Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-11
DOI:10.22034/cmde.2024.60925.2604

The $(4\alpha - \rho)$ order Sturm-Liouville problem with generalized fractional derivative

Mohammad Jafari[1,2,*] **and Farhad Dastmalchi Saei**[2]
[1]Department of Science, Payame Noor University, Tehran, Iran.
[2]Department of Mathematics Faculty of Science Tabriz Branch, Islamic Azad University, Tabriz, Iran.

Abstract

In this paper we analyze the generalized fractional derivative with two parameters for fourth-order Sturm–Liouville problems. These parameters are α(*the fractional order*) and ρ(*a real number*). In the following, we discuss five different forms of Sturm–Liouville problems, which are solved using the $\rho-$Laplace transform.

Keywords. Fractional derivative, The Laplace transform, Sturm-Liouville problems.
2010 Mathematics Subject Classification. 26A33, 34A08, 33E12.

1. Introduction

Sturm-Liouville problems (SLPs) or eigenvalue problems for ordinary and partial differential equations are part of the foundation of classical applied mathematics and mathematical physics and were introduced about 170 years ago. The SLPs arise naturally in solving technical problems in engineering, physics, and more recently in biology and the social sciences. Examples include accurate estimates of the decay (or growth rates) of solutions resulting from heat conduction, concentration analyses, flows in porous media, etc. For vibration problems, they give fundamental frequencies and overtones of musical instruments. Eigenvalue problems are important for determining the critical mass for nuclear reactions in a given geometry and arise naturally in optimization and in the calculus of variations [2, 14, 16].

Based on the idea of a fractional derivative, fractional calculus extends the scope of classical calculus. In a letter to Leibniz, L'Hôpital first introduced the idea of a derivative by asking how $\frac{d^n f}{dx^n}$ could be defined in the case where $n = \frac{1}{2}$. The concept of fractional derivative has been defined in a number of ways, and new definitions are continuously being proposed. The definitions of fractional derivatives by Riemann-Liouville, Gerbashian-Caputo, Grunwald-Letnikov, and Rich-Fisher are the most widely used ones.

Due to the rapid spread of the theory of fractional derivatives and the importance of Sturm-Liouville problems in differential equations, serious studies have been carried out in this area, which have led to significant results. It is important to remember that fractional Sturm-Liouville problems (FSLPs) have been thoroughly studied over the past few decades. The FSLPs arise in many areas, including mechanics, electricity, biology, chemistry, control theory, and economics [3, 4, 9, 11, 12, 15].

In this study, fourth-order Sturm-Liouville problems are considered in five different forms with a new generalized fractional derivative (GFSL). This is a fractional derivative with two parameters. The parameters are α, fractional order, and ρ, a real number. As we will see shortly, α is used to change the structure of the solutions, while ρ is only used to move them.

Below we find the eigenvalues and eigenfunctions of the problems. Furthermore, we observe that when α tends to 1, no matter what ρ is, all solutions agree with each other. On the other hand, if both α and ρ tend to 1, the solutions correspond to solving a fourth-order Sturm-Liouville problem.

Received: 15 March 2024 ; Accepted: 26 January 2025.
∗ Corresponding author. Email: m_jafari@pnu.ac.ir .

The outline of this paper is as we now describe. In section 2, we discuss those concepts and results from the theory of generalized fractional derivatives that will be use in our main results. In the third section, we find solutions to five different forms of GFSL problems, and then we provide analytic solutions to these problems. Finally we present a brief conclusion that describes our achievements.

2. Preliminaries

Definition 2.1. [13]. Suppose that α denote a positive real number and a positive integer, such that $n-1 \leq \alpha < n,\ n \in \mathbb{N}$. Then, we define the Riemann–Liouville fractional derivative of a function f by

$$ {}_a\mathscr{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. $$

Definition 2.2. [13]. Let α and n be as in the previous definition. For a function f, we define the Caputo fractional derivative by

$$ {}_a^C\mathscr{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. $$

Definition 2.3. [13]. Given $z, \beta \in \mathbb{C}$ and α satisfying $\Re(\alpha) > 0$, we define the two-parameter Mittag–Leffler function by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{2.1} $$

Definition 2.4. [8]. Suppose that $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\rho \in \mathbb{R}^+$. For a function f, we define the generalized left and right fractional integrals of order α by

$$ {}_a\mathscr{I}_t^{\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.2} $$

and

$$ {}_t\mathscr{I}_b^{\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.3} $$

respectively.

Definition 2.5. [7]. We define the generalized left and right fractional derivatives of f, in the sense of Riemann–Liouville, by

$$ {}_a\mathscr{D}_t^{\alpha,\rho} f(t) = \gamma^n {}_a\mathscr{I}_t^{n-\alpha,\rho} f(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.4} $$

and

$$ {}_t\mathscr{D}_b^{\alpha,\rho} f(t) = (-\gamma)^n {}_t\mathscr{I}_b^{n-\alpha,\rho} f(t) = \frac{(-\gamma)^n}{\Gamma(n-\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.5} $$

respectively. In these definitions, $\rho > 0$, $\gamma = t^{1-\rho}\frac{d}{dt}$ and $n-1 \leq \alpha < n$.

