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Investigation of Optical Solitons in a Weakly Nonlocal Schrdinger Equation with Parabolic Nonlinearity

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Abstract

The weakly nonlinear Schrdinger equation (NLSE) describes wave phenomena in media characterized by weakly nonlinear dispersion. Widely applicable across diverse fields such as plasma waves, water waves, fiber optics, and Bose-Einstein condensates, the NLSE serves as a versatile model. This study focuses on investigating various solutions for the weakly nonlocal NLSE with parabolic law nonlinearity. Utilizing the Nucci reduction method (NRM), exact solutions, including dark and bright solitons and other traveling wave solutions, are extracted. This method proves valuable for identifying nonlocal symmetries of differential equations, serving as an efficient mathematical tool for nonlinear problem-solving in engineering and related domains. Furthermore, a first integral for the model is derived through the reduction method. These findings are crucial for understanding soliton wave propagation in weakly nonlocal media with parabolic law nonlinearity, providing insights into wave dynamics for the proposed model. Finally, density 2D and 3D plots are presented to visually depict the physical behavior of some obtained solutions within the governing model.

Keywords. Schrödinger model, Parabolic law, First integral, Soliton solution, Nucci method.

2010 Mathematics Subject Classification. 35C07, 35C08, 35C09.

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1. Introduction

Partial Differential Equations (PDEs) serve as powerful mathematical tools for modeling and understanding diverse phenomena in science and engineering. These equations involve multiple variables and their partial derivatives, making them well-suited for describing complex physical processes such as heat conduction, fluid dynamics, and electromagnetic fields. The investigation of PDEs encompasses various aspects, including analytical and numerical methods for solving these equations. Analytically, researchers explore techniques like separation of variables, integral transforms, and series solutions to obtain exact solutions or gain insights into the behavior of solutions. On the other hand, numerical methods, such as finite difference, finite element, and spectral methods, provide computational tools to approximate solutions for PDEs in cases where analytical solutions are challenging or impossible to obtain. Additionally, the study of stability, existence, and uniqueness of solutions, as well as the development of new solution methods, continues to be a vibrant area of research within the broader field of partial differential equations. Understanding and harnessing the mathematical intricacies of PDEs play a crucial role in advancing our comprehension of the natural world and technological applications.

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Numerous scholars have focused their efforts on determining exact solutions for NPDEs, with these equations finding applications across diverse scientific and technological domains, including but not limited to mathematical physics, fluid dynamics, optical fibers, and economics [10, 14, 27].

In recent years, various methodologies have been developed for determining precise solutions to NPDEs. These approaches include the Lie symmetry method [16, 18, 20, 26, 30], the Kudryashov method [21, 22, 33], the sine-Gordon expansion method [6], the invariant subspace method [13, 28], the Sardar subequation method [3, 11], and others [1, 5, 8, 9, 29]. These techniques contribute to the exploration of exact solutions for a wide range of PDEs.

The NLSE is a complex PDE that plays a crucial role in various fields of physics and engineering, describing the evolution of complex wave packets. It arises in different contexts, including nonlinear optics, plasma physics, fluid dynamics, and condensed matter physics. The NLSE and its variants have significant importance in understanding and predicting the behavior of wave-like phenomena in diverse real-world applications.

• NLSE in Nonlinear Optics:

In nonlinear optics, the NLSE governs the propagation of intense laser beams through nonlinear media. This equation describes the interactions among optical waves, considering effects such as self-focusing, self-phase modulation, and optical solitons. Optical solitons, which are stable, localized wave packets that can maintain their shape during propagation, find applications in long-distance communication systems. The NLSE helps optimize and control these phenomena in the design of optical communication systems and laser technologies.

• Bose-Einstein Condensates (BECs):

The NLSE also appears in the study of ultra-cold atomic gases, particularly in the context of Bose-Einstein condensates. In this scenario, the NLSE describes the dynamics of the macroscopic wave function of the condensate. Understanding the NLSE for BECs is crucial for investigating phenomena such as matter-wave solitons and vortices, which have applications in precision measurements and quantum information processing.

• Plasma Physics:

The NLSE arises in the study of Langmuir waves in plasmas, where it describes the evolution of the electron plasma wave. Nonlinear effects become significant in high-intensity laser-plasma interactions and can lead to the generation of harmonics and other phenomena. This has applications in areas such as controlled nuclear fusion research and the development of high-power particle accelerators.

• Fiber Optics and Communication:

NLSE is fundamental in understanding the behavior of optical pulses in fiber optic communication systems. Fiber optic channels exhibit nonlinear effects such as self-phase modulation and cross-phase modulation, which can distort transmitted signals. The NLSE is essential for modeling and mitigating these effects, ensuring the reliability and efficiency of long-distance communication networks.

• Water Waves and Oceanography:

Variants of the NLSE are used to model the propagation of water waves in oceans and other bodies of water. Nonlinear effects, such as wave steepening and wave breaking, can be described using these equations. Understanding these phenomena is crucial for predicting and mitigating the impact of tsunamis, storm surges, and other oceanic events.

