Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-12
DOI:10.22034/cmde.2024.61598.2673

Investigation of optical solitons in a weakly nonlocal schrödinger equation with parabolic nonlinearity

Mir Sajjad Hashemi[1], **Mohammad Mirzazadeh**[2], **Hamood Ur Rehman**[3], **Ahmed H. Arnous**[4], **Mustafa Bayram**[5], and **Mostafa Eslami**[6,*]

[1]Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran.

[2]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar-Vajargah, Iran.

[3]Department of Mathematics, University Of Okara, Okara, Pakistan.

[4]Department of Physics and Engineering Mathematics, Higher Institute of Engineering,El Shorouk Academy-11837, Cairo, Egypt.

[5]Computer Engineering, Biruni University, Istanbul, Turkey.

[6]Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.

Abstract

The weakly nonlinear Schrödinger equation (NLSE) describes wave phenomena in media characterized by weakly nonlinear dispersion. As a versatile framework, it finds application across diverse fields such as plasma waves, water waves, fiber optics, and Bose-Einstein condensates, and this study focuses on investigating various solutions for the weakly nonlocal NLSE with parabolic law nonlinearity. By employing the Nucci reduction method (NRM), we extract exact solutions, including dark and bright solitons and other traveling wave solutions, are extracted. This technique is particularly valuable for identifying nonlocal symmetries of differential equations, providing an efficient analytical tool for nonlinear problem-solving in engineering and related domains. Furthermore, we derive a first integral through the reduction method. These results are essential for understanding soliton wave propagation in weakly nonlocal media with parabolic law nonlinearity, providing insights into wave dynamics for the proposed model. Finally, two- and three-dimensional density plots are presented to illustrate the physical behavior of some obtained solutions within the governing model.

Keywords. Schrödinger model, Parabolic law, First integral, Soliton solution, Nucci method.

2010 Mathematics Subject Classification. 35C07, 35C08, 35C09.

1. INTRODUCTION

Partial Differential Equations (PDEs) serve as powerful mathematical tools for modeling diverse phenomena in science and engineering. These equations incorporating multiple independent variables and their partial derivatives, including heat conduction, fluid dynamics, and electromagnetic fields.

The investigation of PDEs involves a wide range of approaches. It includes both analytical and numerical methods for solving these equations, each providing different insights depending on the nature and complexity of the problem. Analytical techniques such as separation of variables, integral transforms, and series solutions are used to obtain exact solutions or develop a fundamental understanding of solution behavior, and when analytical solutions are intractable, numerical methods such as finite difference, finite element, and spectral methods provide computational tools to approximate solutions for PDEs. Additionally, the study of stability, existence, and uniqueness of solutions, as well as the development of new solution methods, continues to be a vibrant area of research within the broader field of partial differential equations. A deep understanding of the mathematical intricacies of PDEs is therefore crucial for advancing our comprehension of natural systems and driving technological innovation.

Received: 10 May 2024; Accepted: 05 February 2024.

$*$ Corresponding author. Email: mostafa.eslami@umz.ac.ir.

Numerous scholars have focused on obtaining exact solutions for NPDEs find applications across diverse scientific and technological domains, including but not limited to mathematical physics, fluid dynamics, optical fibers, and economics [10, 14, 27], and in recent years various methodologies have been developed for determining precise solutions to these equations. These approaches include the Lie symmetry method [16, 18, 20, 26, 30], the Kudryashov method [21, 22, 33], the sine-Gordon expansion method [6], the invariant subspace method [13, 28], the Sardar subequation method [3, 11], and others [1, 5, 8, 9, 29]. These techniques contribute to the exploration of exact solutions for a wide range of PDEs.

The NLSE is a complex PDE that plays a crucial role in various fields of physics and engineering, describing the evolution of complex wave packets. It arises in different contexts, including nonlinear optics, plasma physics, fluid dynamics, and condensed matter physics. The NLSE and its variants have significant importance in understanding and predicting the behavior of wave-like phenomena in diverse real-world applications.

- NLSE in Nonlinear Optics:

In nonlinear optics, the NLSE governs the propagation of intense laser beams through nonlinear media. This equation describes the interactions among optical waves, considering effects such as self-focusing, self-phase modulation, and optical solitons. Optical solitons, which are stable, localized wave packets that can maintain their shape during propagation, find applications in long-distance communication systems. The NLSE helps optimize and control these phenomena in the design of optical communication systems and laser technologies.

- Bose-Einstein Condensates (BECs):

The NLSE also appears in the study of ultra-cold atomic gases, particularly in the context of Bose-Einstein condensates. In this scenario, the NLSE describes the dynamics of the macroscopic wave function of the condensate. Understanding the NLSE for BECs is crucial for investigating phenomena such as matter-wave solitons and vortices, which have applications in precision measurements and quantum information processing.

- Plasma Physics:

The NLSE arises in the study of Langmuir waves in plasmas, where it describes the evolution of the electron plasma wave. Nonlinear effects become significant in high-intensity laser-plasma interactions and can lead to the generation of harmonics and other phenomena. This has applications in areas such as controlled nuclear fusion research and the development of high-power particle accelerators.

- Fiber Optics and Communication:

As a fundamental model for describing the propagation of optical pulses in fiber optic communication systems, the NLSE has been extensively applied to account for nonlinear effects such as self-phase modulation and cross-phase modulation, which can distort transmitted signals, making it essential for modeling and mitigating these phenomena to ensure the reliability and efficiency of long-distance communication networks.

- Water Waves and Oceanography:

Variants of the NLSE are used to model the propagation of water waves in oceans and other bodies of water. Nonlinear effects, such as wave steepening and wave breaking, can be described using these equations. Understanding these phenomena is crucial for predicting and mitigating the impact of tsunamis, storm surges, and other oceanic events.

- Biophysics:

NLSE variants are employed in the study of biological systems,such as modeling the propagation of nerve impulses, where the NLSE describes the nonlinear dynamics of excitable media, providing insights into the behavior of electrical signals in biological tissues and neural networks.

