



Power series solutions of fractional Lotka-Volterra equation

Jicheng Yu^{1,*} and Yuqiang Feng^{1,2}

¹School of Science, Wuhan University of Science and Technology, Wuhan 430081, Hubei, China.

²Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan 430081, Hubei, China.

Abstract

In this paper, the power series method is applied to the fractional Lotka-Volterra equation, one of the most famous competition models in demography and economics. We obtain some power series solutions of the governing equation and prove their convergence. In addition, we analyze the various types of competitive roles depicted by this model through the truncated graphs of these power series solutions. From the graphs, we can find that the fractional order affects the speed of population growth or decrease, and this effect can be seen as continuous with respect to the order.

Keywords. Power series method, Fractional differential equations, Lotka-Volterra model.

2010 Mathematics Subject Classification. 34A08, 65Q30, 40A05.

1. INTRODUCTION

Nonlinear differential equations are important tools for describing complex phenomena in fields such as physics, engineering, biology, and economics. One of them, the well-known population model, namely the logistic equation, is given by

$$u'(t) = Au(t) - Bu^2(t), \quad (1.1)$$

which is introduced by Verhulst in 1838 [22]. Due to the introduction of a nonlinear term, Eq. (1.1) effectively quenches the unbounded growth in the human population model proposed by Malthus [14] and alleviates the pessimistic prediction of human fate brought about by Malthus' prophecy [1, 23]. In addition, the predator-prey population model is used to describe the growth rate of competing biological species. The most famous competition models are given by Lotka and Volterra, who introduced two coupling constants C_1 and C_2 to describe the interaction between two competitors and transformed the simple logistic equation into the following Volterra-Lotka coupled differential equations:

$$\begin{aligned} u' &= A_1u - B_1u^2 + C_1uv, \\ v' &= A_2v - B_2v^2 + C_2vu, \end{aligned} \quad (1.2)$$

where u and v are the populations of two different biological species. This system of equations contains all fundamental parameters that impact the rate of growth. Among them, A_i represents the reproductive capacity of each species, B_i the niche capacity limitations related to the niche size, and C_i the interaction with other species. The signs of C_1 and C_2 determine the type of competitive roles shown in Table 1 [16].

Fractional differential equations (FDEs), due to nonlocal and memory effects of fractional derivative and integral [7, 12, 19, 20], are successfully used in various fields of natural science and engineering [6, 24–29]. Moreover, in recent years, various forms of fractional logistic and Lotka-Volterra equations have been studied by many scholars using different analytical and numerical methods [4, 5, 8, 11, 13, 15, 17, 18, 21]. Especially, Pareek et al. [18] obtained the

Received: 05 May 2024; Accepted: 20 January 2025.

* Corresponding author. Email: yjcheng@126.com.

TABLE 1. Signs of C_1 and C_2 determine the type of competitive roles.

C_1C_2	Type	Explanation
– –	Pure competition	Occurs when both species suffer from each other's existence
+ –	Predator-prey	Occurs when one of them serves as direct food to the other
+ +	Mutualism	Occurs in case of symbiosis or a win-win situation
+ 0	Commensalism	Occurs when one benefits from the existence of the other, who nevertheless remains unaffected
– 0	Amensalism	Occurs when one suffers from the existence of the other, who is impervious to what is happening
0 0	Neutralism	Occurs if there is no interaction whatsoever

analytical approximate solution in the form of a convergent infinite series for the fractional deterministic Lotka-Volterra model by the homotopy analysis method. Manohar et al. [15] used the fractional reduced differential transform method to solve two cases of the Lotka-Volterra model with the Caputo fractional order derivatives numerically. In this paper, we study the following coupled fractional Volterra-Lotka equations:

$$\begin{aligned} D^\alpha u &= A_1 u - B_1 u^2 + C_1 uv, \\ D^\alpha v &= A_2 v - B_2 v^2 + C_2 vu, \end{aligned} \quad (1.3)$$

by the power series method, which is applied by Area and Nieto to study fractional logistic and Allee logistic equations [2, 3]. We extended their work by applying this method to the coupled fractional Volterra-Lotka equations, and obtained some novel analytical solutions that differ from the results in [4, 5, 8, 11, 13, 15, 17, 18, 21]. In addition, we used these solutions with different fractional orders to comparatively analyze all the types of competitive roles in Table 1.