• Biophysics: NLSE variants are employed in the study of biological systems, such as modeling the propagation of nerve impulses. The NLSE can describe the nonlinear dynamics of excitable media, providing insights into the behavior of electrical signals in biological tissues.

Moreover, the NLSE [4, 12, 23], which is an essential fully-integrated nonlinear dispersive partial differential equation (PDE), has found extensive application in elucidating diverse phenomena like deep water waves, rogue waves, plasmas, and nonlinear optics, including atomic physics. The NLSE serves as a comprehensive description of nonlinear dispersive processes. Zhou et al. [34] considered the weakly nonlocal NLSE having PL nonlinearity with external potential as

$$i\frac{\partial\psi}{\partial t}+\lambda_1\frac{\partial^2\psi}{\partial x^2}+\left(\lambda_2\frac{\partial^2|\psi|^2}{\partial x^2}+\lambda_3|\psi|^2+\lambda_4|\psi|^4\right)\psi+\lambda_5\psi=0, \tag{1.1}$$

and acquired its exact solutions by the robust and direct methods. Our main aim of the present study is to review the above weakly nonlocal NLSE having PL nonlinearity and creating its soliton solutions.

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A lot of researchers investigated this model such as Hadi Rezazadeh *et. al.* [7] constructed singular-1 and dark soliton by using simple equation method. With different techniques, Kamyar *et. al.* [19] explored the bright and dark solution of nonlocal NLSE with PL nonlinearity. Yue Kang *et. al.* [32] proposed the discrimination system to achieve the various types of soliton such as kink soliton, triangular soliton, and bright soliton. In the current study, NRM [25] is applied to enumerate the different solutions that comprise of periodic, exponential, dark and bright solutions. This method, initially introduced in [25], was developed to find nonlocal symmetries of differential equations. The applied method is powerful and efficient for finding the solutions of NPDEs in different fields and obtained results could be useful to explain the propagation of optical solitons in parabolic medium.

The study of Schrdinger equations with nonlinearity is an important area of research in mathematical physics. In recent years, the weakly nonlocal Schrdinger equation with parabolic law nonlinearity has gained significant attention due to its numerous applications in various fields such as quantum mechanics, nonlinear optics, and fluid dynamics. In this paper, we present novel exact solutions for this weakly nonlocal Schrdinger equation with parabolic law nonlinearity, derived using a NRM. The solutions we obtain, which include dark soliton, bright soliton, and traveling wave solutions, have not been previously reported in the literature and demonstrate the complex dynamics of the considered equation. Graphical descriptions, reveal the dynamical behavior of these solutions, underscoring their uniqueness compared to other existing techniques.

Our findings have significant implications for the study of nonlinear phenomena in physical systems. The ability to obtain exact solutions to such complex equations is crucial in understanding the underlying physics and designing new experiments. Additionally, using Maple, our results are verified by back-substitution in the original equations, affirming the robustness of our approach. The suggested NRM is not only direct and simple but also suitable for constructing new results, paving the way for future applications on NLSEs with dual power law and perturbed NLSEs with Kerr law. This is particularly useful for identifying solitons in photorefractive and polymer materials. The proposed technique holds potential for further applications in natural science models, aiding in the investigation of other mathematical challenges and characterizing the behavior of nonlinear models.

The remaining sections of this manuscript are organized as follows: In Section 2, we delve into the fundamental concept of the governing equation and the resulting reduced ordinary differential equation (ODE). Section 3 explores the application of NRM to the model. The outcomes, discussions, and a physical interpretation related to the weakly nonlinear Schrdinger equation (NLSE) are presented in Section 4. Finally, Section 5 encompasses the conclusion of this study.

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2. Mathematical Analysis

Transforming the Schrödinger equation is a key method for reducing it to a set of ordinary differential equations (ODEs). This simplification process is important as it makes the equation easier to solve and analyze. These transformations can greatly aid in solving the Schrödinger equation and provide deeper insight into the behavior of quantum systems. Begin by employing the subsequent complex transformation:

$$\psi(x,t) = \Psi(\zeta)e^{i(\kappa x+\mu t)}, \qquad \zeta = x + vt, \tag{2.1}$$

where κ, v and μ indicate wave number, speed and frequency respectively. By using the above transformation, the WNSE having parabolic law nonlinearity is reduced in the following equation

$$\lambda_1 \frac{d^2\Psi(\zeta)}{d\zeta^2} - \left(\kappa^2\lambda_1 + \mu\right)\Psi(\zeta) + 2\lambda_2 \frac{d^2\Psi(\zeta)}{d\zeta^2}\Psi^2(\zeta) + 2\lambda_2\left(\frac{d\Psi(\zeta)}{d\zeta}\right)^2 \Psi(\zeta) + \lambda_3\Psi^3(\zeta) + \lambda_4\Psi^5(\zeta) = 0, \tag{2.2}$$

where $v = -2\kappa\lambda_1$ the shows the speed of wave. Analytical methods for solving ODEs play a fundamental role in mathematics and its applications. These methods involve finding exact solutions to differential equations without relying on numerical approximations. One of the most common techniques is separation of variables, which involves isolating the dependent and independent variables on opposite sides of the equation and then integrating both sides. Another widely used method is the method of integrating factors, particularly useful for linear first-order ODEs. It involves multiplying both sides of the equation by a carefully chosen integrating factor to simplify the integration process. Additionally, there are techniques like variation of parameters, where a particular solution is sought as a linear combination of known solutions to a related homogeneous equation. These special functions provide solutions that arise in specific physical and mathematical contexts. Overall, analytical methods for solving ODEs allow us to obtain exact solutions, leading to a deeper understanding of the underlying phenomena and facilitating further analysis and applications. In the upcoming section, our aim is to explore the first integral and solutions of equation (2.2) using NRM.

