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Optical solitons and stability analysis for the improved Eckhaus equations

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Abstract

In this article, the propagation of modulated waves in one and two dimensional systems are analyzed by investigating the improved Eckhaus models analytically. Along with additional dimensions, dissipative factors, nonlocal effects, and higher-order nonlinear elements, the enhanced Eckhaus equation expands the original Eckhaus equation. The investigation of the governing models' optical soliton solutions, including periodic, dark, brilliant, and singular solitons, is the focus of this article. This is done by obtaining a novel optical solution using the tanh-coth approach. Another type that incorporates nonlinearity and modulation effects in both spatial dimensions, and includes an extra spatial dimension, is the $(2 + 1)$-dimensional enhanced Eckhaus model. These equations are effective resources for examining a wide range of one- and two-dimensional system physical phenomena, including pattern generation, wave interaction, and soliton dynamics. Analyzing these equations can be challenging due to their higher dimensionality and nonlinear nature and numerical methods are often used to obtain solutions for specific cases or conditions. Consequently, trigonometric function solutions, hyperbolic function output and exponential functions solution with Independent parameters are acquired.3D and 2D contour plots of some solutions of the nonlinear model are specified. These governing equations have some applications in domains like nonlinear optics, condensed matter physics and fluid dynamics.

Keywords. The improved Eckhaus model, Optical solitons, Exact solutions, Tanh-coth method, Stability analysis, Nonlinear dynamics.
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1. Introduction

The inspection of new wave solution structures to nonlinear partial differential equations (NPDEs) performs a significant role in the analysis of nonlinear physical processes. Such structures characterizes the evolution of a system that reveals nonlinear nature [2, 3, 40–42, 48, 49, 53]. Such models emerge in numerous fields of (engineering and science) such as, nonlinear optics, hydrodynamic stability, fluid dynamics, materials science and natural philosophy. NPDEs describe a wide array of complicated fact, including chaos, structuring method and solitary wave propagation [12, 13, 15, 34, 39, 51]. Some analytical strategies have been used to examine nonlinear phenomenon emerging in various contemporary fields of research involving perturbation hypothesis, numerical methods and variational methods. Certain general kinds of NPDEs include the Korteweg-de Vries equation [17], the nonlinear Schrödinger equation [14], the reaction-diffusion equations [31] and the Navier-Stokes equations [19]. These equations describe a vast variety of physical processes, namely the behavior of fluids, the dynamics of decomposition and dissemination of waves [11, 22, 43].

The original Eckhaus equation is a one-dimensional equation that describes the propagation of modulated waves in a medium. However, in many cases, the dynamics of wave propagation are affected by other factors such as nonlinearities, dissipation, nonlocal interactions or higher dimensions [20, 23, 27, 35]. To account for these factors, an improved Eckhaus equation has been developed that includes higher-order nonlinear terms, dissipative terms,

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nonlocal effects, and higher-dimensions. The $(2 + 1)$-dimensional enhanced Eckhaus equation is an extra variation of the enhanced Eckhaus equation that considers an additional spatial dimension and combines nonlinearity and modulation effects in both spatial directions. Due to their higher dimensionality and non-linearity, these equations can be challenging. Several different numerical techniques, including spectral techniques, grid-based approaches, and finite difference techniques, are widely used to analyze the dynamics and identify solutions for specific scenarios [6, 26, 30, 36, 37].

The enhanced Eckhaus equation is a useful tool for studying the dynamics of modulated light waves in nonlinear settings, especially in optical fibers [18]. It makes it possible to research the production of solitons and modulation instability, two topics with important applications in signal processing and optical communication [7, 25, 28, 47]. Some outstanding studies have examined and provided a range of information on improved and $(2 + 1)$-dimensional improved Eckhaus equations, including their derivation, analytical and numerical solutions, and applications in many domains such as optics and oceanography [16, 54, 55]. Furthermore covered are related topics like the inverse scattering transform and soliton dynamics, which are necessary to understand the occurrence of nonlinear waves in such systems. The dynamical technique and system of complex acoustic events help to verify the stability of solutions for NPDEs [32].