Definition 2.6. [5]. We define the generalized left and right fractional derivatives of f, in the sense of Caputo, by

$$ {}_a^C\mathscr{D}_t^{\alpha,\rho} f(t) = {}_a\mathscr{I}_t^{n-\alpha,\rho} \gamma^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} \gamma^n f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.6} $$

and

$$ {}_t^C\mathscr{D}_b^{\alpha,\rho} f(t) = {}_t\mathscr{I}_b^{n-\alpha,\rho} (-\gamma)^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} (-\gamma)^n f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{2.7} $$

respectively. Here $\rho \in \mathbb{R}^+$, $\gamma = t^{1-\rho}\frac{d}{dt}$ and $\alpha \in \mathbb{C}, \Re(\alpha) > 0$.

In this definition, if $\rho = 1$ we obtain the Caputo fractional derivative, and if $\rho \to 0$ we obtain the Caputo–Hadamard fractional derivative.

Theorem 2.7. *[6] For a function $f : [0, \infty] \to \mathbb{R}$, define the ρ-Laplace transform by*

$$\mathscr{L}_\rho\{f(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}}, \tag{2.8}$$

where $\rho > 0$. Then, the integral converges for any value of s.

Theorem 2.8. *[1] Assume that for a function $f : [0, \infty] \to \mathbb{R}$, the ρ-Laplace transform exists. Then*

$$\mathscr{L}_\rho\{f(t)\}(s) = \mathscr{L}\{f((\rho t)^{\frac{1}{\rho}})\}(s), \tag{2.9}$$

in which $\mathscr{L}\{f\}$ denotes the Laplace transform of f.

Definition 2.9. [6]. For functions f and g, we define the ρ-convolution by

$$(f *_\rho g)(t) = \int_0^1 f\Big((t^\rho - s^\rho)^{\frac{1}{\rho}}\Big) g(s) \frac{ds}{s^{1-\rho}}. \tag{2.10}$$

Theorem 2.10. *[6]. (The ρ-convolution theorem)*

$$\mathscr{L}_\rho\{f *_\rho g\} = \mathscr{L}_\rho\{f\}\mathscr{L}_\rho\{g\}. \tag{2.11}$$

Theorem 2.11. *[6]. Assume that $\alpha > 0$, $f \in AC^n_\gamma[0, a]$ for every positive number a, and that $\gamma^k f$ is of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$ for each $k \in \{0, 1, \dots, n\}$. Then*

$$\mathscr{L}_\rho\{({}_0^C D_t^{\alpha,\rho})(t)\}(s) = s^\alpha \mathscr{L}_\rho\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\Big(\gamma^k f\Big)(0), \tag{2.12}$$

where $s > c$.

Theorem 2.12. *[6]. Assume that $\alpha > 0$, $f \in AC^n_\gamma[0, a]$ for every positive number a, and that ${}_aI_t^{n-k-\alpha,\rho} f$ is of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$ for each $k \in \{0, 1, \dots, n-1\}$. Then*

$$\mathscr{L}_\rho\{({}_0D_t^{\alpha,\rho})(t)\}(s) = s^\alpha \mathscr{L}_\rho\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\Big({}_a\mathscr{I}_t^{n-k-\alpha,\rho} f\Big)(0), \ \ s > c. \tag{2.13}$$

Theorem 2.13. *[6]. Assume that $f \in AC^{n-1}_\gamma[0, \infty)$, and that for each $i \in \{0, 1, ..., n-1\}$, $\gamma^i f$ is of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$. If $\gamma^n f$ is piecewise continuous on $[0, T]$, then its ρ-Laplace transform exists for $s > c$, and*

$$\mathscr{L}_\rho\{(\gamma^n f)(t)\}(s) = s^n \mathscr{L}_\rho\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\Big(\gamma^n f\Big)(0). \tag{2.14}$$

Lemma 2.14. *[10]. Suppose that $\Re(\alpha) > 0$ and $|\frac{\lambda}{s^\alpha}| < 1$. Then,*

- $\mathscr{L}_\rho\{E_\alpha(-\lambda(\frac{t^\rho}{\rho})^\alpha)\} = \frac{s^\alpha}{s(s^\alpha+\lambda)}$,
- $\mathscr{L}_\rho\{1 - E_\alpha(-\lambda(\frac{t^\rho}{\rho})^\alpha)\} = \frac{\lambda}{s(s^\alpha+\lambda)}$,
- $\mathscr{L}_\rho\{(\frac{t^\rho}{\rho})^{\beta-1} E_{\alpha,\beta}(\lambda(\frac{t^\rho}{\rho})^\alpha)\} = \frac{s^{\alpha-\beta}}{(s^\alpha-\lambda)}$.