3. Nucci's reduction method

Autonomous systems of ODEs can be rephrased as systems of first-order ODEs by choosing one dependent variable as the new independent variable. During this transformation, specific variables are eliminated to form a combination of first and second-order equations. The detection of point symmetries for this mixed system can be automated without the need for a detailed assumption about the structure of the symmetry. Since the coefficient function associated with the original independent variable only appears as its derivative in the simplified system, nonlocal symmetries for this variable are converted into local symmetries for the reduced system, and their calculation can be performed using an algorithm. This approach is commonly known as NRM.

In this study, new exact solutions and corresponding first integrals for Eq. (2.2) are determined using NRM. The NRM was recently applied to study different types of differential equations in [15, 17, 24]. All calculations are extracted by using the software Maple. To apply the proposed reduction method, a change of variables, as described in [2, 31], is assumed:

$$\Phi_1(\zeta) = \Psi(\zeta), \qquad \Phi_2(\zeta) = \Psi'(\zeta). \tag{3.1}$$

It is well-known that, an ODE can be rewritten as an equivalent dynamical system, which is a more general framework for analyzing the system's behavior. The equivalent dynamical system represents the evolution of the system through time by specifying a set of state variables and a set of equations that describe how those variables change over time. By reformulating the ODE in this way, we can analyze the stability, equilibrium, and bifurcations of the system, gaining a more complete understanding of its behavior. Additionally, the equivalent dynamical system allows us to apply a range of mathematical and computational tools to study the system's properties, making it a valuable tool in many areas of science and engineering.

Therefore, by using the assumptions (3.1), Eq. (2.2) can be transformed into the following dynamical system:

$$\begin{cases} \Phi_1' = \Phi_2, \\ \Phi_2' = \dfrac{\left(\kappa^2\lambda_1 + \mu\right)\Phi_1 - 2\,\lambda_2\Phi_1\Phi_2^2 - \lambda_3\Phi_1^3 - \lambda_4\Phi_1^5}{2\,\lambda_2\Phi_1^2 + \lambda_1}. \end{cases} \tag{3.2}$$

If we select Φ_1 as the new independent variable, the system (3.2) transforms into the following form:

$$\frac{d\Phi_2(\Phi_1)}{d\Phi_1} = \frac{\left(\kappa^2\lambda_1 + \mu\right)\Phi_1 - 2\,\lambda_2\Phi_1\left(\Phi_2\left(\Phi_1\right)\right)^2 - \lambda_3\Phi_1^3 - \lambda_4\Phi_1^5}{\left(2\,\lambda_2\Phi_1^2 + \lambda_1\right)\Phi_2\left(\Phi_1\right)}. \tag{3.3}$$

We adopt two different approaches to address Eq. (3.3). Initially, Maple furnishes an implicit solution, supplemented by a few specific explicit solutions. Subsequently, we derive the solutions for Eq. (3.3) in specific formats, allowing us to proceed with our methodology in the ensuing scenarios:

Case 1.

In this instance, we employ the Maple software directly to integrate Eq. (3.3). The exact solution can be articulated as follows:

$$\Phi_2(\Phi_1) = -\frac{\sqrt{6\left(2\,\Phi_1^2\lambda_2 + \lambda_1\right)\left(-2\,\Phi_1^6\lambda_4 + 6\,\kappa^2\Phi_1^2\lambda_1 - 3\,\Phi_1^4\lambda_3 + 6\,\mu\,\Phi_1^2 + 6\,R_1\right)}}{6(2\,\Phi_1^2\lambda_2 + \lambda_1)}, \tag{3.4}$$

where R_1 is an arbitrary constant. A first integral of a differential equation is a function that remains constant along the solution curves of the equation. In other words, it is a quantity that is conserved as the system evolves through time. First integrals are essential in understanding the behavior of many physical systems because they provide a way to simplify the analysis of the system's behavior. For instance, if the first integral is a conserved quantity such as energy, then it can be used to predict the long-term behavior of the system. Additionally, first integrals can help to reveal the existence of symmetries in the system, which can aid in the development of numerical methods for