Converting PDEs into ODEs has been demonstrated to be a viable method in recent years for generating exact solutions for non-linear wave equations [5]. Various methodologies are documented in the literature for obtaining precise solutions, such as the (G'/G)-expansion method [52]. The approaches used in this study include the hyperbolic tangent and extended hyperbolic tangent methods [1], the Jacobi elliptic function method [29], the homogeneous balancing method [50], the first-integral method [24], the sine-cosine method [10], the exp-function method [21], and several others. This paper has investigated Soliton-based solutions for the enhanced Eckhaus problem using Hirota's bilinear approach [56]. The article gives explicit soliton solutions and examines their features, such as interactions and stability. These mathematical approaches offer essential tools for analyzing and comprehending the dynamics of the enhanced Eckhaus equation. Researchers can get precise soliton solutions, investigate the parameter space of the equation, analyze stability features, and gain an understanding of the production and interactions of modulated waves in nonlinear media [4, 33, 38, 46].

This study focuses on the use of the tanh-coth approach to get novel solutions for the enhanced Eckhaus equation and the enhanced $(2 + 1)$-dimensional Eckhaus equation. The purpose of this endeavor is to find the optical Soliton solutions to the governing equations by careful observation and analysis. This strategy is employed to enhance the calculations that were previously accomplished. The answers obtained would be highly desired in elucidating specific intriguing physical phenomena to physicists.

The motivation behind our work lies in the need to extend our understanding of the intricate behavior presented by the improved Eckhaus model and its (2+1)-dimensional Eckhaus model. The tanh-coth method is applied to extract exact optical solutions and by stability analysis, we illuminate the mechanisms used for governing soliton dynamics. Our stability analysis proves that the obtained solutions are not only mathematically valid but also effective against perturbations, which is important for their real world applications. Our research gives to contribute to the growing body of knowledge in nonlinear dynamics and enhance computational methods that improve the analysis of complex mathematical models.

The paper is divided into the following sections. In the upcoming section, the approach of employing a systematic procedure to get precise solutions for NPDEs is explained. Section 3 provides applications of these methods to showcase the newly derived solutions of the enhanced Eckhaus problem and the $(2+1)$-dimensional enhanced Eckhaus equation. Ultimately, the findings and deductions are presented in the concluding section of this document.

2. The tanh-coth method

Assume the NLPDE has the following set up,

$$H(\Psi, \Psi_t, \Psi_x, \Psi_{tt}, \Psi_{xx}, \Psi_{xt}, ...) = 0, \tag{2.1}$$

where H is a polynomial function and Ψ is a wave function in variables x, y and t. Introduce the following transformation,

$$\Psi = \Psi(x,t) = \chi(\zeta), \quad \zeta = \alpha x - \upsilon t. \tag{2.2}$$

By shifting Eq. (2.2) into Eq. (2.1), an ODE of the following form is obtained,

$$J(\Psi, \Psi', \Psi'', \Psi''', ...) = 0, \tag{2.3}$$

where the sign $'$ expresses the derivatives of $\Psi(\zeta)$ with respect to ζ. By [8], Wazwaz has abridged for utilizing tanh-coth technique. This technique has the following steps:
Step 1. By employing a wave variable $\zeta = \alpha x - \upsilon t$, the Eq. (2.3) is anciently unified since all expressions presumes derivatives from that integration parameters are considered zeros.Standard tanh method has created by (Malfliet 1992), such as tanh or coth are accomplished as novel variable and derivatives of tanh or coth are presented by tanh or coth. Considering a novel independent variable

$$V = \tanh\gamma\zeta, \quad or \quad V = \coth\gamma\zeta, \quad \zeta = bx - \upsilon t. \tag{2.4}$$

where γ is a wave number, yields to the alteration of derivatives:

$$\begin{aligned} \frac{d}{d\zeta} &= \gamma(1 - V^2)\frac{d}{dV}, \\ \frac{d^2}{d\zeta^2} &= -2\gamma^2 V(1 - V^2)\frac{d}{dV} + \gamma^2(1 - V^2)^2\frac{d^2}{dV^2}. \end{aligned} \tag{2.5}$$

