mathematical Analysis, *3*(1) (2008), 27-40.

[19] T. S. Gadjiev, Y. Rustamov, and A. Yangaliyeva, *The Behaviour of Solutions to Degenerate Double Nonlinear Parabolic Equations*, Proceedings of the Fourteenth International Conference on Management Science and Engineering Management. ICMSEM 2020. Advances in Intelligent Systems and Computing, vol 1190. Springer, Cham. (1190), (2020), 447-459.

[20] G. Gilbert and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed. Springer-Verlag, 1983.

[21] V. Guliyev, T. Gadjiev, and S. Galandarova, *Dirichlet boundary value problems for uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces*, Electron. J. Qual. Theory Differ. Equ., *71* (2017), 1-17.

[22] V. S. Guliyev, T. S. Gadjiev, and A. Serbetci, *The Dirichlet problem for the uniformly higher-order elliptic equations in generalized weighted Sobolev-Morrey spaces*, Adv. arXiv preprint arXiv:1911.01653, *6*.

[23] V. S. Guliyev and L. Softova, *Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*, Potential Anal., *38*(4) (2013), 843-862.

[24] T. Mengesha and N. C. Phuc, *Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains*, J. Differential Equations, *250*(5) (2011), 2485–2507.

[25] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Anal., Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., (1991), 183–189.

[26] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., *43* (1938), 126–166.

[27] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr., *166* (1994), 95–103.

[28] N. C. Phuc, *Nonlinear Muckenhoupt-Wheeden type bounds on Reifenberg flat domains, with applications to quasi-linear Riccati type equations*, Adv. Math., *250* (2014), 387–419.

[29] L. G. Softova, *The Dirichlet problem for elliptic equations with VMO coefficients in generalized Morrey spaces*, Advances in harmonic analysis and operator theory,, Oper. Theory Adv. Appl., 229, Birkhäuser/Springer Basel AG, Basel, (2013), 371-386.