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solving the differential equation. Finding first integrals can be a challenging task, but many techniques are available to simplify the process, such as the use of Lie symmetry methods, which exploit the symmetry properties of the differential equation to find the first integral. In the current study, the first integral of Eq. (2.2), derived by the reduction technique, can be expressed as follows:

$$\frac{\psi^6(\zeta)\,\lambda_4}{3}+\frac{\psi^4(\zeta)\,\lambda_3}{2}+\left(-\kappa^2\lambda_1+2\,\lambda_2\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi(\zeta)\right)^2-\mu\right)\psi^2(\zeta)+\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi(\zeta)\right)^2\lambda_1=R_1.$$

By treating Φ_1 as a dependent variable and inserting Eq. (3.4) into the initial equation of (3.2), a deduction can be made:

$$\frac{d\Phi_1(\zeta)}{d\zeta}=-\frac{\sqrt{6\,(2\,\Phi_1^2\lambda_2+\lambda_1)\,(-2\,\Phi_1^6\lambda_4+6\,\kappa^2\Phi_1^2\lambda_1-3\,\Phi_1^4\lambda_3+6\,\mu\,\Phi_1^2+6\,R_1)}}{6(2\,\Phi_1^2\lambda_2+\lambda_1)},\tag{3.5}$$

with exact solution in the implicit form:

$$\zeta+\int\frac{\sqrt{6}\,(2\,{\Phi_1}^2\lambda_2+\lambda_1)\,d\Phi_1}{\sqrt{(2\,{\Phi_1}^2\lambda_2+\lambda_1)\,(-2\,{\Phi_1}^6\lambda_4+6\,\kappa^2{\Phi_1}^2\lambda_1-3\,{\Phi_1}^4\lambda_3+6\,\mu\,{\Phi_1}^2+6\,R_1)}}+R_2=0,\tag{3.6}$$

where R_2 represents a constant with arbitrary value. To derive explicit solutions from (3.6), we adopt the assumptions $\mu=-\lambda_1\kappa^2$, $\lambda_2=\frac{\lambda_4\lambda_1}{3\lambda_3}$, and $R_1=0$. Then, implicit solution (3.6) can be written as

$$\zeta-\frac{\sqrt{2}\Phi_1(\zeta)}{\sqrt{-2\lambda_4\,\Phi_1^6(\zeta)-3\lambda_3\,\Phi_1^4(\zeta)}}\sqrt{\frac{\lambda_1\,(2\lambda_4\,\Phi_1^2(\zeta)+3\,\lambda_3)}{\lambda_3}}+R_2=0.\tag{3.7}$$

Hence, solving this equation with respect to $\Phi_1(\zeta)$, implies

$$\Phi_1(\zeta)=\Psi(\zeta)=\pm\frac{\sqrt{-2\,\lambda_3\lambda_1}}{\lambda_3\,(\zeta+R_2)}.\tag{3.8}$$

Therefore, the final exact solution is

$$\psi_1(x,t)=\pm\frac{\sqrt{-2\,\lambda_3\lambda_1}}{\lambda_3\,(x-2\kappa\lambda_1 t+R_2)}\times\exp\left(i\left(\kappa x-\kappa^2\lambda_1 t\right)\right).\tag{3.9}$$

The singular soliton solution for the equation in this scenario is illustrated in Figure 1.

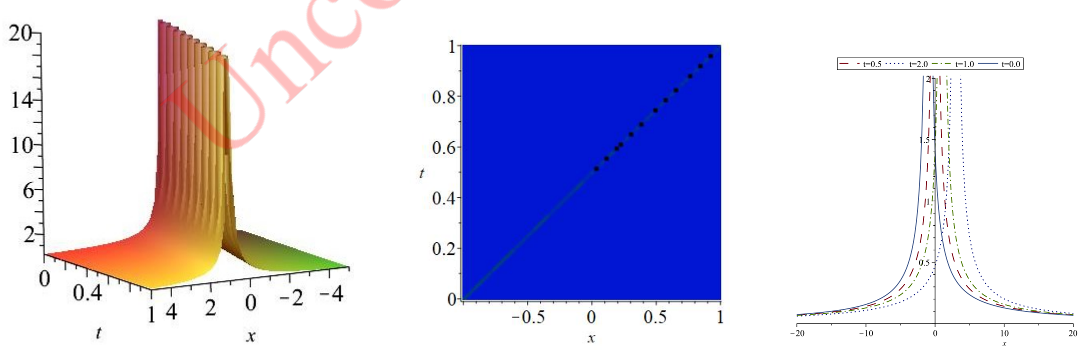

FIGURE 1. 3D, 2D and density plots of $|\psi_1(x,t)|$, with respect to $\lambda_1=\lambda_3=\kappa=R_2=1$.