Moreover, the NLSE [4, 12, 23], which is an essential fully-integrated nonlinear dispersive partial differential equation (PDE), has found extensive application in elucidating diverse phenomena like deep water waves, rogue waves, plasmas, and nonlinear optics, including atomic physics. The NLSE serves as a comprehensive description of nonlinear dispersive processes. Zhou et al. [34] considered the weakly nonlocal NLSE having PL nonlinearity with external potential as

$$i\frac{\partial\psi}{\partial t} + \lambda_1\frac{\partial^2\psi}{\partial x^2} + \left(\lambda_2\frac{\partial^2|\psi|^2}{\partial x^2} + \lambda_3|\psi|^2 + \lambda_4|\psi|^4\right)\psi + \lambda_5\psi = 0, \tag{1.1}$$

and acquired its exact solutions by the robust and direct methods. The primary objective of the present study is to revisit this weakly nonlocal NLSE with parabolic law nonlinearity and construct its soliton solutions.

Numerous researchers have previously investigated this model such as Hadi Rezazadeh *et. al.* [7] constructed singular and dark soliton by using simple equation method. With different techniques, Kamyar *et. al.* [19] explored the bright and dark solution of nonlocal NLSE with PL nonlinearity. Yue Kang *et. al.* [32] proposed the discrimination system to achieve the various types of soliton such as kink soliton, triangular soliton, and bright soliton. In the current study, we apply the Nucci reduction method (NRM) [25] to derive the different solutions that comprise of periodic, exponential, dark and bright solutions. This method, initially introduced in [25], was developed to find nonlocal symmetries of differential equations. The applied method is powerful and efficient for finding the solutions of NPDEs in different fields and the results obtained can provide valuable insights into the propagation of optical solitons in parabolic medium.

The study of Schrödinger equations with nonlinearity is an important area of research in mathematical physics. In recent years, the weakly nonlocal Schrödinger equation with parabolic law nonlinearity has gained significant attention due to its numerous applications in various fields such as quantum mechanics, nonlinear optics, and fluid dynamics. In this paper, we present novel exact solutions for this weakly nonlocal Schrödinger equation with parabolic law nonlinearity, derived using a NRM. The solutions we obtain, which include dark soliton, bright soliton, and traveling wave solutions, are, to the best of our knowledge, previously unreported in the literature and demonstrate the complex dynamics of the considered equation. Graphical descriptions, reveal the dynamical behavior of these solutions, highlighting the uniqueness of the results achieved with this technique compared to existing methods.

Our findings have significant implications for the study of nonlinear phenomena in physical systems. The ability to obtain exact solutions to such complex equations is crucial in understanding the underlying physics and designing new experiments. We have verified all solutions using Maple by back-substituting them into the original equations, confirming their correctness and the robustness of our approach. The proposed Nucci reduction method (NRM) is not only direct and straightforward but also highly effective for constructing new classes of solutions. This paves the way for future applications to more complex systems, such as NLSEs with dual-power law nonlinearity and perturbed NLSEs with Kerr law. Such applications are particularly relevant for identifying solitons in photorefractive and polymer materials. We anticipate that this technique will find further utility in natural science models, aiding in the investigation of other challenging mathematical problems and characterizing the behavior of nonlinear models.

The remaining sections of this manuscript are organized as follows: Section 2 outlines the governing model of the governing equation and the resulting reduced ordinary differential equation (ODE). Section 3 explores the application of NRM to the model. The outcomes, discussions, and a physical interpretation related to the weakly nonlinear Schrödinger equation (NLSE) are presented in section 4. Finally, section 5 encompasses the conclusion of this study.

2. Mathematical analysis

Transforming the Schrödinger equation is a standard method for reducing it to a set of ordinary differential equations (ODEs). This simplification facilitates is important as it makes the equation easier to solve and analyze.These transformations aid in solving the Schrödinger equation and provide insight into quantum system behavior. We begin by applying the following complex transformation:

$$\psi(x,t) = \Psi(\zeta) e^{i(\kappa x + \mu t)}, \qquad \zeta = x + vt, \tag{2.1}$$

where κ, v and μ represent the wave number, speed, and frequency, respectively. Applying this transformation reduces the WNSE with parabolic law nonlinearity to the following equation:

$$\lambda_1 \frac{d^2\Psi(\zeta)}{d\zeta^2} - \left(\kappa^2\lambda_1 + \mu\right)\Psi(\zeta) + 2\lambda_2 \frac{d^2\Psi(\zeta)}{d\zeta^2}\Psi^2(\zeta) + 2\lambda_2 \left(\frac{d\Psi(\zeta)}{d\zeta}\right)^2 \Psi(\zeta) + \lambda_3\Psi^3(\zeta) + \lambda_4\Psi^5(\zeta) = 0, \tag{2.2}$$

where $v = -2\kappa\lambda_1$ represents the wave speed. Consequently, analytical methods for ODEs enable the derivation of exact solutions, which are crucial for a deeper understanding of underlying phenomena and for facilitating further analysis. In the following section, we employ the NRM method to derive the first integral and obtain solutions for Equation (2.2).

3. Nucci's reduction method

Autonomous systems of ODEs can be reformulated as systems of first-order ODEs by designating one dependent variable as the new independent variable. During this transformation, specific variables are eliminated to resulting in a mixed system of first and second-order equations. The detection of point symmetries for this mixed system can be automated without requiring prior assumptions about the structure of the symmetry. Because the coefficient function associated with the original independent variable only appears as its derivative in the simplified system, nonlocal symmetries are converted into local symmetries for the reduced system, and their calculation can be algorithmic. This approach constitutes the NRM method.

This study employs the NRM to determine new exact solutions and their corresponding first integrals for Eq. (2.2). The NRM was recently applied to various types of differential equations in [15, 17, 24]. All computations were performed using Maple software. To apply this reduction method, we implement the following change of variables [2, 31]:

$$\Phi_1(\zeta) = \Psi(\zeta), \qquad \Phi_2(\zeta) = \Psi'(\zeta). \tag{3.1}$$

It is well-known that, an ODE can be rewritten as an equivalent dynamical system, which is a more general framework for analyzing the system's behavior. The equivalent dynamical system represents the evolution of the system through time by specifying a set of state variables and a set of equations that describe how those variables change over time. By reformulating the ODE in this way, we can analyze the stability, equilibrium, and bifurcations of the system, gaining a more complete understanding of its behavior. Additionally, the equivalent dynamical system allows us to apply a range of mathematical and computational tools to study the system's properties, making it a valuable tool in many areas of science and engineering.