Step 2. The (tanh-coth) method allows the exertion of finite expansion

$$\chi(\gamma\zeta) = h(V) = \sum_{r=0}^{N} a_k V^r + \sum_{r=1}^{N} b_k V^{-r} \text{ and } \quad V' = \gamma(1 - V^2). \tag{2.6}$$

Step 3. In the majority of cases, N is +ve integer that will be examined by homogeneous balancing criterion. Putting Eq. (2.6) into the ODE, an algebraic formulation in powers of V is obtained.
Step 4. By gathering all coefficients of similar powers of V^j, the resultant equation will indicate a system of algebraic expression containing the constants a_k, b_k and γ. Inspected these constants by solving the system of algebraic equations to gain a analytic solution $\zeta(x,t)$ in closed formation.

3. The improved Eckhaus equation

Consider the improved Eckhaus equation [29] to investigate the exact solutions

$$\iota q_t + q_{xx} + 2(|q|^2)_{xx} q + |q|^4 q = 0, \tag{3.1}$$

where q=$q(x,t)$ is a (complex-valued) function x is the contiguous variable and also t is time. The expression q_{xx} shows the second contiguous derivative of q, and q_x shows the first contiguous derivative of q.
Applying the wave variable

$$q(x,t) = \chi(\zeta)e^{\iota\varphi}, \qquad \zeta = x - 2\alpha t, \;\; \varphi = \alpha x + \upsilon t, \tag{3.2}$$

where $\chi(\zeta)$ is real function, the parameters α, υ are to be resolved. Shifting Eq. (3.2) into Eq. (3.1), we get following terms

$$\begin{aligned} q_x &= (\chi' + \iota\alpha\chi)e^{\iota\varphi}, \\ q_{xx} &= (\chi'' + 2\iota\alpha\chi' - \alpha^2\chi)e^{\iota\varphi}, \\ (|q|^2)_{xx} &= 2((\chi')^2 + \chi''\chi). \end{aligned} \tag{3.3}$$

By putting all terms in (3.1), we have

$$\chi'' - (\alpha^2 + \upsilon)\chi + 2(\chi^2)''\chi + \chi^5 = 0, \tag{3.4}$$

Hence, the following ODE representation is obtained,

$$\chi'' - (\alpha^2 + \upsilon)\chi + 4(\chi')^2\chi + 4\chi''\chi^2 + \chi^5 = 0, \tag{3.5}$$

where the prime indicates differentiation with respect to ζ. The solution of Eq. (3.5) takes the following form after the balance process N=1,

$$\chi(\gamma\zeta) = h(I) = a_0 + a_1 I, \quad I = \tanh\gamma\zeta \ \text{ or } \ \coth\gamma\zeta. \tag{3.6}$$

To determine the acquired algebraic system with unknowns a_0, a_1, υ and γ. The new solutions of the improved Eckhaus Eq. (3.1) are analyzed in certain solution sets and taking all values in Eq. (2.6) within Eq. (2.4) are as precedes:
Case 1. $\{\gamma = -\frac{3}{4\sqrt{2}}, \upsilon = \frac{9-16\alpha^2}{16}, a_0 = 0, a_1 = -\frac{\sqrt{3}}{2}\}$.
The solution of the improved Eckhaus Eq. (3.1) is

$$q_1(x,t) = \frac{\sqrt{-3}}{2}\tanh\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}, \tag{3.7}$$

$$q_2(x,t) = \frac{\sqrt{-3}}{2}\coth\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}. \tag{3.8}$$

Case 2. $\{\gamma = -\frac{3}{4\sqrt{2}}, \upsilon = \frac{9-16\alpha^2}{16}, a_0 = 0, a_1 = \frac{\sqrt{3}}{2}\}$.
The solution of the improved Eckhaus Eq. (3.1) is

$$q_3(x,t) = -\frac{\sqrt{-3}}{2}\tanh\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}, \tag{3.9}$$

$$q_4(x,t) = -\frac{\sqrt{-3}}{2}\coth\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}. \tag{3.10}$$