Case 2.
In this case, suppose exact solution of (3.3), in the following special form:

$$\Phi_2(\Phi_1)=D_2\Phi_1^2+D_1\Phi_1+D_0.\tag{3.10}$$

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Here, $D_i,\ i=0,1,2$, represent constants that need determination. When Eq. (3.10) is substituted into (3.3), it results in a fifth-order polynomial with respect to Φ_1. By equating the coefficients of $\Phi_1^i,\ i=0,\dots,5$ (denoted by $P_i,\ i=0,\dots,5$), we derive the ensuing system of algebraic equations:

$$\begin{aligned}
&P_5=-6D_2^2\lambda_2-\lambda_4=0,\\
&P_4=-10D_1D_2\lambda_2=0,\\
&P_3=-8D_0D_2\,\lambda_2-4D_1^2\lambda_2-2D_2^2\lambda_1-\lambda_3=0,\\
&P_2=-6D_0D_1\,\lambda_2-3D_1D_2\,\lambda_1=0,\\
&P_1=-2D_0^2\lambda_2-2D_0D_2\,\lambda_1-D_1^2\lambda_1+\kappa^2\lambda_1+\mu=0,\\
&P_0=-D_0D_1\,\lambda_1=0.
\end{aligned}\tag{3.11}$$

Various solutions for the system (3.11) can be examined under the following conditions:

Case 2.1:

$$D_2=\sqrt{-\frac{\lambda_4}{6\lambda_2}},\quad D_1=0,\ D_0=\frac{\lambda_1\lambda_4-3\,\lambda_2\lambda_3}{24\lambda_2^2\sqrt{-\frac{\lambda_4}{6\lambda_2}}},\ \mu=-\frac{16\,\kappa^2\lambda_1\lambda_2^2\lambda_4-\lambda_1^2\lambda_4^2+2\,\lambda_1\lambda_2\lambda_3\lambda_4+3\,\lambda_2^2\lambda_3^2}{16\lambda_4\lambda_2^2}.$$

If we reintroduce Φ_1 as the dependent variable and substitute Eq. (3.10) into the first equation of (3.2), we obtain the following outcome:

$$\frac{d\Phi_1(\zeta)}{d\zeta}=\sqrt{-\frac{\lambda_4}{6\lambda_2}}\Phi_1^2(\zeta)+\frac{\lambda_1\lambda_4-3\,\lambda_2\lambda_3}{24\lambda_2^2\sqrt{-\frac{\lambda_4}{6\lambda_2}}},\tag{3.12}$$

with exact solution:

$$\Phi_1(\zeta)=\Psi(\zeta)=\frac{\sqrt{(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}}{2\lambda_4\lambda_2}\tanh\left(\frac{\sqrt{6\,(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}\,(R_1+\zeta)}{12\lambda_2^2\sqrt{-\frac{\lambda_4}{\lambda_2}}}\right),$$

where R_1, is an arbitrary constant. The final solution is expressed as

$$\begin{aligned}
\psi_2(x,t)=&\frac{\sqrt{(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}}{2\lambda_4\lambda_2}\tanh\left(\frac{\sqrt{6\,(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}\,(R_1+x-2\kappa\lambda_1t)}{12\lambda_2^2\sqrt{-\frac{\lambda_4}{\lambda_2}}}\right)\\
&\times\exp\left(i\left(\kappa x-\frac{16\,\kappa^2\lambda_1\lambda_2^2\lambda_4-\lambda_1^2\lambda_4^2+2\,\lambda_1\lambda_2\lambda_3\lambda_4+3\,\lambda_2^2\lambda_3^2}{16\lambda_4\lambda_2^2}t\right)\right).
\end{aligned}\tag{3.13}$$

The equation's dark soliton, which corresponds to this case, is seen in Figure 2.

Case 2.2:

$$\begin{aligned}
&D_2=-\frac{4D_0\,\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}}{2\lambda_1},\quad D_1=0,\ D_0=D_0,\\
&\mu=2D_0^2\lambda_2-D_0\left(4D_0\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}\right)-\kappa^2\lambda_1,\\
&\lambda_4=\frac{3\lambda_2}{\lambda_1}\left(\frac{-4D_0\lambda_2\left(4D_0\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}\right)}{\lambda_1}+\lambda_3\right).
\end{aligned}$$

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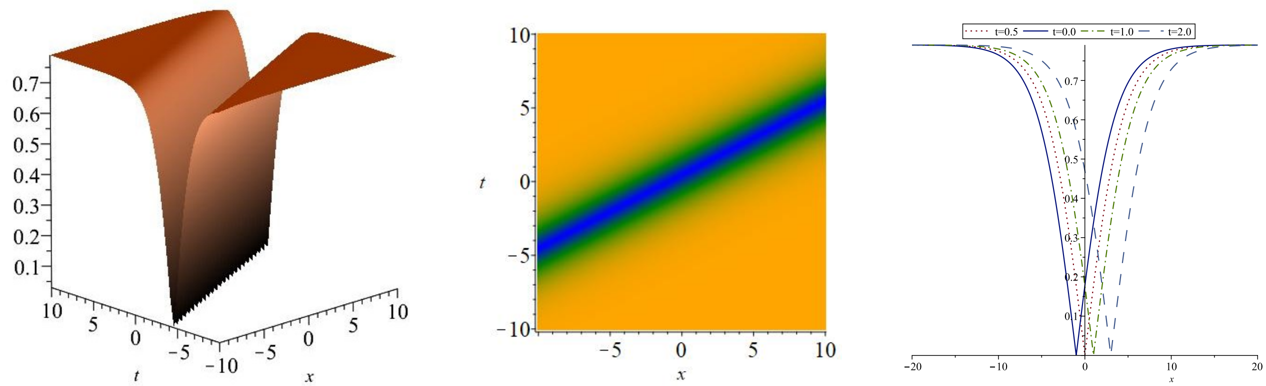

FIGURE 2. 3D, 2D and density plots of $|\psi_2(x,t)|$, with respect to $\lambda_1 = \lambda_3 = \lambda_4 = \kappa = R_1 = 1$, and $\lambda_2 = 2$.