Therefore, by using the assumptions (3.1), Eq. (2.2) can be transformed into the following dynamical system:

$$\begin{cases} \Phi_1' = \Phi_2, \\ \Phi_2' = \dfrac{\left(\kappa^2\lambda_1 + \mu\right)\Phi_1 - 2\,\lambda_2\Phi_1\Phi_2^2 - \lambda_3\Phi_1^3 - \lambda_4\Phi_1^5}{2\,\lambda_2\Phi_1^2 + \lambda_1}. \end{cases} \tag{3.2}$$

If we select Φ_1 as the new independent variable, the system (3.2) transforms into the following form:

$$\frac{d\Phi_2(\Phi_1)}{d\Phi_1} = \frac{\left(\kappa^2\lambda_1 + \mu\right)\Phi_1 - 2\,\lambda_2\Phi_1\left(\Phi_2\left(\Phi_1\right)\right)^2 - \lambda_3\Phi_1^3 - \lambda_4\Phi_1^5}{\left(2\,\lambda_2\Phi_1^2 + \lambda_1\right)\Phi_2\left(\Phi_1\right)}. \tag{3.3}$$

We adopt two different approaches to address Eq. (3.3). Initially, Maple furnishes an implicit solution, supplemented by a few specific explicit solutions.Second, we derive solutions to Eq. (3.3) in specific forms, which enables the application of our methodology in the following cases:

Case 1.

Here, Eq. (3.3) is integrated directly using Maple. The exact solution is given by:

$$\Phi_2(\Phi_1) = -\frac{\sqrt{6\left(2\,\Phi_1^2\lambda_2 + \lambda_1\right)\left(-2\,\Phi_1^6\lambda_4 + 6\,\kappa^2\Phi_1^2\lambda_1 - 3\,\Phi_1^4\lambda_3 + 6\,\mu\,\Phi_1^2 + 6\,R_1\right)}}{6(2\,\Phi_1^2\lambda_2 + \lambda_1)}, \tag{3.4}$$

where R_1 is an arbitrary constant. A first integral of a differential equation is a function that remains constant its solution curves. It represents a conserved quantity as the system evolves. First integrals are essential for understanding physical systems, as they simplify the analysis of system behavior. For example, a first integral representing a conserved quantity like energy can be used to predict the system's long-term behavior. Furthermore, first integrals can help to reveal the existence of symmetries in the system, which can inform in the development of numerical methods for solving the differential equation. Although finding first integrals can be challenging, several techniques simplify the process. For instance, Lie symmetry methods exploit the differential equation's symmetry properties to derive them. In this study, the first integral for Eq. (2.2), obtained via the reduction technique, is given by:

$$\frac{\psi^6\left(\zeta\right)\lambda_4}{3} + \frac{\psi^4\left(\zeta\right)\lambda_3}{2} + \left(-\kappa^2\lambda_1 + 2\,\lambda_2\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi\left(\zeta\right)\right)^2 - \mu\right)\psi^2\left(\zeta\right) + \left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi\left(\zeta\right)\right)^2\lambda_1 = R_1.$$

Treating Φ_1 as the dependent variable and substituting Eq. (3.4) into the first equation of system (3.2) yields:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = -\frac{\sqrt{6\left(2\,\Phi_1^2\lambda_2 + \lambda_1\right)\left(-2\,\Phi_1^6\lambda_4 + 6\,\kappa^2\Phi_1^2\lambda_1 - 3\,\Phi_1^4\lambda_3 + 6\,\mu\,\Phi_1^2 + 6\,R_1\right)}}{6(2\,\Phi_1^2\lambda_2 + \lambda_1)}, \tag{3.5}$$

this leads to an exact implicit solution:

$$\zeta+\int\frac{\sqrt{6}\left(2\,{\Phi_1}^2\lambda_2+\lambda_1\right)d\Phi_1}{\sqrt{\left(2\,{\Phi_1}^2\lambda_2+\lambda_1\right)\left(-2\,{\Phi_1}^6\lambda_4+6\,\kappa^2{\Phi_1}^2\lambda_1-3\,{\Phi_1}^4\lambda_3+6\,\mu\,{\Phi_1}^2+6\,R_1\right)}}+R_2=0, \tag{3.6}$$

where R_2 is an arbitrary constant. To obtain explicit solutions from (3.6), we impose the parameter constraints $\mu=-\lambda_1\kappa^2$, $\lambda_2=\frac{\lambda_4\lambda_1}{3\lambda_3}$, and $R_1=0$. Under these constraints, the implicit solution (3.6) simplifies to:

$$\zeta-\frac{\sqrt{2}\Phi_1\left(\zeta\right)}{\sqrt{-2\lambda_4\,\Phi_1^6\left(\zeta\right)-3\lambda_3\,\Phi_1^4\left(\zeta\right)}}\sqrt{\frac{\lambda_1\left(2\lambda_4\,\Phi_1^2\left(\zeta\right)+3\,\lambda_3\right)}{\lambda_3}}+R_2=0. \tag{3.7}$$

Solving this equation for $\Phi_1(\zeta)$, yields:

$$\Phi_1(\zeta)=\Psi(\zeta)=\pm\frac{\sqrt{-2\,\lambda_3\lambda_1}}{\lambda_3\left(\zeta+R_2\right)}. \tag{3.8}$$

Consequently, the final exact solution is given by:

$$\psi_1(x,t)=\pm\frac{\sqrt{-2\,\lambda_3\lambda_1}}{\lambda_3\left(x-2\kappa\lambda_1t+R_2\right)}\times\exp\left(i\left(\kappa x-\kappa^2\lambda_1t\right)\right). \tag{3.9}$$

The singular soliton solution described by this scenario is depicted in Figure 1.