Case 3. $\{\gamma = \frac{3}{4\sqrt{2}}, \upsilon = \frac{9-16\alpha^2}{16}, a_0 = 0, a_1 = -\frac{\sqrt{3}}{2}\}$.
The solution of the improved Eckhaus Eq. (3.1) is

$$q_5(x,t) = -\frac{\sqrt{-3}}{2}\tanh\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}, \tag{3.11}$$

$$q_6(x,t) = -\frac{\sqrt{-3}}{2}\coth\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}. \tag{3.12}$$

Case 4. $\{\gamma = \frac{3}{4\sqrt{2}}, \upsilon = \frac{9-16\alpha^2}{16}, a_0 = 0, a_1 = -\frac{\sqrt{3}}{2}\}$.

The solution of improved Eckhaus Eq. (3.1) is

$$q_7(x,t) = -\frac{\sqrt{-3}}{2}\tanh\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}, \tag{3.13}$$

$$q_8(x,t) = -\frac{\sqrt{-3}}{2}\coth\left(\frac{3(x-2\alpha t)}{4\sqrt{2}}\right)e^{\iota(x-2\alpha t)}. \tag{3.14}$$

4. The $(2+1)$-dimensional improved Eckhaus equation

The $(2+1)$-dimensional enhanced Eckhaus equation is discussed in this section [44].

$$\iota q_t + q_{xx} - q_{yy} + 2(|q|^2)_{xx}q + |q|^4 q = 0. \tag{4.1}$$

Applying the wave variable

$$q(x,y,t) = e^{\iota\varphi}\chi(\zeta), \quad \zeta = x + cy + dt, \quad \varphi = \alpha x + \beta y + \upsilon t, \tag{4.2}$$

where $\chi(\zeta)$ is a real function and the $\alpha, \beta, \upsilon, c, d$ are real parameters. Inserting Eq. (4.2) into Eq. (4.1), we obtain the relation $d = -2(\alpha - c\beta)$, then Eq. (4.1) has following nonlinear ODE

$$(1-c^2)\chi'' + (\beta^2 - \alpha^2 - \upsilon)\chi + 2(\chi^2)''\chi + \chi^5 = 0, \tag{4.3}$$

Hence, the following ODE representation is obtained,

$$(1-c^2)\chi'' + (\beta^2 - \alpha^2 - \upsilon)\chi + 4(\chi')^2\chi + 4\chi''\chi^2 + \chi^5 = 0. \tag{4.4}$$

where the prime indicates differentiation with respect to ζ. The solution of Eq. (4.4) takes the following form after the balance process N=1,

$$\chi(\gamma\zeta) = h(I) = a_0 + a_1 I, \quad I = \tanh\gamma\zeta, \ \text{ or } \ \coth\gamma\zeta. \tag{4.5}$$

To determine the acquired algebraic system with unknowns a_0, a_1, υ and γ. The new solutions of $(2+1)$-dimensional improved Eckhaus Eq. (4.1) are analyzed in certain solution sets and taking all values in Eq. (2.6) within Eq. (2.4) are as precedes:

Case 1.
$\{\gamma = -\frac{3+2c-c^2}{\sqrt{32+64c-32c^2}}, \upsilon = \frac{1}{16}(9+12c-2c^2-4c^3+c^4-16\alpha^2+16\beta^2), a_0 = 0, a_1 = -\frac{1}{2}\sqrt{-3-2c+c^2}\}$.
The solution of the Eq. (4.1) is found as

$$q_1(x,t) = \left(\frac{1}{2}\sqrt{-3-2c+c^2}\tanh\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}, \tag{4.6}$$

$$q_2(x,t) = \left(\frac{1}{2}\sqrt{-3-2c+c^2}\coth\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}. \tag{4.7}$$