3. The main results

The aim of this section is to study five different forms of GFSL problems. These are determined by the following fractional Sturm–Liouville operators.

$$
\begin{aligned}
&L_1 = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}) + q(t),\\
&L_2 = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}(\gamma^2) + q(t),\\
&L_3 = -{}_0\mathscr{D}_t^{2\alpha,\rho}({}_t\mathscr{D}_b^{2\alpha,\rho}) + q(t),\\
&L_4 = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}({}_t\mathscr{D}_b^{2\alpha,\rho}) + q(t),\\
&L_5 = -{}_0\mathscr{D}_t^{2\alpha,\rho}(\gamma^2) + q(t).
\end{aligned}
$$

Theorem 3.1. *Let $0 < \alpha \leq 1$. Consider the GFSL problem*

$$L_1 y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\Big) + q(t)y(t) = \lambda y(t), \tag{3.1}$$

with initial conditions

$$y(0) = c_1,\ \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}y\Big)(0) = c_2,\ \Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) = c_3,\ \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0) = c_4. \tag{3.2}$$

Here, $y \in AC_\gamma^n[a,b]$, $n \in \mathbb{R}$ and $q : [0,n] \to \mathbb{R}$ is continuous. Then, the solution to the GFSL problem described in (3.1) and (3.2) is

$$
\begin{aligned}
y(t) =& c_1 E_{4\alpha,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_2\Big(\frac{t^\rho}{\rho}\Big)^{\alpha} E_{4\alpha,\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big)\\
&+ c_3\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha} E_{4\alpha,2\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_4\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha} E_{4\alpha,3\alpha+1}\Big(-\lambda(\frac{t^\rho}{\rho})^{4\alpha}\Big)\\
&+ \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha-1} E_{4\alpha,4\alpha}\Big(-\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha}\Big) \times q(\tau)y(\tau,\lambda)\frac{d\tau}{\tau^{1-\rho}}.
\end{aligned} \tag{3.3}
$$

Proof. Apply the ρ-Laplace transform to the both sides of (3.1). Then, using the initial conditions (3.2) and Theorem 2.11 we can write

$$
\begin{aligned}
&-\mathscr{L}_\rho\Big\{{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\Big)\Big\} + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^\alpha\mathscr{L}_\rho\Big\{{}_0^C\mathscr{D}_t^{\alpha,\rho}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\Big)\Big\} + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho})y\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^\alpha\Big[s^\alpha\mathscr{L}_\rho\{{}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\} + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0)\Big] + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^{2\alpha}\mathscr{L}_\rho\{{}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\} + s^{2\alpha-1}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^{3\alpha}\mathscr{L}_\rho\{{}_0^C\mathscr{D}_t^{\alpha,\rho}y(t)\} + s^{3\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}y\Big)(0) + s^{2\alpha-1}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0)\\
&+\mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^{4\alpha}\mathscr{L}_\rho\{y(t)\} + s^{4\alpha-1}y(0) + s^{3\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}y\Big)(0) + s^{2\alpha-1}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0)\\
&+s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&-s^{4\alpha}\mathscr{L}_\rho\{y(t)\} + s^{4\alpha-1}c_1 + s^{3\alpha-1}c_2 + s^{2\alpha-1}c_3 + s^{\alpha-1}c_4 + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\},\\
&\mathscr{L}_\rho\{y(t)\} = c_1\frac{s^{4\alpha-1}}{s^{4\alpha}+\lambda} + c_2\frac{s^{3\alpha-1}}{s^{4\alpha}+\lambda} + c_3\frac{s^{2\alpha-1}}{s^{4\alpha}+\lambda} + c_4\frac{s^{\alpha-1}}{s^{4\alpha}+\lambda} + \frac{1}{s^{4\alpha}+\lambda}\mathscr{L}_\rho\{q(t)y(t)\}.
\end{aligned}
$$

Now, to obtain (3.3), we just need to apply the inverse ρ-Laplace transform to the both sides of the last equation. □

Theorem 3.2. *Let $0 < \alpha \le 1$. Consider the GFSL problem*

$$L_2 y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big(\gamma^2 y(t)\Big) + q(t)y(t) = \lambda y(t), \tag{3.4}$$

with initial conditions

$$y(0) = c_5, \ \Big(\gamma y\Big)(0) = c_6, \ \Big(\gamma^2 y\Big)(0) = c_7, \ \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) = c_8, \tag{3.5}$$

in which $q(t)$ and $y(t)$ are as in Theorem 3.1. Then, the solution to the GFSL problem described in (3.4) and (3.5) is as follows.

$$\begin{aligned} y(t) &= c_8 E_{2\alpha+2,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_6\Big(\frac{t^\rho}{\rho}\Big)E_{2\alpha+2,2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) \\ &+ c_7\Big(\frac{t^\rho}{\rho}\Big)^2 E_{2\alpha+2,3}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_8\Big(\frac{t^\rho}{\rho}\Big)^{\alpha+2} E_{2\alpha+2,\alpha+3}\Big(-\lambda(\frac{t^\rho}{\rho})^{2\alpha+2}\Big) \\ &+ \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{2\alpha+2} E_{2\alpha+2,2\alpha+2}\Big(-\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{2\alpha+2}\Big) q(\tau)y(\tau,\lambda)\frac{d\tau}{\tau^{1-\rho}}. \end{aligned} \tag{3.6}$$