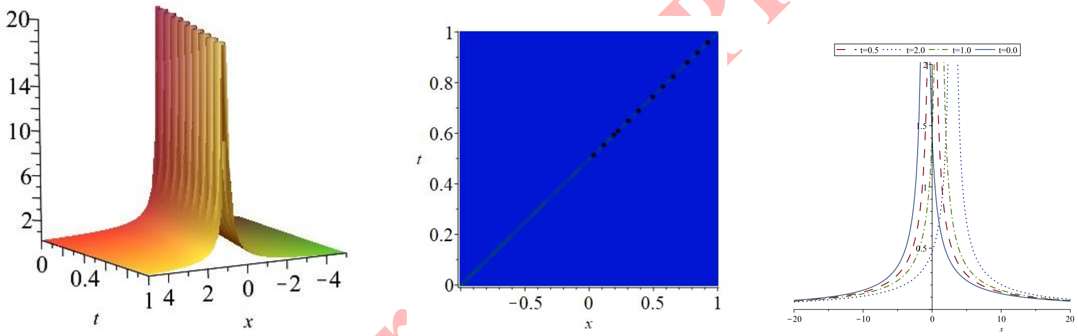

FIGURE 1. 3D, 2D, and density plots of $|\psi_1(x,t)|$, with respect to $\lambda_1=\lambda_3=\kappa=R_2=1$.

Case 2.

For this case, we assume an exact solution to Eq. (3.3) of the following special form:

$$\Phi_2(\Phi_1)=D_2\Phi_1^2+D_1\Phi_1+D_0. \tag{3.10}$$

Here, D_i, $i=0,1,2$, are constants to be determined. Substituting Eq. (3.10) into Eq. (3.3) yields a fifth-order polynomial in Φ_1. Equating the coefficients of Φ_1^i, $i=0,\ldots,5$ (denoted by P_i, $i=0,\ldots,5$), we derive the ensuing system of algebraic equations:

$$\begin{aligned}
&P_5=-6D_2^2\lambda_2-\lambda_4=0,\\
&P_4=-10D_1D_2\lambda_2=0,\\
&P_3=-8D_0D_2\,\lambda_2-4D_1^2\lambda_2-2D_2^2\lambda_1-\lambda_3=0,\\
&P_2=-6D_0D_1\,\lambda_2-3D_1D_2\,\lambda_1=0,\\
&P_1=-2D_0^2\lambda_2-2D_0D_2\,\lambda_1-D_1^2\lambda_1+\kappa^2\lambda_1+\mu=0,\\
&P_0=-D_0D_1\,\lambda_1=0.
\end{aligned} \tag{3.11}$$

The system (3.11) admits solutions under the following conditions:

Case 2.1:

$$D_2=\sqrt{-\frac{\lambda_4}{6\lambda_2}},\quad D_1=0,\ D_0=\frac{\lambda_1\lambda_4-3\,\lambda_2\lambda_3}{24\lambda_2^2\sqrt{-\frac{\lambda_4}{6\lambda_2}}},\ \mu=-\frac{16\,\kappa^2\lambda_1\lambda_2^2\lambda_4-\lambda_1^2\lambda_4^2+2\,\lambda_1\lambda_2\lambda_3\lambda_4+3\,\lambda_2^2\lambda_3^2}{16\lambda_4\lambda_2^2}.$$

If we reintroduce Φ_1 as the dependent variable and substitute Eq. (3.10) into the first equation of (3.2), gives:

$$\frac{d\Phi_1(\zeta)}{d\zeta}=\sqrt{-\frac{\lambda_4}{6\lambda_2}}\Phi_1^2(\zeta)+\frac{\lambda_1\lambda_4-3\,\lambda_2\lambda_3}{24\lambda_2^2\sqrt{-\frac{\lambda_4}{6\lambda_2}}}, \tag{3.12}$$

which has the exact solution:

$$\Phi_1(\zeta)=\Psi(\zeta)=\frac{\sqrt{(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}}{2\lambda_4\lambda_2}\tanh\left(\frac{\sqrt{6\,(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}\,(R_1+\zeta)}{12\lambda_2^2\sqrt{-\frac{\lambda_4}{\lambda_2}}}\right),$$

where R_1 is an arbitrary constant. Thus the final solution is

$$\begin{aligned}\psi_2(x,t)=&\frac{\sqrt{(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}}{2\lambda_4\lambda_2}\tanh\left(\frac{\sqrt{6\,(\lambda_1\lambda_4-3\,\lambda_2\lambda_3)\,\lambda_4\lambda_2}\,(R_1+x-2\kappa\lambda_1 t)}{12\lambda_2^2\sqrt{-\frac{\lambda_4}{\lambda_2}}}\right)\\&\times\exp\left(i\left(\kappa x-\frac{16\,\kappa^2\lambda_1\lambda_2^2\lambda_4-\lambda_1^2\lambda_4^2+2\,\lambda_1\lambda_2\lambda_3\lambda_4+3\,\lambda_2^2\lambda_3^2}{16\lambda_4\lambda_2^2}t\right)\right).\end{aligned}\tag{3.13}$$

The equation's dark soliton, which corresponds to this case, is seen in Figure 2.

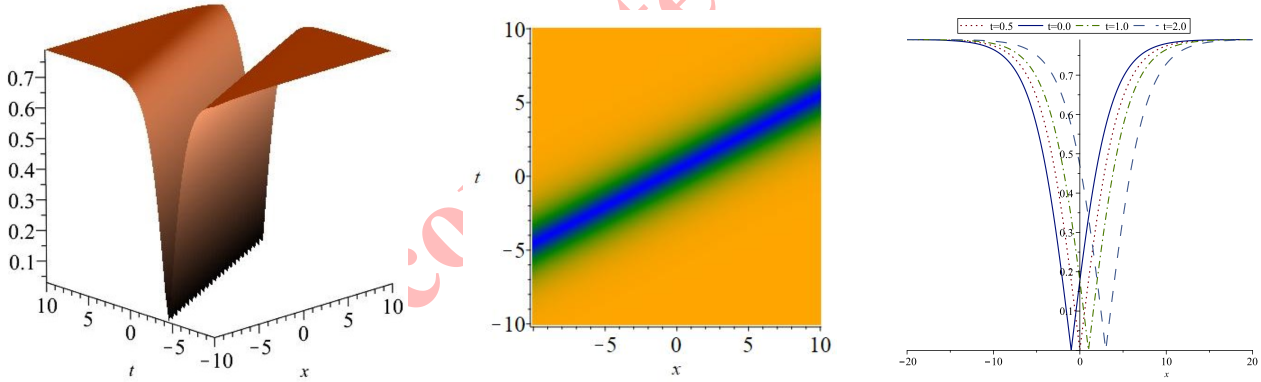

FIGURE 2. 3D, 2D, and density plots of $|\psi_2(x,t)|$, with respect to $\lambda_1=\lambda_3=\lambda_4=\kappa=R_1=1$, and $\lambda_2=2$.