Case 2.
$\{\gamma = -\frac{3+2c-c^2}{\sqrt{32+64c-32c^2}}, \upsilon = \frac{1}{16}(9+12c-2c^2-4c^3+c^4-16\alpha^2+16\beta^2), a_0 = 0, a_1 = \frac{1}{2}\sqrt{-3-2c+c^2}\}$.
The solution of the Eq. (4.1) is observed as

$$q_3(x,t) = \left(-\frac{1}{2}\sqrt{-3-2c+c^2}\tanh\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}, \tag{4.8}$$

$$q_4(x,t) = -\left(\frac{1}{2}\sqrt{-3-2c+c^2}\coth\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}. \tag{4.9}$$

Case 3.
$\{\gamma = \frac{3+2c-c^2}{\sqrt{32+64c-32c^2}}, \upsilon = \frac{1}{16}(9+12c-2c^2-4c^3+c^4-16\alpha^2+16\beta^2), a_0 = 0, a_1 = -\frac{1}{2}\sqrt{-3-2c+c^2}\}$.
The solution of the Eq. (4.1) is given as

$$q_5(x,t) = -\left(\frac{1}{2}\sqrt{-3-2c+c^2}\tanh\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}, \tag{4.10}$$

$$q_6(x,t) = -\left(\frac{1}{2}\sqrt{-3-2c+c^2}\coth\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}. \tag{4.11}$$

Case 4.
$\{\gamma = \frac{3+2c-c^2}{\sqrt{32+64c-32c^2}}, \upsilon = \frac{1}{16}(9+12c-2c^2-4c^3+c^4-16\alpha^2+16\beta^2), a_0 = 0, a_1 = \frac{1}{2}\sqrt{-3-2c+c^2}\}$.
The solution of the Eq. (4.1) is of the form

$$q_7(x,t) = \left(\frac{1}{2}\sqrt{-3-2c+c^2}\tanh\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}, \tag{4.12}$$

$$q_8(x,t) = \left(\frac{1}{2}\sqrt{-3-2c+c^2}\coth\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)e^{\iota(x+cy+t(\alpha-c\beta))}. \tag{4.13}$$

5. Stability analysis

As the solution of improved $(2+1)$-dimensional Eckhaus Eq. (4.1) is the hamiltonian structure. By the momentum for Eq. (4.1) is represented as [45],

$$P=\frac{1}{2}\int_{-\infty}^{\infty} q^2(\zeta)d\zeta, \tag{5.1}$$

where P is the momentum and q is the electric potential. The stability state of a solitary wave is

$$\frac{\partial P}{\partial \lambda}>0, \tag{5.2}$$

here λ is the frequency. Substituting Eq. (4.6) into Eq. (5.1), following value of momentum P is obtained,

$$P=\int_{-20}^{20}\kappa\left(\frac{1}{2}\sqrt{-3-2c+c^2}\tanh\left(\frac{(3+2c-c^2)(x+cy-2t(\alpha-c\beta))}{\sqrt{32+64c-32c^2}}\right)\right)^2 d\zeta. \tag{5.3}$$

This implies the following expression after evaluation of the above integral,

$$P=\frac{\kappa}{4}(-3c-2c^2+c^3)[-\text{sech}\left(\frac{(-3-2c+c^2)(20+cy-4\alpha+4c\beta)}{4\sqrt{2+4c-2c^2}}\right)^2 + \text{sech}\left(\frac{(-3-2c+c^2)(-20-4\alpha+cy+4c\beta)}{4\sqrt{2+4c-2c^2}}\right)^2].$$

Now,take $y=2$ and utilizing the stability state examined in Eq. (5.2), it is obtained that

$$\frac{\kappa}{4}(-3c-2c^2+c^3)[-\text{sech}\left(\frac{(-3-2c+c^2)(20+2c-4\alpha+4c\beta)}{4\sqrt{2+4c-2c^2}}\right)^2 + \text{sech}\left(\frac{(-3-2c+c^2)(-20-4\alpha+2c+4c\beta)}{4\sqrt{2+4c-2c^2}}\right)^2]>0.$$

Therefore, Eq. (4.1) is stable nonlinear model provided that the above inequality holds.