It should be noted that D^α denotes the following Caputo fractional derivative for an absolutely continuous function $f(t)$:

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad 0 < t, \quad 0 < \alpha < 1. \quad (1.4)$$

From the definition, we can get that the Caputo fractional derivative of a constant is zero, and that of a power function is

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \quad \gamma > -1. \quad (1.5)$$

For other properties of the Caputo fractional derivative, we can refer to the monographs [7, 12, 19, 20]. The Gamma function $\Gamma(z) = \int_a^\infty e^{-z} t^{z-1} dt$ is extensively used in this paper, and we can refer to some other important special functions and their applications in [9].

The motivation of this paper is to extend the classical Volterra-Lotka coupled differential equations to their corresponding fractional order forms, which can more accurately characterize the genetic effects and long-range dependencies of population density changes. The main contributions of our work is the first application of power series method to fractional Volterra-Lotka coupled differential equations and obtaining some analytical solutions with potential application value. The rest of this paper is organized as follows. In section 2, we apply the power series method to obtain some convergent power series solutions and analyze the dynamic properties of these solutions graphically for the six types of competitive roles in Table 1. In section 3, the conclusions and outlooks are presented.

2. METHODS AND RESULTS

In this section, we obtained some convergent power series solutions of Eq. (1.3) using the power series method, and discussed all types of competitive roles listed in Table 1 through the graphs of these solutions.



Let us assume

$$u(t) = \sum_{n=0}^{\infty} a_n(t^\alpha)^n, \quad v(t) = \sum_{n=0}^{\infty} b_n(t^\alpha)^n, \tag{2.1}$$

then

$$u^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n a_j a_{n-j} \right) (t^\alpha)^n, \quad v^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n b_j b_{n-j} \right) (t^\alpha)^n, \tag{2.2}$$

$$uv = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n a_j b_{n-j} \right) (t^\alpha)^n, \quad vu = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n b_j a_{n-j} \right) (t^\alpha)^n. \tag{2.3}$$

From Eq. (1.5), we can get

$$D^\alpha u = \sum_{n=0}^{\infty} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} a_{n+1} (t^\alpha)^n, \quad D^\alpha v = \sum_{n=0}^{\infty} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} b_{n+1} (t^\alpha)^n. \tag{2.4}$$

Substituting (2.1)–(2.4) into Eq. (1.3) and equating the coefficients of different powers of t^α yields that

$$\begin{aligned} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} a_{n+1} &= A_1 a_n - B_1 \sum_{j=0}^n a_j a_{n-j} + C_1 \sum_{j=0}^n a_j b_{n-j}, \\ \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} b_{n+1} &= A_2 b_n - B_2 \sum_{j=0}^n b_j b_{n-j} + C_2 \sum_{j=0}^n b_j a_{n-j}, \end{aligned} \tag{2.5}$$

from which we can obtain the explicit expressions of a_n and b_n . For $k \geq 0$, we have

$$\begin{aligned} a_{n+1} &= \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \left(A_1 a_n - B_1 \sum_{j=0}^n a_j a_{n-j} + C_1 \sum_{j=0}^n a_j b_{n-j} \right), \\ b_{n+1} &= \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \left(A_2 b_n - B_2 \sum_{j=0}^n b_j b_{n-j} + C_2 \sum_{j=0}^n b_j a_{n-j} \right), \end{aligned} \tag{2.6}$$

with the initial conditions $a_0 = u(0)$ and $b_0 = v(0)$. Therefore, the power series solutions of Eq. (1.3) are

$$\begin{aligned} u(t) &= a_0 + \sum_{n=0}^{\infty} \frac{\Gamma(n\alpha+1)(t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)} \left(A_1 a_n - B_1 \sum_{j=0}^n a_j a_{n-j} + C_1 \sum_{j=0}^n a_j b_{n-j} \right), \\ v(t) &= b_0 + \sum_{n=0}^{\infty} \frac{\Gamma(n\alpha+1)(t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)} \left(A_2 b_n - B_2 \sum_{j=0}^n b_j b_{n-j} + C_2 \sum_{j=0}^n b_j a_{n-j} \right). \end{aligned} \tag{2.7}$$