Case 2.2:

$$\begin{aligned}&D_2=-\frac{4D_0\,\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}}{2\lambda_1},\quad D_1=0,\ D_0=D_0,\\&\mu=2D_0^2\lambda_2-D_0\left(4D_0\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}\right)-\kappa^2\lambda_1,\\&\lambda_4=\frac{3\lambda_2}{\lambda_1}\left(\frac{-4D_0\lambda_2\left(4D_0\lambda_2-\sqrt{16D_0^2\lambda_2^2-2\,\lambda_1\lambda_3}\right)}{\lambda_1}+\lambda_3\right).\end{aligned}$$

Upon reintroducing Φ_1 as the dependent variable and inserting Eq. (3.10) into the initial equation of (3.2), the resulting expression is as follows:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = -\frac{4D_0\,\lambda_2 - \sqrt{16D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}}{2\lambda_1}\Phi_1^2\,(\zeta) + D_0. \tag{3.14}$$

The ODE of first order is

$$\Phi_1\,(\zeta) = \Psi(\zeta) = \frac{\sqrt{2D_0\,\lambda_1\left(4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}}{4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}}\tanh\left(\frac{\sqrt{2D_0\,\lambda_1\left(4D_0\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}\,(R_1+\zeta)}{2\lambda_1}\right),$$

where R_1 is a constant. Therefore the final solution can be described as

$$\begin{aligned}\psi_3(x,t) = {} & \frac{\sqrt{2D_0\,\lambda_1\left(4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}}{4\,D_0\,\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}} \\ & \times \tanh\left(\frac{\sqrt{2D_0\,\lambda_1\left(4D_0\lambda_2 - \sqrt{16\,D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right)}\,(R_1 + x - 2\kappa\lambda_1 t)}{2\lambda_1}\right) \\ & \times \exp\left(i\left(\kappa x + \left[2D_0^2\lambda_2 - D_0\left(4D_0\lambda_2 - \sqrt{16D_0^2\lambda_2^2 - 2\,\lambda_1\lambda_3}\right) - \kappa^2\lambda_1\right]t\right)\right).\end{aligned} \tag{3.15}$$

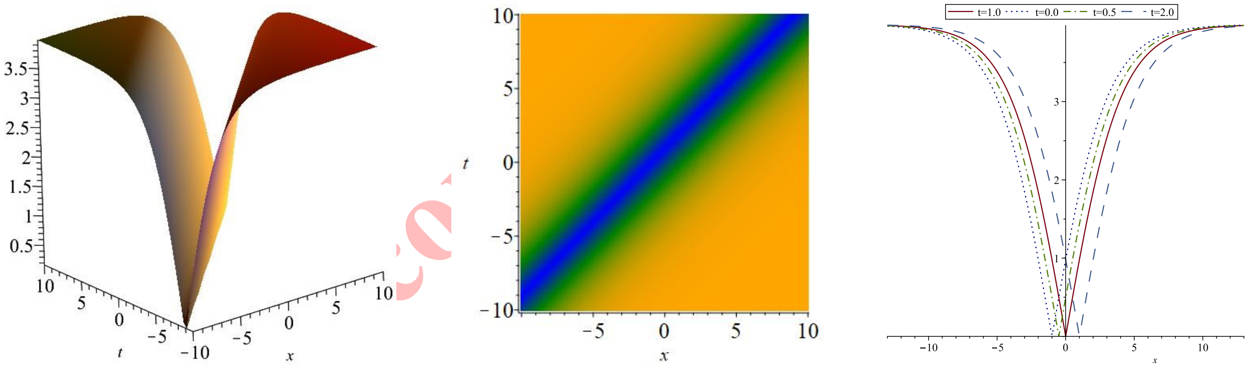

FIGURE 3. 3D, 2D, and density plots of $|\psi_3(x,t)|$, with respect to $\lambda_1 = \lambda_3 = \lambda_4 = D_0 = R_1 = 1$, $\lambda_2 = 2$, and $\kappa = 0.5$.

The dark soliton of the equation, corresponding to the (3.15), is shown in Figure 3.

Case 2.3:

$$D_2 = 0, \quad D_1 = \sqrt{\frac{\lambda_3}{4\lambda_2}}, \quad D_0 = 0, \quad \mu = -\frac{\left(4\,\kappa^2\lambda_2 + \lambda_3\right)\lambda_1}{4\lambda_2}, \quad \lambda_4 = 0.$$

If we express Φ_1 as a dependent variable, substituting Eq. (3.10) into the first equation of (3.2) yields:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = \sqrt{\frac{\lambda_3}{4\lambda_2}}\Phi_1\,(\zeta)\,. \tag{3.16}$$

The solution to this first-order ODE is:

$$\Phi_1(\zeta) = \Psi(\zeta) = R_1 \exp\left(\frac{\sqrt{-\frac{\lambda_3}{\lambda_2}}}{2}\zeta\right),$$

where R_1 is an arbitrary constant. The final solution is:

$$\psi_4(x,t) = R_1 \exp\left(\frac{\sqrt{-\frac{\lambda_3}{\lambda_2}}}{2}(x - 2\kappa\lambda_1 t)\right) \times \exp\left(i\left(\kappa x - \frac{(4\kappa^2\lambda_2 + \lambda_3)\lambda_1}{4\lambda_2}t\right)\right). \tag{3.17}$$

Traveling wave solutions and corresponding density plots of (3.17), in the real of Eq. (3.17) and its corresponding density plots, for both the real and imaginary parts, are shown in Figure 4.