Upon reintroducing Φ_1 as the dependent variable and inserting Eq. (3.10) into the initial equation of (3.2), the resulting expression is as follows:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = -\frac{4D_0\,\lambda_2 - \sqrt{16D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}}{2\lambda_1}\Phi_1^2(\zeta) + D_0. \tag{3.14}$$

The ODE of first order is $\Phi_1(\zeta) = \Psi(\zeta) =$

$$\frac{\sqrt{2D_0\,\lambda_1\left(4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}}{4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}}\tanh\left(\frac{\sqrt{2D_0\,\lambda_1\left(4D_0\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}\,(R_1+\zeta)}{2\lambda_1}\right),$$

where R_1, is a constant. Therefore the final solution can be described as

$$\begin{aligned}\psi_3(x,t) &= \frac{\sqrt{2D_0\,\lambda_1\left(4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}}{4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}} \\ &\times \tanh\left(\frac{\sqrt{2D_0\,\lambda_1\left(4D_0\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}\,(R_1 + x - 2\kappa\lambda_1 t)}{2\lambda_1}\right) \\ &\times \exp\left(i\left(\kappa x + \left[2D_0^2\lambda_2 - D_0\left(4D_0\lambda_2 - \sqrt{16D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right) - \kappa^2\lambda_1\right]t\right)\right).\end{aligned} \tag{3.15}$$

The dark soliton of the equation, corresponding to the (3.15), is shown in Figure 3.

Case 2.3:

$$D_2 = 0,\quad D_1 = \sqrt{\frac{\lambda_3}{4\lambda_2}},\ D_0 = 0,\ \mu = -\frac{\left(4\,\kappa^2\lambda_2 + \lambda_3\right)\lambda_1}{4\lambda_2},\ \lambda_4 = 0.$$

If we express Φ_1 as a dependent variable, substituting Eq. (3.10) into the initial equation of (3.2) yields:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = \sqrt{\frac{\lambda_3}{4\lambda_2}}\Phi_1(\zeta). \tag{3.16}$$

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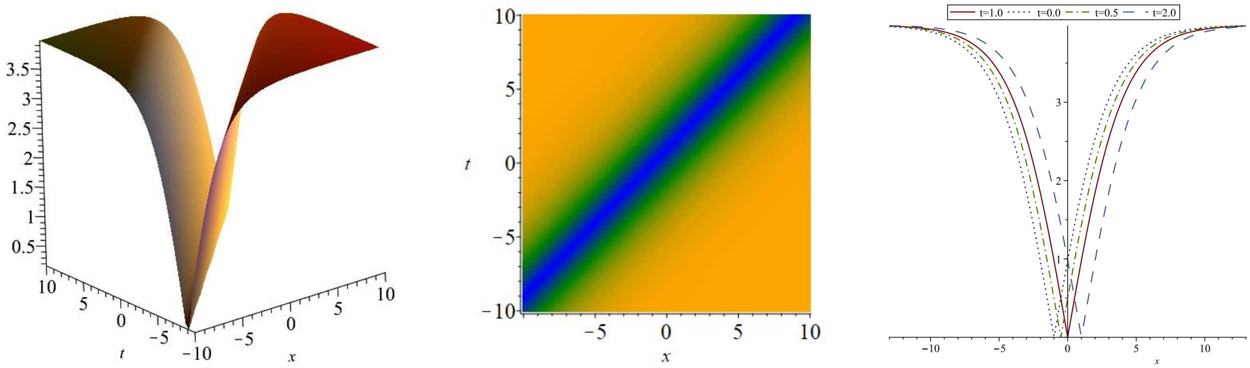

FIGURE 3. 3D, 2D and density plots of $|\psi_3(x,t)|$, with respect to $\lambda_1 = \lambda_3 = \lambda_4 = D_0 = R_1 = 1,\ \lambda_2 = 2$, and $\kappa = 0.5$.

So, the first order ODE is written as

$$\Phi_1(\zeta) = \Psi(\zeta) = R_1 \exp\left(\frac{\sqrt{-\frac{\lambda_3}{\lambda_2}}}{2}\zeta\right),$$

where R_1, is constant and final result is given below

$$\psi_4(x,t) = R_1 \exp\left(\frac{\sqrt{-\frac{\lambda_3}{\lambda_2}}}{2}(x - 2\kappa\lambda_1 t)\right) \times \exp\left(i\left(\kappa x - \frac{(4\kappa^2\lambda_2 + \lambda_3)\lambda_1}{4\lambda_2}t\right)\right). \tag{3.17}$$

Traveling wave solutions and corresponding density plots of (3.17), in the real and imaginary senses, are demonstrated in Figure 4.