Proof. Apply the ρ-Laplace transform to the both sides of (3.4). Then, using the initial conditions (3.5), Theorem 2.11 and Theorem 2.13 we can write

$$\begin{aligned} &-\mathscr{L}_\rho\Big\{{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big(\gamma^2 y(t)\Big)\Big\} + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &-s^\alpha\mathscr{L}_\rho\Big\{{}_0^C\mathscr{D}_t^{\alpha,\rho}\Big(\gamma^2 y(t)\Big)\Big\} + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}. \end{aligned}$$

Using the Laplace transform,

$$\begin{aligned} &-s^\alpha\Big[s^\alpha\mathscr{L}_\rho\{\gamma^2 y(t)\} + s^{\alpha-1}\Big(\gamma^2 y\Big)(0)\Big] + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &-s^{2\alpha}\mathscr{L}_\rho\{\gamma^2 y(t)\} + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) + s^{\alpha-1}\Big(\gamma^2 y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &-s^{2\alpha}\Big[s^2\mathscr{L}_\rho\{y(t)\} - s\Big(\gamma^0 y\Big)(0) - \Big(\gamma y\Big)(0)\Big] + s^{2\alpha-1}\Big(\gamma^2 y\Big)(0) + s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) \\ &+\mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &-s^{2\alpha+2}\mathscr{L}_\rho\{y(t)\} + s^{2\alpha+1}y(0) + s^{2\alpha}\Big(\gamma y\Big)(0) + s^{2\alpha-1}\Big(\gamma^2 y\Big)(0) \\ &+ s^{\alpha-1}\Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &-s^{2\alpha+2}\mathscr{L}_\rho\{y(t)\} + s^{2\alpha+1}c_5 + s^{2\alpha}c_6 + s^{2\alpha-1}c_7 + s^{\alpha-1}c_8 + \mathscr{L}_\rho\{q(t)y(t)\} = \lambda\mathscr{L}_\rho\{y(t)\}, \\ &\mathscr{L}_\rho\{y(t)\} = c_5\frac{s^{2\alpha+1}}{s^{2\alpha+2}+\lambda} + c_6\frac{s^{2\alpha}}{s^{2\alpha+2}+\lambda} + c_7\frac{s^{2\alpha-1}}{s^{2\alpha+2}+\lambda} + c_8\frac{s^{\alpha-1}}{s^{2\alpha+2}+\lambda} + \frac{1}{s^{2\alpha+2}+\lambda}\mathscr{L}_\rho\{q(t)y(t)\}. \end{aligned}$$

Now, to obtain (3.6), we just need to apply the inverse ρ-Laplace transform to the both sides of the last equation. □

Theorem 3.3. *Consider the GFSL problem*

$$L_3 y(t) = -{}_0\mathscr{D}_t^{2\alpha,\rho}\Big({}_0\mathscr{D}_t^{2\alpha,\rho}y(t)\Big) + q(t)y(t) = \lambda y(t), \ 0 < \alpha \le 1, \tag{3.7}$$

with initial conditions

$$\begin{aligned} &\Big({}_0\mathscr{I}_t^{1-\alpha}y\Big)(0) = c_9, \ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho}y\Big)(0) = c_{10}, \\ &\Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) = c_{11}, \ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{3\alpha,\rho}y\Big)(0) = c_{12}, \end{aligned} \tag{3.8}$$

where $y \in AC_\gamma^n[a,b]$, $n \in \mathbb{R}$ and $q:[0,n] \to \mathbb{R}$ is continuous. Then, the solution to the problem described in (3.7) and (3.8) is

$$\begin{aligned} y(t) &= c_9\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1} E_{4\alpha,\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{10}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha-1} E_{4\alpha,2\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) \\ &+ c_{11}\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha-1} E_{4\alpha,3\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{12}\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha-1} E_{4\alpha,4\alpha}\Big(-\lambda(\frac{t^\rho}{\rho})^{4\alpha}\Big) \\ &+ \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha-1} E_{4\alpha,4\alpha}\Big(-\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha}\Big) q(\tau) y(\tau,\lambda)\frac{d\tau}{\tau^{1-\rho}}. \end{aligned} \tag{3.9}$$