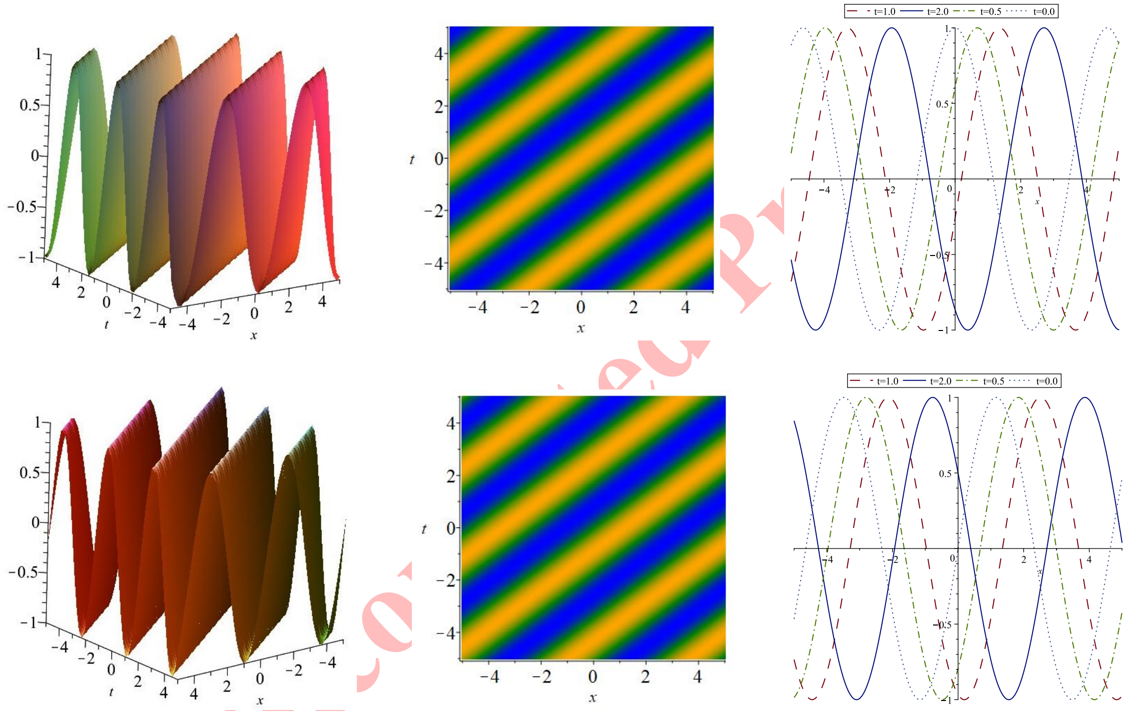

FIGURE 4. 3D, 2D, and density plots of $Re(\psi_4(x,t))$, and $Im(\psi_4(x,t))$, with respect to $\lambda_1 = \lambda_3 = \kappa = D_0 = R_1 = 1$, and $\lambda_2 = 2$.

Case 2.4:

$$D_2 = \sqrt{-\frac{\lambda_4}{6\lambda_2}}, \quad D_1 = 0, \ D_0 = D_0, \ \mu = -3\,\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\kappa^2\lambda_2 + 2D_0^2\lambda_4 - 2D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_3 + \kappa^2\lambda_3\right),$$

$$\lambda_1 = 3\,\frac{\lambda_2}{\lambda_4}\left(8D_0\sqrt{-\frac{\lambda_4}{6\lambda_2}}\lambda_2 + \lambda_3\right).$$

Regarding Φ_1 as the dependent variable and substituting Eq. (3.10) into the first equation of (3.2) gives:

$$\frac{d\Phi_1(\zeta)}{d\zeta} = \sqrt{-\frac{\lambda_4}{6\lambda_2}}\Phi_1^2 + D_0. \tag{3.18}$$

Hence, the exact solution of the ODE is

$$\Phi_1(\zeta) = \Psi(\zeta) = \frac{\sqrt[4]{6}}{\sqrt{-\frac{\lambda_4}{\lambda_2}}} \sqrt{D_0 \sqrt{-\frac{\lambda_4}{\lambda_2}}} \tan\left(\frac{\sqrt{D_0 \sqrt{-\frac{\lambda_4}{\lambda_2}}}}{\sqrt[4]{6}} (R_1 + \zeta) \right),$$

where R_1 is an arbitrary constant. Thus, the final solution is

$$\psi_5(x,t) = \frac{\sqrt[4]{6}}{\sqrt{-\frac{\lambda_4}{\lambda_2}}} \sqrt{D_0 \sqrt{-\frac{\lambda_4}{\lambda_2}}} \tan\left(\frac{\sqrt{D_0 \sqrt{-\frac{\lambda_4}{\lambda_2}}}}{\sqrt[4]{6}} \left(R_1 + x - 6\kappa \frac{\lambda_2}{\lambda_4} \left(8D_0 \sqrt{-\frac{\lambda_4}{6\lambda_2}} \lambda_2 + \lambda_3 \right) t \right) \right)$$
$$\times \exp\left(i \left(\kappa x + 3\frac{\lambda_2}{\lambda_4} \left(8D_0 \sqrt{-\frac{\lambda_4}{6\lambda_2}} \kappa^2 \lambda_2 + 2D_0^2 \lambda_4 - 2D_0 \sqrt{-\frac{\lambda_4}{6\lambda_2}} \lambda_3 + \kappa^2 \lambda_3 \right) t \right) \right). \tag{3.19}$$

The periodic solution corresponding to Eq. (3.19) is shown in Figure 5.