Theorem 2.1. *The power series solutions (2.7) are convergent in a neighborhood of $t = 0$.*

Proof. Considering Eqs. (2.6), we can obtain

$$\begin{aligned} |a_{n+1}| &\leq \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \left(|A_1| |a_n| + |B_1| \sum_{j=0}^n |a_j| |a_{n-j}| + |C_1| \sum_{j=0}^n |a_j| |b_{n-j}| \right), \\ |b_{n+1}| &\leq \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \left(|A_2| |b_n| + |B_2| \sum_{j=0}^n |b_j| |b_{n-j}| + |C_2| \sum_{j=0}^n |b_j| |a_{n-j}| \right). \end{aligned} \tag{2.8}$$

From the properties of the Gamma function, $\Gamma(\xi)$ is an increasing function for $\xi \geq 2$. That is to say, if a positive integer N_α satisfies $N_\alpha \alpha + 1 \geq 2$ and $(N_\alpha - 1)\alpha + 1 < 2$, then for $n \geq N_\alpha$,

$$\Gamma((n+1)\alpha+1) > \Gamma(n\alpha+1) > \Gamma(2) = 1. \tag{2.9}$$



So $\frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} < 1$ for arbitrary $n \geq N_\alpha$. When $n < N_\alpha$, the inequality $0.5 < \Gamma(n\alpha + 1) < 1$ holds, that is, $\frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} < 2$. Therefore, Eqs. (2.8) can be written as

$$\begin{aligned}
 |a_{n+1}| &\leq M \left(|a_n| + \sum_{j=0}^n |a_j||a_{n-j}| + \sum_{j=0}^n |a_j||b_{n-j}| \right), \\
 |b_{n+1}| &\leq M \left(|b_n| + \sum_{j=0}^n |b_j||b_{n-j}| + \sum_{j=0}^n |b_j||a_{n-j}| \right),
 \end{aligned}
 \tag{2.10}$$

where $M = \max\{2|A_i|, 2|B_i|, 2|C_i|\}$.

Consider another power series

$$P(t) = \sum_{n=0}^{\infty} p_n(t^\alpha)^n, \quad Q(t) = \sum_{n=0}^{\infty} q_n(t^\alpha)^n,
 \tag{2.11}$$

where $p_0 = |a_0|$, $q_0 = |b_0|$, and

$$p_{n+1} = M \left(p_n + \sum_{j=0}^n p_j p_{n-j} + \sum_{j=0}^n p_j q_{n-j} \right), \quad n \geq 0,
 \tag{2.12}$$

$$q_{n+1} = M \left(q_n + \sum_{j=0}^n q_j q_{n-j} + \sum_{j=0}^n q_j p_{n-j} \right), \quad n \geq 0.
 \tag{2.13}$$

Therefore, it is easily seen that $|a_n| \leq p_n$ and $|b_n| \leq q_n$ for $n = 0, 1, 2, \dots$, that is, the power series (2.11) are the majorant series of (2.1). We next show that the power series (2.11) are convergent. By simple calculation, we can get

$$P(t) = p_0 + Mt^\alpha (P(t) + P^2(t) + P(t)Q(t)),
 \tag{2.14}$$

$$Q(t) = q_0 + Mt^\alpha (Q(t) + Q^2(t) + Q(t)P(t)).
 \tag{2.15}$$

Consider the following implicit function with respect to the independent variable t :

$$F(t, P, Q) = P - p_0 - Mt^\alpha (P + P^2 + PQ),
 \tag{2.16}$$

$$G(t, P, Q) = Q - q_0 - Mt^\alpha (Q + Q^2 + QP),
 \tag{2.17}$$

which are analytic in a neighborhood of $(0, p_0, q_0)$, and $F(0, p_0, q_0) = 0$, $G(0, p_0, q_0) = 0$, $\frac{\partial(F,G)}{\partial(P,Q)}|_{(0,p_0,q_0)} = 1 \neq 0$. Therefore, by the implicit function theorem, the power series (2.11) are analytic in a neighborhood of $t = 0$. It implies that the power series solution (2.1) is convergent in a neighborhood of $t = 0$. This completes the proof. \square