6. Results and discussion

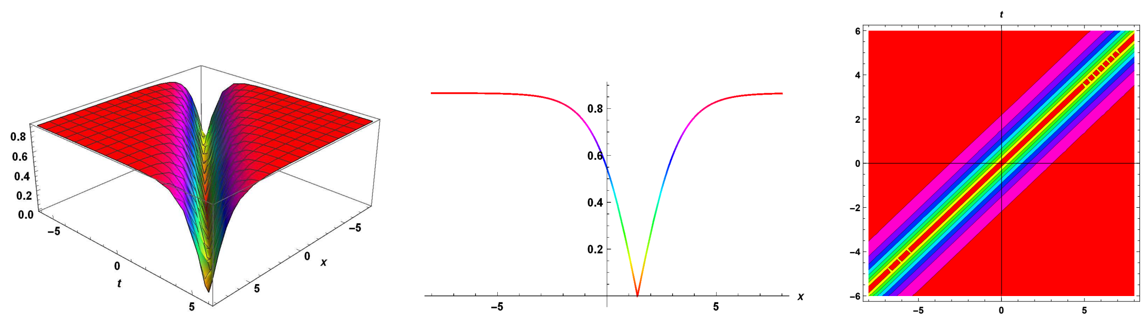
In this section, the improved Eckhaus and the improved $(2+1)$-dimensional Eckhaus model often possess solutions that exhibits complex and diverse behavior, including solitons, solitary waves, breathers and other localized structures. These solutions has displayed stability, instability or exhibit interesting dynamics under various conditions. Some general effects that are arise from solutions to these nonlinear equations such as solitons, solitary waves, scattering, stability, instability, nonlinear superposition and chaotic behavior. These nonlinear equations has admitted soliton solutions, which can arise due to the balance between dispersion and nonlinearity. By employing the particular parameters of these equations, the solutions are located optical soliton including periodic, dark, bright and singular solitons of this governing equations. We selected the improved Eckhaus model from existing literature due to its extensive investigation in past studies. This existing work has offered a solid foundation, and as a result, we faced minimal difficulties in modeling the equations.

In our study, we employ the tanh-coth method to extract exact optical solutions for both the improved Eckhaus model and its (2+1)-dimensional Eckhaus model, distinguishing our work from previous research. Notably, while prior studies have utilized methods such as the improved tanh method and the Bernoulli subequation method, which are often limited to (1+1)-dimensional equations while our approach expands the dimensional scope by investigating the both (1+1) and (2+1) equations . Additionally, we have also provide a stability analysis to ensure the effectiveness of the obtained solutions. By demonstrating the effectiveness of the tanh-coth method, our study offers valuable insights into the dynamics of nonlinear systems to highlights the practical applications of these solutions in fields such as industrial studies, biomedical development,and space plasma physics.

However, we acknowledge that one potential risk associated with our methodology is the possibility of encountering failures in the applied methods. While our selected technique has been proven effective in various contexts, there are possibilities in which the obtained solutions may not satisfy constraint or stability conditions, highlights the necessity of careful verification and validation of our results to ensure their reliability.