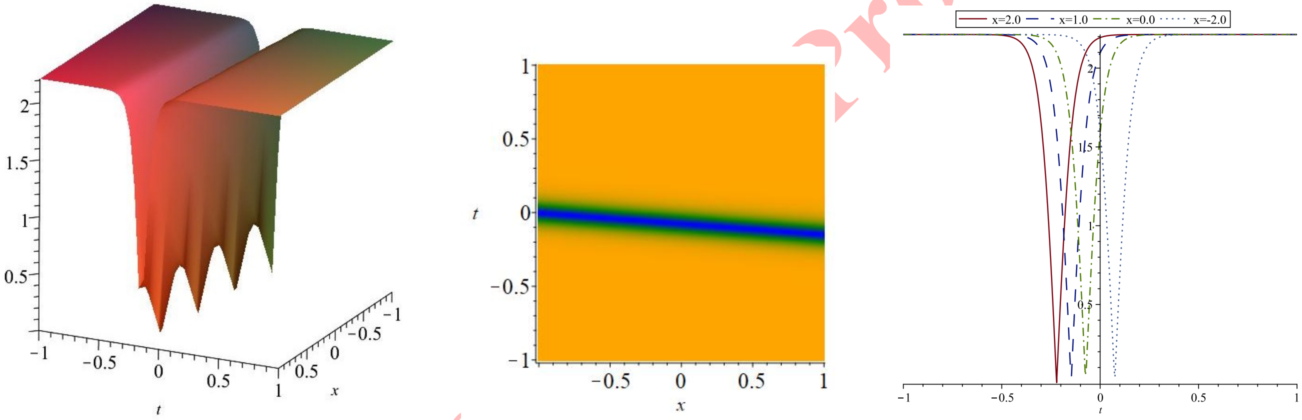

FIGURE 5. 3D, 2D, and density plots of $|\psi_5(x,t)|$, with respect to $\lambda_3 = \lambda_4 = R_1 = 1$, $\lambda_2 = -1$, $D_0 = -2$, and $\kappa = 0.3$.

4. DISCUSSION AND RESULTS

This paper successfully derived and recovered optical soliton solutions to the dispersive concatenation model with linear chromatic dispersion and self-phase modulation. Previous studies have established a foundation by developing concatenation models combining established equations such as the NLSE, LPDE model, and Sasa-Satsuma equation. This work extended these models by incorporating higher-order dispersive effects, introducing equations like the SHE and quintic-order NLSE. Using the improved modified extended tanh function method, various novel solutions were obtained. These solutions include bright, dark, and singular solitons. Furthermore, other mathematical solutions such as Weierstrass elliptic, exponential, rational, and singular periodic solutions. 3D and 2D plots of selected solutions are provided to illustrate the nature of the wave propagation. These solutions are presented with appropriately chosen parameter values. The obtained solitons confirm that a balance is maintained between the nonlinearity and dispersion. These solutions will soliton solutions are highly relevant for the communications industry, as they can propagate over long distances while maintaining their shape and speed, which is a critical property for signal integrity.

5. Conclusion

In this paper, we study a weakly nonlocal NLSE with PL nonlinearity, and NRM is successfully applied to explore the exact solutions. method yielded a variety of solutions, including dark, bright, and singular solitons, as well as traveling wave solutions. The dynamical behavior of the attained solutions is illustrated through 3D and density plots in Figures 1–5 The current work shows that solution extracted by present method are novel and different as compared to some other existing techniques in literature [19, 23]. Lastly, using Maple, our results are verified by back substitution in the original equations. The suggested approach is not only direct and simple but also suitable for constructing new results. In future, NRM can be applied on NLSE having dual power law and perturbed NLSE with kerr law which are very useful to identify the solitons in photo-refractive and polymer materials. It is observed that the proposed technique can potentially be used to implement further models that develop in the fields of natural science that may be adopt to investigate the other mathematical challenges and generated results can be applied to characterize the behavior of nonlinear models.The IMETS was successfully applied in this work to investigate dispersive optical solitons. Many solitons and other solutions were extracted. These solutions including {bright, dark and singular} solitary solutions, Weierstrass elliptic and singular periodic solutions. Moreover, graphical representations in both 2D and 3D of some of the recovered solutions are presented to illustrate the characteristics of the propagating wave. These solutions provide an explanation for a wide range of fascinating and challenging physical phenomena due to the NLSE model's applicability in several scientific domains, including wave-guides and optical fibers. The retrieved solutions in this research study are novel, and the model was not previously investigated using the proposed methodology. The approach's success, convenience of use, and efficacy show the method's applicability for dealing with nonlinear optical problems. With all of these features, it will undoubtedly enrich the literature. The results are thus tremendously promising and lead to the avenues of further research in this arena. Later, the model will be studied with differential group delay followed by the consideration of the model with dispersion-flattened fibers. Additionally, a stochastic version of the model could be investigated to analyze the effect of noise on the soliton solutions.

References

[1] M. J. Ablowitz, *Nonlinear dispersive waves: asymptotic analysis and solitons*, Cambridge University Press, *47* (2011).

[2] A. R. Z. U. Akbulut, M. Mirzazadeh, M. S. Hashemi, K. Hosseini, S. Salahshour, and C. Park, *Triki–Biswas model: Its symmetry reduction, Nucci's reduction and conservation laws*, International Journal of Modern Physics B, *37*(07) (2023), 2350063.

[3] L. Akinyemi, U. Akpan, P. Veeresha, H. Rezazadeh, and M. Inc, *Computational techniques to study the dynamics of generalized unstable nonlinear Schrödinger equation*, Journal of Ocean Engineering and Science, (2022).

[4] L. Akinyemi, M. Şenol, M. Mirzazadeh, and M. Eslami, *Optical solitons for weakly nonlocal Schrödinger equation with parabolic law nonlinearity and external potential*, Optik, *230* (2021), 166281.

[5] G. Akram and H. U. Rehman, *Numerical solution of eighth order boundary value problems in reproducing Kernel space*, Numerical Algorithms, *62*(3) (2013), 527-540.

[6] K. K. Ali, A. M. Wazwaz, and M. S. Osman, *Optical soliton solutions to the generalized nonautonomous nonlinear Schrödinger equations in optical fibers via the sine-Gordon expansion method*, Optik, *208* (2020), 164132.

[7] M. O. Al-Amr, H. Rezazadeh, K. K. Ali, and A. Korkmazki, *N1-soliton solution for Schrödinger equation with competing weakly nonlocal and parabolic law nonlinearities*, Communications in Theoretical Physics, *72*(6) (2020), 065503.

[8] T. An, N. H. M. Shahen, S. N. Ananna, M. F. Hossain, and T. Muazu, *Exact and explicit travelling-wave solutions to the family of new 3D fractional WBBM equations in mathematical physics*, Results in Physics, *19* (2020), 103517.