What follows is an analysis of the types of competitive role listed in Table 1 based on the obtained power series solutions. To draw some truncated graphs of the power series solutions (2.7), the first few terms of them are given below:

$$\begin{aligned}
 a_1 &= \frac{1}{\Gamma(\alpha + 1)} \left(A_1 a_0 - B_1 a_0^2 + C_1 a_0 b_0 \right), \\
 b_1 &= \frac{1}{\Gamma(\alpha + 1)} \left(A_2 b_0 - B_2 b_0^2 + C_2 b_0 a_0 \right),
 \end{aligned}
 \tag{2.18}$$

$$\begin{aligned}
 a_2 &= \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \left(A_1 a_1 - 2B_1 a_0 a_1 + C_1 (a_0 b_1 + b_0 a_1) \right), \\
 b_2 &= \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \left(A_2 b_1 - 2B_2 b_0 b_1 + C_2 (b_0 a_1 + a_0 b_1) \right),
 \end{aligned}
 \tag{2.19}$$



$$\begin{aligned}
 a_3 &= \frac{\Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)} \left(A_1 a_2 - B_1(2a_0 a_2 + a_1^2) + C_1(a_0 b_2 + a_1 b_1 + a_2 b_0) \right), \\
 b_3 &= \frac{\Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)} \left(A_2 b_2 - B_2(2b_0 b_2 + b_1^2) + C_2(b_0 a_2 + b_1 a_1 + b_2 a_0) \right),
 \end{aligned}
 \tag{2.20}$$

$$\begin{aligned}
 a_4 &= \frac{\Gamma(3\alpha + 1)}{\Gamma(4\alpha + 1)} \left(A_1 a_3 - 2B_1(a_0 a_3 + a_1 a_2) + C_1(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) \right), \\
 b_4 &= \frac{\Gamma(3\alpha + 1)}{\Gamma(4\alpha + 1)} \left(A_2 b_3 - 2B_2(b_0 b_3 + b_1 b_2) + C_2(b_0 a_3 + b_1 a_2 + b_2 a_1 + b_3 a_0) \right),
 \end{aligned}
 \tag{2.21}$$

$$\begin{aligned}
 a_5 &= \frac{\Gamma(4\alpha + 1)}{\Gamma(5\alpha + 1)} \left(A_1 a_4 - B_1(2a_0 a_4 + 2a_1 a_3 + a_2^2) + C_1(a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0) \right), \\
 b_5 &= \frac{\Gamma(4\alpha + 1)}{\Gamma(5\alpha + 1)} \left(A_2 b_4 - B_2(2b_0 b_4 + 2b_1 b_3 + b_2^2) + C_2(b_0 a_4 + b_1 a_3 + b_2 a_2 + b_3 a_1 + b_4 a_0) \right).
 \end{aligned}
 \tag{2.22}$$

Assuming that the initial values and related parameters are fixed, we mainly focus on how the signs of parameters C_i determine the interaction between two species. The following graphs represent the corresponding types of competitive roles in Table 1. They are truncated graphs of the first six terms ($n = 0, \dots, 5$) of power series solutions (2.7) in these types. Figures 1 and 2 illustrate the dynamic behavior for the power series solution (2.7) for mutualism and pure competition between two species with the same initial and parameter values, respectively. It can be seen that the fractional order affects the speed of population growth or decrease, and this effect can be seen as continuous with respect to order α . Therefore, we can choose an appropriate fractional order model based on real-world observation data, rather than just having a single first-order model. Figures 3 to 6 reflect the trend of population changes for two species with different initial and parameter values under the remaining four types in Table 1. From these graphs, it can be seen that although the given initial and parameter values are not actual data, they roughly reflect the trend of changes in the population of two species for the six types of competitive role in Table 1.