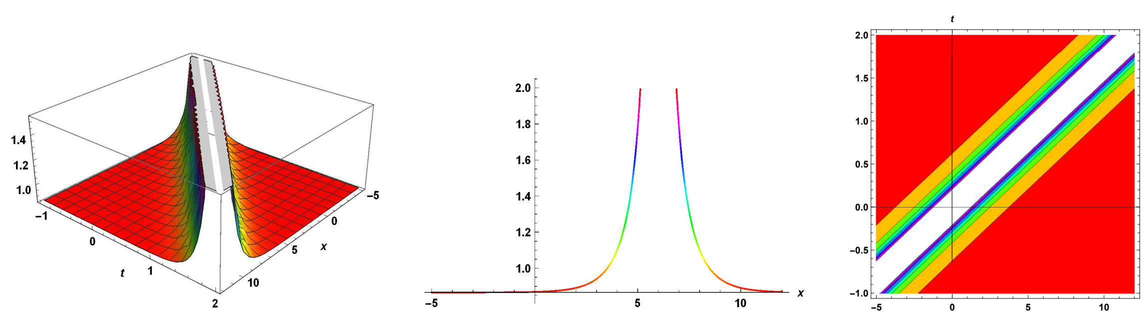
One can distinguish the new solutions of the Eckhaus improved model and improved Eckhaus model are depicted. A variety of some graphical illustrations of 3D, 2D and contour plots with arbitrary free constants are observed in tanh-coth method. For the improved Eckhaus Eq. (3.4), Figure 1 shows $|q_3(x,t)|$ indicated in Case 2 with free parameters $\alpha = 0.2, \upsilon = 3, \lambda = -2, \varepsilon = 1$ and Figure 2 illustrates $|q_6(x,t)|$ displayed in Case 3 with free parameters $\alpha = 3, \upsilon = 2, \lambda = -1, \varepsilon = 1$. For the $(2+1)$-dimensional improved Eckhaus Eq. (4.12), Figure 3 shows $|q_1(x,t)|$ indicated in Case 1 with free parameters $\alpha = 0.3, \beta = 1.2, \upsilon = 1, c = 0.4, y = -0.1, \lambda = -2, \varepsilon = 2$, and Figure 4 illustrated $|q_4(x,t)|$ displayed in Case 2 with free parameters $\alpha = 0.1, \beta = 2.2, \upsilon = 1, c = 0.5, y = -0.2, \lambda = -1, \varepsilon = 1$.

The physical implications of singular and dark solitons are significant across different fields. Dark solitons are characterized by localized drops in amplitude and are significant in fluid dynamics for modeling wave depressions, in optical fibers for distortion-free data transmission and in Bose-Einstein condensates (BECs) as density dips to offer insights into quantum fluid dynamics. They also appear in plasma physics as electric field voids and help in controlled plasma wave propagation. In contrast, singular solitons show extreme phenomena like shock waves or intensity spikes, found in nonlinear optics , fluid mechanics, and plasma systems (ion-acoustic shock waves). These solitons are useful for understanding critical breakdown phenomena and the dynamics of extreme nonlinear systems such as potential applications in cosmology for modeling singularities like black holes.



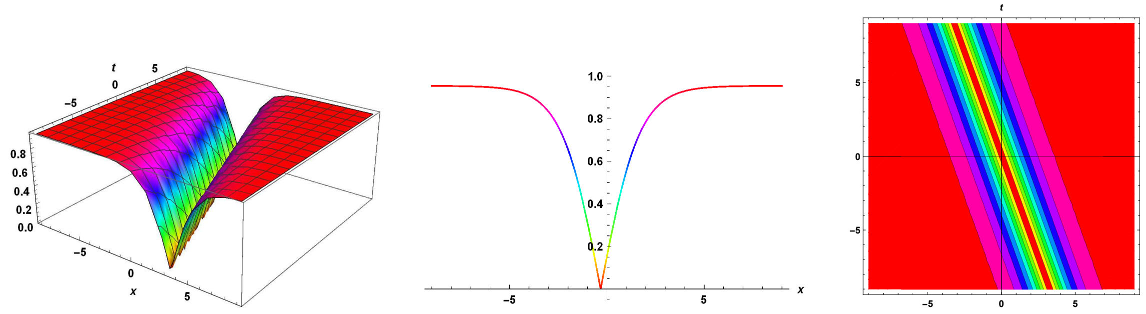
(a) (b) (c)

Figure 1. The 3D, 2D, and contour plots of $q_3(x,t)$ in (3.9) are generated for the parameter values $\alpha = 0.2, \upsilon = 3, \lambda = -2, \varepsilon = 1$. Also, -8 $\leq x \leq$ 8 and -6 $\leq t \leq$ 6.