Proof. The proof can be completed in the same way as the proof of Theorem 3.1. □

Theorem 3.4. *Consider the GFSL problem with initial conditions:*

$$L_4 y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big({}_0\mathscr{D}_t^{2\alpha,\rho} y(t)\Big) + q(t)y(t) = \lambda y(t), \tag{3.10}$$

$$\begin{aligned} &\Big({}_0\mathscr{I}_t^{1-\alpha} y\Big)(0) = c_{13},\ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho} y\Big)(0) = c_{14}, \\ &\Big({}_0\mathscr{D}_t^{2\alpha,\rho} y\Big)(0) = c_{15},\ \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}{}_0\mathscr{D}_t^{2\alpha,\rho} y\Big)(0) = c_{16}, \end{aligned} \tag{3.11}$$

where $y \in AC_\gamma^n[a,b]$, $n \in \mathbb{R}$ and $q:[0,n] \to \mathbb{R}$ is continuous. Then, the solution to the problem described in (3.10) and (3.11) is

$$\begin{aligned} y(t) &= c_{13}\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1} E_{4\alpha,\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{14}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha-1} E_{4\alpha,2\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) \\ &+ c_{15}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha} E_{4\alpha,2\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{16}\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha} E_{4\alpha,3\alpha+1}\Big(-\lambda(\frac{t^\rho}{\rho})^{4\alpha}\Big) \\ &+ \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha-1} E_{4\alpha,4\alpha}\Big(-\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{4\alpha}\Big) q(\tau) y(\tau,\lambda)\frac{d\tau}{\tau^{1-\rho}}. \end{aligned} \tag{3.12}$$

Proof. This is the same as the proofs of Theorem 3.1 and Theorem 3.3, and uses Theorem 2.11 and Theorem 2.12. □

Theorem 3.5. *Let $0 < \alpha \le 1$. Consider the GFSL problem*

$$L_5 y(t) = -{}_0\mathscr{D}_t^{2\alpha,\rho}\Big(\gamma^2 y(t)\Big) + q(t)y(t) = \lambda y(t), \tag{3.13}$$

with initial conditions

$$y(0) = c_{17},\ \Big(\gamma y\Big)(0) = c_{18},\ \Big(\mathscr{I}_t^{1-\alpha}\gamma^2 y\Big)(0) = c_{19},\ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) = c_{20}, \tag{3.14}$$

in which $q(t)$ and $y(t)$ are as in Theorem 3.1. Then, the solution to the GFSL problem described in (3.13) and (3.14) is as follows.

$$\begin{aligned} y(t) &= c_{17} E_{2\alpha+2,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_{18}\Big(\frac{t^\rho}{\rho}\Big) E_{2\alpha+2,2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) \\ &+ c_{19}\Big(\frac{t^\rho}{\rho}\Big)^{\alpha+1} E_{2\alpha+2,\alpha+2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_{20}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+1} E_{2\alpha+2,2\alpha+2}\Big(-\lambda(\frac{t^\rho}{\rho})^{2\alpha+1}\Big) \\ &+ \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{2\alpha+1} E_{2\alpha+2,2\alpha+2}\Big(-\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{2\alpha+2}\Big) q(\tau) y(\tau,\lambda)\frac{d\tau}{\tau^{1-\rho}}. \end{aligned} \tag{3.15}$$

Proof. The proof can be completed in the same way as the proof of Theorem 3.1. □

Note that the solutions (3.3), (3.6), (3.9), (3.12), and (3.15) correspond to the solution of the following fourth-order Sturm–Liouville problem in which both α and ρ are equal to 1.

$$y^{(4)}(t) + q(t)y(t) = \lambda y(t), \tag{3.16}$$

$$y(0) = A,\, y'(0) = B,\; y''(0) = C,\; y'''(0) = D, \tag{3.17}$$

$$\begin{aligned} y(t) &= AE_{4,1}(-\lambda t^4) + BtE_{4,2}(-\lambda t^4) + Ct^2E_{4,3}(-\lambda t^4) + Dt^3E_{4,4}(-\lambda t^4) \\ &+ \int_0^t \Big(t-\tau\Big)^3 E_{4,4}\Big(-\lambda(t-\tau)^4\Big)q(\tau)y(\tau,\lambda)d\tau. \end{aligned} \tag{3.18}$$

Furthermore, it is easy to see that all these representations will be the same when $\alpha = 1$, regardless of the value of ρ.

4. Results

Now, we consider the homogeneous parts of the problems we have already discussed, and we compare their analytic solutions. The homogeneous part of (3.1) is

$$L_1y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y(t)\Big) = \lambda y(t),\; 0<\alpha\le 1, \tag{4.1}$$

$$y(0) = c_1,\; \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}y\Big)(0) = c_2,\; \Big({}_0^C\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) = c_3,\; \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}({}_0^C\mathscr{D}_t^{2\alpha,\rho}y)\Big)(0) = c_4. \tag{4.2}$$

The analytic solution to the problem described in (4.1) and (4.2) is

$$\begin{aligned} y(t) &= c_1E_{4\alpha,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_2\Big(\frac{t^\rho}{\rho}\Big)^{\alpha}E_{4\alpha,\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) \\ &+ c_3\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha}E_{4\alpha,2\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_4\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha}E_{4\alpha,3\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big). \end{aligned} \tag{4.3}$$