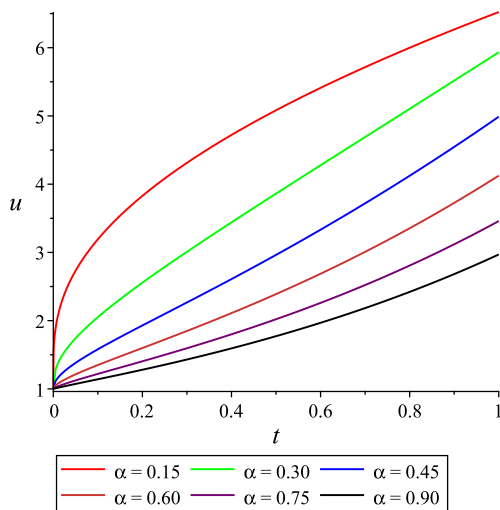


FIGURE 1. Mutualism between two species with $a_0 = b_0 = 1$, $A_i = B_i = C_i = 1$ for different fractional orders.

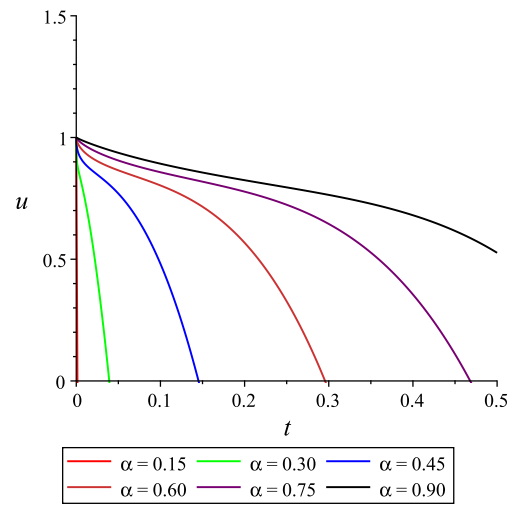


FIGURE 2. Pure competition between two species with $a_0 = b_0 = 1$, $A_i = B_i = 1$, $C_i = -1$ for different fractional orders.



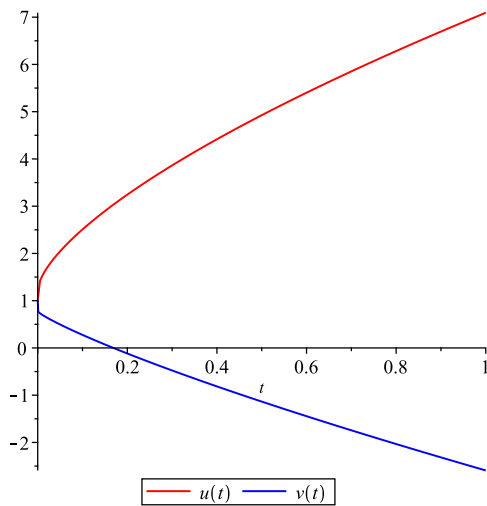


FIGURE 3. Predator-prey between two species with $a_0 = b_0 = 1$, $A_i = B_i = 1$, $C_1 = 1$, $C_2 = -1$ for $\alpha = 0.15$.

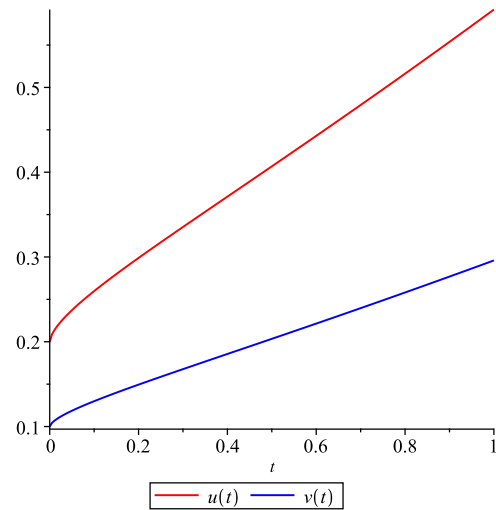


FIGURE 4. Commensalism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_1 = 1$, $C_2 = 0$ for $\alpha = 0.60$.

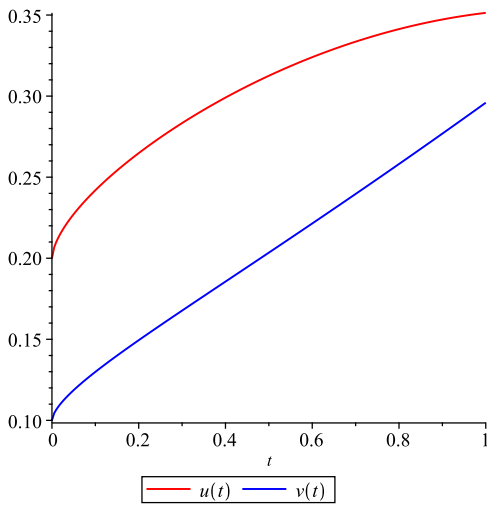


FIGURE 5. Amensalism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_1 = -1$, $C_2 = 0$ for $\alpha = 0.60$.