Case 2.4:

$$D_2 = \sqrt{-\frac{\lambda_4}{6\lambda_2}},\quad D_1 = 0,\ D_0 = D_0,\ \mu = -3\,\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\kappa^2\lambda_2 + 2D_0^2\lambda_4 - 2D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_3 + \kappa^2\lambda_3\right),$$
$$\lambda_1 = 3\,\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_2 + \lambda_3\right).$$

If we regard Φ_1 as the dependent variable and insert Eq. (3.10) into the initial equation of (3.2), the result will be:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = \sqrt{-\frac{\lambda_4}{6\lambda_2}}\Phi_1^2 + D_0. \tag{3.18}$$

Hence, the exact solution of ODE is

$$\Phi_1(\zeta) = \Psi(\zeta) = \frac{\sqrt[4]{6}}{\sqrt{-\frac{\lambda_4}{\lambda_2}}}\sqrt{D_0\sqrt{-\frac{\lambda_4}{\lambda_2}}}\tan\left(\frac{\sqrt{D_0\sqrt{-\frac{\lambda_4}{\lambda_2}}}}{\sqrt[4]{6}}(R_1 + \zeta)\right),$$

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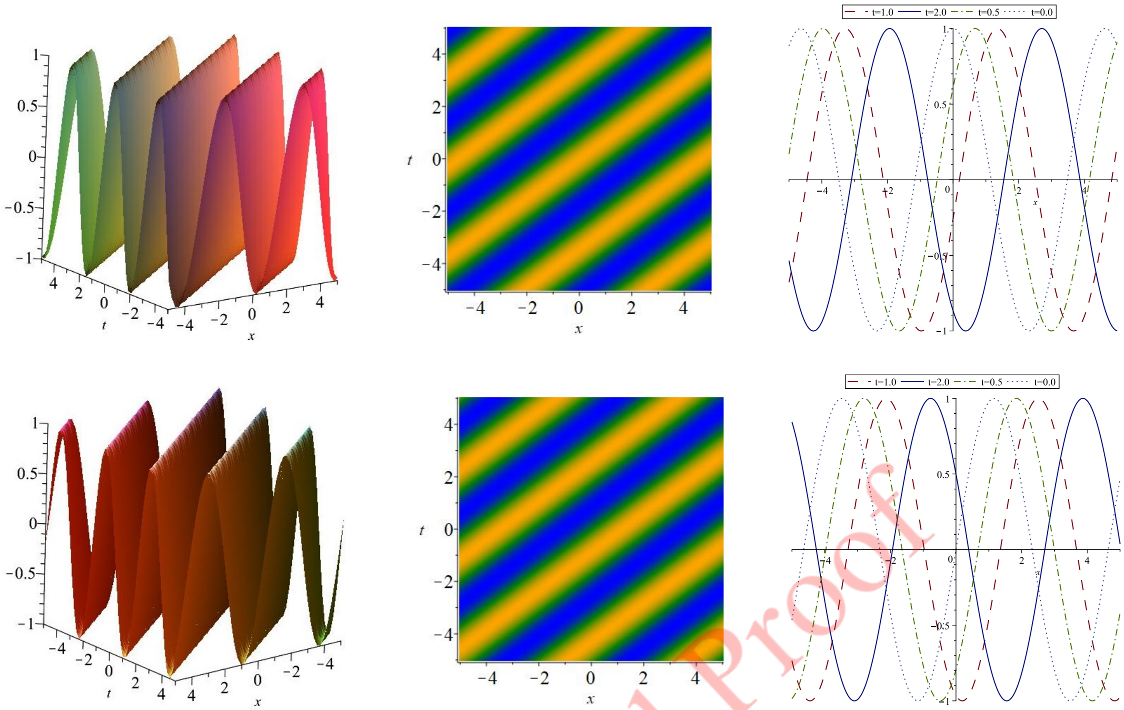

FIGURE 4. 3D, 2D and density plots of $Re(\psi_4(x,t))$, and $Im(\psi_4(x,t))$, with respect to $\lambda_1 = \lambda_3 = \kappa = D_0 = R_1 = 1$, and $\lambda_2 = 2$.

where R_1, is any constant. The final solution can be expressed as

$$\psi_5(x,t) = \frac{\sqrt[4]{6}}{\sqrt{-\frac{\lambda_4}{\lambda_2}}}\sqrt{D_0\sqrt{-\frac{\lambda_4}{\lambda_2}}}\tan\left(\frac{\sqrt{D_0\sqrt{-\frac{\lambda_4}{\lambda_2}}}}{\sqrt[4]{6}}\left(R_1 + x - 6\kappa\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_2 + \lambda_3\right)t\right)\right)$$
$$\times \exp\left(i\left(\kappa x + 3\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\kappa^2\lambda_2 + 2D_0^2\lambda_4 - 2D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_3 + \kappa^2\lambda_3\right)t\right)\right). \tag{3.19}$$

The periodic solution of the equation, corresponding to the (3.19), is shown in Figure 5.