The homogeneous part of (3.4) is

$$L_2y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big(\gamma^2y(t)\Big) = \lambda y(t),\; 0<\alpha\le 1, \tag{4.4}$$

$$y(0) = c_5,\; \Big(\gamma y\Big)(0) = c_6,\; \Big(\gamma^2y\Big)(0) = c_7,\; \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}(\gamma^2y)\Big)(0) = c_8. \tag{4.5}$$

The analytic solution to the problem described in (4.4) and (4.5) is

$$\begin{aligned} y(t) &= c_5E_{2\alpha+2,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_6\Big(\frac{t^\rho}{\rho}\Big)E_{2\alpha+2,2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) \\ &+ c_7\Big(\frac{t^\rho}{\rho}\Big)^{2}E_{2\alpha+2,3}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_8\Big(\frac{t^\rho}{\rho}\Big)^{\alpha+2}E_{2\alpha+2,\alpha+3}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big). \end{aligned} \tag{4.6}$$

The homogeneous part of (3.7) is

$$L_3y(t) = -{}_0\mathscr{D}_t^{2\alpha,\rho}\Big({}_0\mathscr{D}_t^{2\alpha,\rho}y(t)\Big) = \lambda y(t),\; 0<\alpha\le 1, \tag{4.7}$$

$$\begin{aligned} &\Big({}_0\mathscr{I}_t^{1-\alpha}y\Big)(0) = c_9,\; \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho}y\Big)(0) = c_{10}, \\ &\Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{2\alpha,\rho}y\Big)(0) = c_{11},\; \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{3\alpha,\rho}y\Big)(0) = c_{12}. \end{aligned} \tag{4.8}$$

The analytic solution to the problem described in (4.7) and (4.8) can be found as

$$\begin{aligned} y(t) &= c_9\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{4\alpha,\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{10}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha-1}E_{4\alpha,2\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) \\ &+ c_{11}\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha-1}E_{4\alpha,3\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{12}\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha-1}E_{4\alpha,4\alpha}\Big(-\lambda(\frac{t^\rho}{\rho})^{4\alpha}\Big). \end{aligned} \tag{4.9}$$

The homogeneous part of (3.10) is

$$L_4 y(t) = -{}_0^C\mathscr{D}_t^{2\alpha,\rho}\Big({}_0\mathscr{D}_t^{2\alpha,\rho} y(t)\Big) = \lambda y(t),\ 0<\alpha\le 1, \tag{4.10}$$

$$\Big({}_0\mathscr{I}_t^{1-\alpha} y\Big)(0) = c_{13},\ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho} y\Big)(0) = c_{14},$$
$$\Big({}_0\mathscr{D}_t^{2\alpha,\rho} y\Big)(0) = c_{15}, \Big({}_0^C\mathscr{D}_t^{\alpha,\rho}{}_0\mathscr{D}_t^{2\alpha,\rho} y\Big)(0) = c_{16}. \tag{4.11}$$

The analytic solution to the problem described in (4.10) and (4.11) is

$$y(t) = c_{13}\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1} E_{4\alpha,\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{14}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha-1} E_{4\alpha,2\alpha}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big)$$
$$+ c_{15}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha} E_{4\alpha,2\alpha+1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{4\alpha}\Big) + c_{16}\Big(\frac{t^\rho}{\rho}\Big)^{3\alpha} E_{4\alpha,3\alpha+1}\Big(-\lambda(\frac{t^\rho}{\rho})^{4\alpha}\Big). \tag{4.12}$$

Finally, the homogeneous part of (3.13) is

$$L_5 y(t) = -{}_0\mathscr{D}_t^{2\alpha,\rho}\Big(\gamma^2 y(t)\Big) = \lambda y(t),\ 0<\alpha\le 1, \tag{4.13}$$

$$y(0) = c_{17},\ \Big(\gamma y\Big)(0) = c_{18},$$
$$\Big(\mathscr{I}_t^{1-\alpha}\gamma^2 y\Big)(0) = c_{19},\ \Big(\mathscr{I}_t^{1-\alpha}{}_0\mathscr{D}_t^{\alpha,\rho}(\gamma^2 y)\Big)(0) = c_{20}, \tag{4.14}$$

in which $q(t)$ and $y(t)$ are as in Theorem 3.1. The following is the solution to the GFSL problem described in (3.13) and (3.14).

$$y(t) = c_{17}E_{2\alpha+2,1}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big) + c_{18}\Big(\frac{t^\rho}{\rho}\Big)E_{2\alpha+2,2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big)$$
$$+ c_{19}\Big(\frac{t^\rho}{\rho}\Big)^{\alpha+1}E_{2\alpha+2,\alpha+2}\Big(-\lambda\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+2}\Big)$$
$$+ c_{20}\Big(\frac{t^\rho}{\rho}\Big)^{2\alpha+1}E_{2\alpha+2,2\alpha+2}\Big(-\lambda(\frac{t^\rho}{\rho})^{2\alpha+1}\Big). \tag{4.15}$$

5. Conclusion

In this paper, we found integral representations for the solutions of certain fourth-order GFSL problems, and we obtained their analytic solutions. We could find the eigenfunctions and eigenvalues, and we examined the eigenvalues with respect to different values of α and ρ. Also, we observed that when α approached 1, the classic solution was obtained.