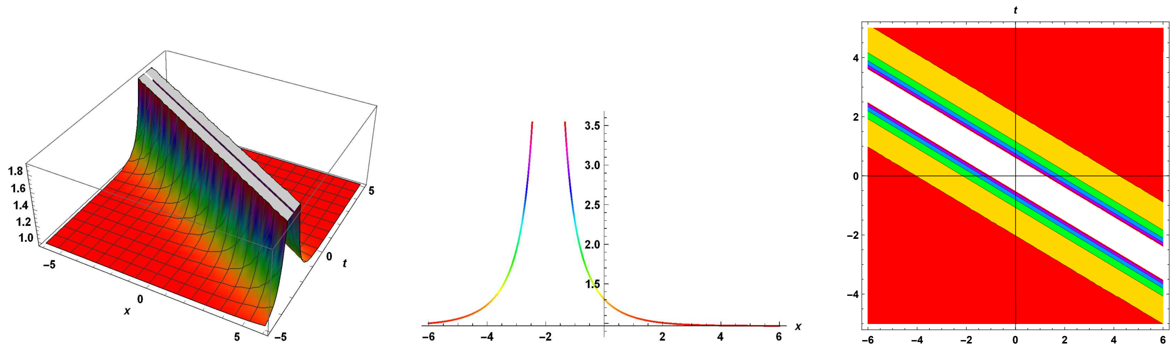
(a) (b) (c)

Figure 2. The 3D, 2D, and contour plots of $q_6(x,t)$ in (3.12) are generated for the parameter values $\alpha = 3, \upsilon = 2, \lambda = -1, \varepsilon = 1$, with $-5 \leq x \leq 12$ and $-1 \leq t \leq 2$.



(a) (b) (c)

FIGURE 3. The 3D, 2D, and contour plots of $q_1(x,t)$ in (4.6) are generated for the parameter values $\alpha = 0.3, \beta = 1.2, \upsilon = 1, c = 0.4, y = -0.1, \lambda = -2, \varepsilon = 2$, with $-9 \leq x \leq 9$ and $-9 \leq t \leq 9$.



(a) (b) (c)

FIGURE 4. The 3D, 2D, and contour plots of $q_4(x,t)$ in (4.9) are generated for the parameter values $\alpha = 0.1, \beta = 2.2, \upsilon = 1, c = 0.5, y = -0.2, \lambda = -1, \varepsilon = 1$, with $-6 \leq x \leq 6$ and $-5 \leq t \leq 5$.

7. Conclusion

Our research successfully derives exact optical solutions for the improved Eckhaus model and its (2+1)-dimensional counterpart using the tanh-coth method. The soliton solutions identified, including optical, singular, combined, dark and singular solitons, illustrate the rich dynamics and behaviors of nonlinear systems. The stability analysis further proves the validation of these results and ensures their implications in real world contexts. However, we acknowledge a significant limitation of our work is the absence of periodic solutions., while our study offers valuable knowledge into the nature of solitons, the investigation of periodic solutions remains an important area for future investigation. We acknowledge that extending our study to include periodic solution to enhance our understanding of the improved Eckhaus model and its implications across various scientific fields. We also intend to explore the stochastic and fractional versions of the governing model. By exploring these extensions, we will find the new dynamics and soliton behaviors that may arise due to stochastic influences and fractional derivatives . This future work will not only deepen our understanding of nonlinear dynamics but also broaden the applicability of our findings to more complex systems encountered in various scientific and engineering fields. Our results contribute to the knowledge on nonlinear dynamics and provide a foundation for future in both fractional and stochastic domains.

Declarations

Availability of data and materials Not applicable.
Conflict of interest The authors declare that they have no conflict of interest.
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Authors' contributions

(1) Hina Zulfiqar: Methodology, visualization.
(2) Kalim U. Tariq: Conceptualization, supervision, resources.
(3) Hamood Ur Rehman: Scientific computation, review and editing.
(4) Taiba Kouser: validation, project administration.
(5) Waqar Ahmad: Formal analysis and investigation, writing original draft.

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