4. Discussion and Results

This paper successfully ventured and recovered optical soliton solutions to the dispersive concatenation model with linear chromatic dispersion and self-phase modulation. Earlier investigations have laid the groundwork by introducing concatenation models combining established equations such as the NLSE, LPDE model, and Sasa-Satsuma equation. We have extended these models to incorporate higher-order dispersive effects, introducing equations like the SHE and quintic-order NLSE. Studying was conducted with the aid of the improved modified extended tanh function method, various and novel solutions were raised. These solutions including bright, dark and singular solitons. In addition, other mathematical solutions such as Weierstrass elliptic, exponential, rational and singular periodic solutions were constructed. 3D and 2D graphical representations were illustrated for some selected solutions to show the nature of the propagated waves. These solutions were presented by setting the parameters with appropriate values. The obtained solitons proved that a dedicated balance occurred between the non linear and dispersion terms. These solutions will

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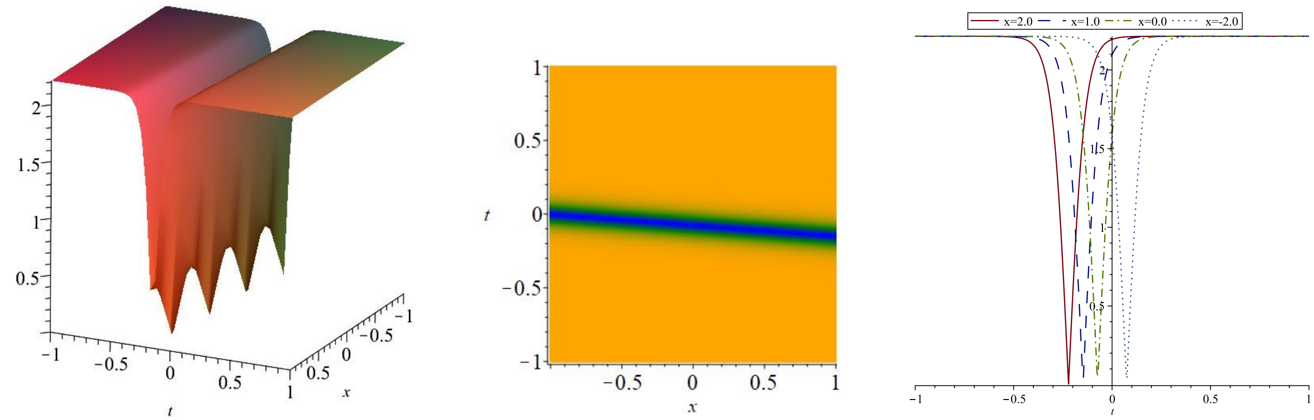

FIGURE 5. 3D, 2D and density plots of $|\psi_5(x,t)|$, with respect to $\lambda_3 = \lambda_4 = R_1 = 1,\ \lambda_2 = -1,\ D_0 = -2$, and $\kappa = 0.3$.

help in the development of the communication industry as these types of solutions can propagate to very long distances maintaining their shapes and speeds.

5. Conclusion

In this paper, we study a weakly nonlocal NLSE with PL nonlinearity, and NRM is successfully applied to explore the exact solutions. This approach offers solutions in the form of dark soliton, bright soliton, and traveling wave solutions. Graphical descriptions of density and 3D plots of selected solutions are carried out in Figures 1-5 reveals the dynamical behavior of the attained solutions. The current work shows that solution extracted by present method are novel and different as compared to some other existing techniques in literature [19, 23]. Lastly, using Maple, our results are verified by back substitution in the original equations. The suggested approach is not only direct and simple but also suitable for constructing new results. In future, NRM can be applied on NLSE having dual power law and perturbed NLSE with kerr law which are very useful to identify the solitons in photo-refractive and polymer materials. It is observed that the proposed technique can potentially be used to implement further models that develop in the fields of natural science that may be adopt to investigate the other mathematical challenges and generated results can be applied to characterize the behavior of nonlinear models.The IMETS was successfully applied in this work to investigate dispersive optical solitons. Many solitons and other solutions were extracted. These solutions including {bright, dark and singular} solitary solutions, Weierstrass elliptic and singular periodic solutions. Moreover, graphical representations in both 2D and 3D of some of the recovered solutions are presented to illustrate the characteristics of the propagating wave. These solutions provide an explanation for a wide range of fascinating and challenging physical phenomena due to the NLSE models applicability in several scientific domains, including wave-guides and optical fibers. The retrieved solutions in this research study are novel, and the model was not previously investigated using the proposed methodology. The approachs success, convenience of use, and efficacy show the methods applicability for dealing with nonlinear optical problems. With all of these features, it will undoubtedly enrich the literature. The results are thus tremendously promising and lead to the avenues of further research in this arena. Later, the model will be studied with differential group delay followed by the consideration of the model with dispersion-flattened fibers. In addition, the stochastic model can be investigated to show the effect of the noise on the extracted solutions.

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