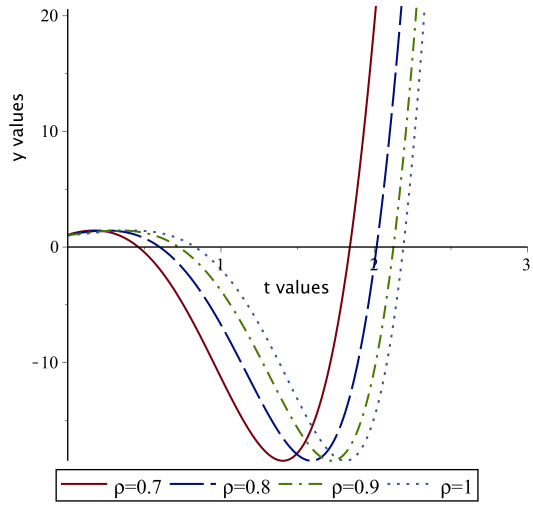

FIGURE 1. Eigenvalue of (4.3): α=0.9 and $c_1 = c_2 = c_3 = c_4 = 1$

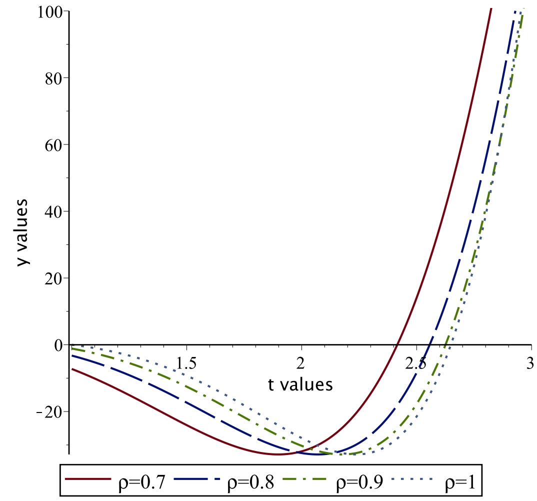

FIGURE 2. Eigenvalue of (4.3): α=1 and $c_1 = c_2 = c_3 = c_4 = 1$

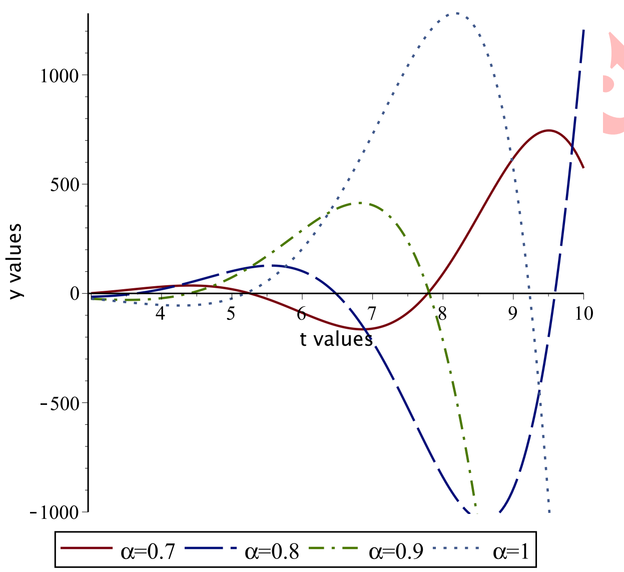

FIGURE 3. Eigenvalue of (4.3): $\rho = 0.8$ and $c_1 = c_2 = c_3 = c_4 = 1$

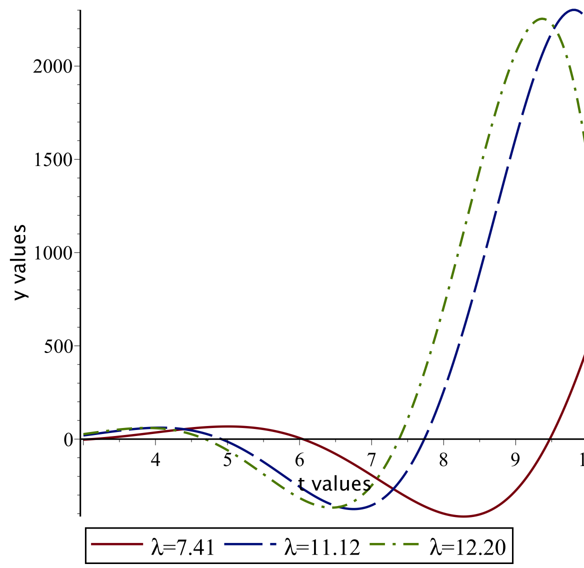

FIGURE 4. Eigenfunction of (4.3): $\alpha = 0.75$ and $\rho = 0.7$, while $c_1 = c_2 = c_3 = c_4 = 1$