[9] S. N. Ananna, P. P. Gharami, T. An, and M. Asaduzzaman, *The improved modified extended tanh-function method to develop the exact travelling wave solutions of a family of 3D fractional WBBM equations*, Results in Physics, *41* (2022), 105969.

[10] M. I. Asjad, N. Ullah, H. U. Rehman, and D. Baleanu, *Optical solitons for conformable space-time fractional non-linear model*, (2022).

[11] H. M. Baskonus, M. S. Osman , H. U. Rehman, M. Ramzan, M. Tahir, and S. Ashraf, *On pulse propagation of soliton wave solutions related to the perturbed Chen–Lee–Liu equation in an optical fiber*, Optical and Quantum Electronics, *53* (2021), 1-17.

[12] M. D. Feit, J. A. Fleck Jr, and A. Steiger, *Solution of the Schrödinger equation by a spectral method*, Journal of Computational Physics, *47*(3) (1982), 412-433.

[13] R. K. Gazizov and A. A. Kasatkin, *Construction of exact solutions for fractional order differential equations by the invariant subspace method*, Computers & Mathematics with Applications, *66*(5) (2013), 576-584.

[14] V. Gokulakrishnan and R. Srinivasan, *Impulsive effects on stabilization of stochastic nonlinear reaction-diffusion systems with time delays and boundary feedback control*, J Math Comput Sci, *28*(4) (2023), 350-362.

[15] M. S. Hashemi, *A novel approach to find exact solutions of fractional evolution equations with non-singular kernel derivative*, Chaos, Solitons & Fractals, *152* (2021), 111367.

[16] M. S. Hashemi and D. Baleanu, *Lie symmetry analysis of fractional differential equations*, Chapman and Hall/CRC, (2020).

[17] M. S. Hashemi, M. Inc, B. Kilic, and A. Akgül, *On solitons and invariant solutions of the Magneto-electro-elastic circular rod*, Waves in Random and Complex Media, *26*(3) (2016), 259-271.

[18] M. S. Hashemi, M. C. Nucci, and S. Abbasbandy, *Group analysis of the modified generalized Vakhnenko equation*, Communications in Nonlinear Science and Numerical Simulation, *18*(4) (2013), 867-877.

[19] K. Hosseini, S. Salahshour, and M. Mirzazadeh, *Bright and dark solitons of a weakly nonlocal Schrödinger equation involving the parabolic law nonlinearity*, Optik, *227*(2021), 166042.

[20] L. Gagnon and P. Winternitz, *Lie symmetries of a generalised nonlinear Schrodinger equation: I. The symmetry group and its subgroups*, Journal of Physics A: Mathematical and General, *21*(7) (1988), 1493.

[21] M. A. Iqbal, Y. Wang, M. M. Miah, and M. S. Osman, *Study on date–Jimbo–Kashiwara–Miwa equation with conformable derivative dependent on time parameter to find the exact dynamic wave solutions*, Fractal and Fractional, *6*(1) (2021), 4.

[22] N. A. Kudryashov, *Highly dispersive solitary wave solutions of perturbed nonlinear Schrödinger equations*, Applied Mathematics and Computation, *371* (2020), 124972.

[23] W. Liu, Y. Zhang, Z. Luan, Q. Zhou, M. Mirzazadeh, M. Ekici, and A. Biswas, *Dromion-like soliton interactions for nonlinear Schrödinger equation with variable coefficients in inhomogeneous optical fibers*, Nonlinear Dynamics, *96* (2019), 729-736.

[24] M. Mirzazadeh, A. Sharif, M. S. Hashemi, A. Akgül, and S. M. El Din, *Optical solitons with an extended (3+ 1)-dimensional nonlinear conformable Schrödinger equation including cubic–quintic nonlinearity*, Results in Physics, *49* (2023), 106521.

[25] M. C. Nucci and P. G. L. Leach, *The determination of nonlocal symmetries by the technique of reduction of order*, Journal of mathematical analysis and applications, *251*(2)(2000), 871-884.

[26] M. S. Osman, D. Baleanu, A. R. Adem, K. Hosseini, M. Mirzazadeh, and M. Eslami, *Double-wave solutions and Lie symmetry analysis to the (2+ 1)-dimensional coupled Burgers equations*, Chinese Journal of Physics, *63* (2020), 122-129.

[27] M. N. Rasoulizadeh, M. J. Ebadi, Z. Avazzadeh, and O. Nikan, *An efficient local meshless method for the equal width equation in fluid mechanics*, Engineering Analysis with Boundary Elements, *131* (2021), 258-268.

[28] H. U. Rehman, I. Iqbal, S. Subhi Aiadi, N. Mlaiki, and M. S. Saleem, *Soliton solutions of Klein–Fock–Gordon equation using Sardar subequation method*, Mathematics, *10*(18)(2022), 3377.

[29] H. U. Rehman, N. Ullah, and M. A. Imran, *Highly dispersive optical solitons using Kudryashov's method*, Optik, *199*(2019), 163349.

[30] R. Sahadevan and P. Prakash, *On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations*, Chaos, Solitons & Fractals, *104* (2017), 107-120.

[31] F. L. Xia, F. Jarad, M. S. Hashemi, and M. B. Riaz, *A reduction technique to solve the generalized nonlinear dispersive mK (m, n) equation with new local derivative*, Results in Physics, *38* (2022), 105512.

[32] Y. Xie, L. Li, and Y. Kang, *New solitons and conditional stability to the high dispersive nonlinear Schrödinger equation with parabolic law nonlinearity*, Nonlinear Dynamics, *103*(1) (2021), 1011-1021.

[33] A. Zafar, M. Raheel, M. Asif, K. Hosseini, M. Mirzazadeh, and L. Akinyemi, *Some novel integration techniques to explore the conformable M-fractional Schrödinger-Hirota equation*, Journal of Ocean Engineering and Science, *7*(4) (2022), 337-344.

[34] Q. Zhou, D. Yao, S. Ding, Y. Zhang, F. Chen, and X. Liu, *Spatial optical solitons in fifth order and seventh order weakly nonlocal nonlinear media*, Optik, *124*(22) (2013), 5683-5686.