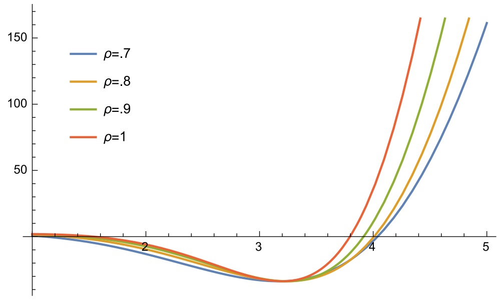

FIGURE 5. Eigenvalue of (4.6): α=0.9 and $c_1 = c_2 = c_3 = c_4 = 1$

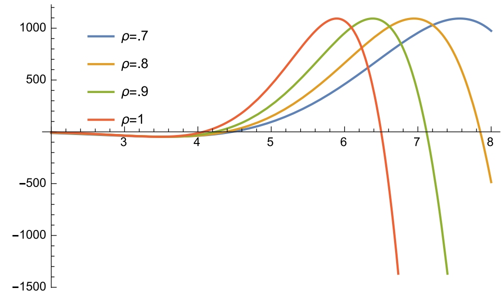

FIGURE 6. Eigenvalue of (4.6): α=1 and $c_1 = c_2 = c_3 = c_4 = 1$

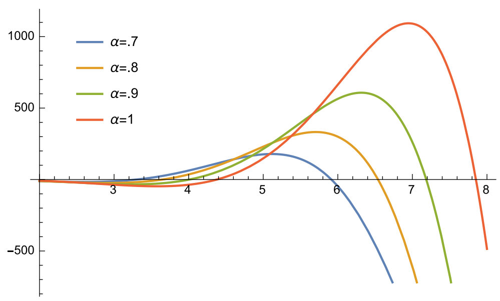

FIGURE 7. Eigenvalue of (4.6): $\rho = 0.8$ and $c_1 = c_2 = c_3 = c_4 = 1$

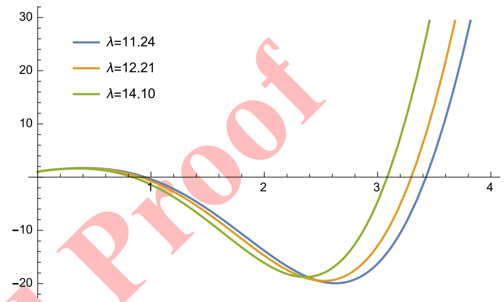

FIGURE 8. Eigenfunction of (4.6): $\alpha = 0.75$ and $\rho = 0.7$, while $c_1 = c_2 = c_3 = c_4 = 1$

References

[1] T. Abdeljawad,*On conformable fractional calculus,* J. Comput. Appl. Math., *279* (2013), 57–66.
[2] A. Afarideh ,F. Dastmalchi Saei ,M. Lakestani, and B. Nemati Saray *Pseudospectral method for solving fractional Sturm-Liouville problem using Chebyshev cardinal functions*, Physica Scripta. IOP Publishing, *96*(12) (2021), 125267.
[3] M. A. AL-Gwaiz, *Sturm-Liouville theory and its applications*, Springer - verlage, London, 2008.
[4] Q. M. AL-Mdallal, T. Abdeljawad, and M. A. Hajji, *Theorical and Computational prespectives on eigenvaluse of fourth-order fractional Sturm–Liouville problem*, International Journal of Computer Science.
[5] F. Jarad, T. Abdeljawad, and D. Baleanu, *On the generalized fractional derivatives and their Caputo modification,* J. Nonlinear Sci. Appl., *10*(5) (2017), 2607–2619.
[6] F. Jarad and T. Abdeljawad, *A modified Laplace transform for certain generalized fractional operators,* Res. Nonlinear Anal., *1*(2) (2018), 88–98.
[7] U. N. Katugampola, *A new approach to generalized fractional derivatives,* Bull. Math. Anal. Appl. *6*(4) (2014), 1–15.
[8] U. N. Katugampola, *New approach to a generalized fractional integral,* Appl. Math. Comput. *218*(3) (2011), 860–865.
[9] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, New York, 1993.
[10] R. Ozarslan, E. Bas, and D. Baleanu, *Representation of solutions for Sturm–Liouville eigenvalue problems with generalized fractional derivative,*
[11] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic, New York, 1974.
[12] I. Podlubny, *Fractional differential equations*, Academic, New York, 1999.
[13] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, *198* (1998).
[14] B. Ronald Guenther, J. W. Lee, *Sturm-Liouville Problems: Theory and Numerical Implementation*, CRC Press, (2018).
[15] S. Kilbas, O. Marichev, *Fractional integrals and derivatives*, Berlin: Gorden and Breach, 1993.
[16] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs, American Mathematical Society, *121* (2005).