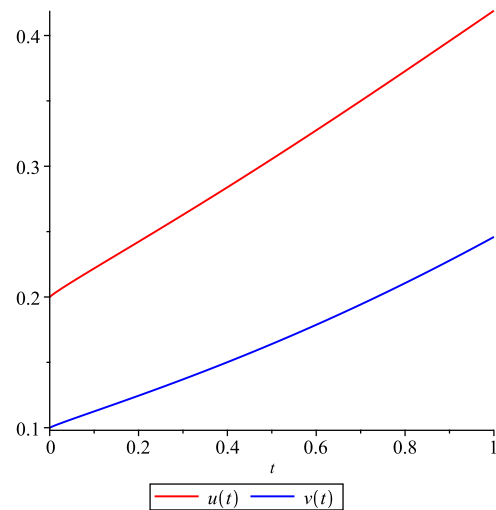


FIGURE 6. Neutralism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_i = 0$ for $\alpha = 0.90$.

3. CONCLUSION

This paper extends the power series method to coupled nonlinear fractional ordinary differential equations. We obtained the power series solution of the fractional Lotka-Volterra equation and proved its convergence. All types of competitive roles represented by the equation were analyzed through graphics. The introduction of fractional order greatly enriches the discussion of the competitive role for the Lotka-Volterra equation. Although the signs of



parameters C_i for the interaction between species determine the types of competitive roles shown in Table 1, this degree of determination is also influenced by the fractional order. For the given two species with the fixed initial values ($u(0)$ and $v(0)$) and parameters (A_i , B_i , and C_i), the fractional order can adjust the degree of conformity between the model and real data. In addition, we believe that the present study will help us to further generalize the power series method to study more nonlinear FDEs. Excitingly, there is a new paper this year that systematically introduces the power series method [10].

REFERENCES

- [1] I. Area, J. Losada, and J. J. Nieto, *A note on the fractional logistic equation*, *Physica A*, 444 (2016), 182–187.
- [2] I. Area and J. J. Nieto, *Power series solution of the fractional logistic equation*, *Physica A*, 573 (2021), 125947.
- [3] I. Area and J. J. Nieto, *On the fractional Allee logistic equation in the Caputo sense*, *Examples Counterexamples*, 4 (2023), 100121.
- [4] S. Das and P. K. Gupta, *A mathematical model on fractional Lotka–Volterra equations*, *J. Theor. Biol.*, 277(1) (2011), 1–6.
- [5] M. D’Ovidio and P. Loreti, *Solutions of fractional logistic equations by Euler’s numbers*, *Physica A*, 506 (2018), 1081–1092.
- [6] Y. Q. Feng and J. C. Yu, *Lie symmetry analysis of fractional ordinary differential equation with neutral delay*, *AIMS Math.*, 6 (2021), 3592–3605.
- [7] R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000.
- [8] M. Izadi, Ş. Yüzbaşı, and W. Adel, *Accurate and efficient matrix techniques for solving the fractional Lotka–Volterra population model*, *Physica A*, 600 (2022), 127558.
- [9] K. Jangid, S. D. Purohit, and D. L. Suthar, *A note on Lambert’s law involving incomplete I-functions*, *J. Sci. Arts*, 22(1) (2022), 91–96.
- [10] M. Jornet, *Power-series solutions of fractional-order compartmental models*, *Comp. Appl. Math.*, 43 (2024), 67.
- [11] A. R. Kanth and S. Devi, *A practical numerical approach to solve a fractional Lotka–Volterra population model with non-singular and singular kernels*, *Chaos Solitons Fractals*, 145 (2021), 110792.
- [12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, New York: Elsevier, 2006.
- [13] S. Kumar, R. Kumar, R. P. Agarwal, and B. Samet, *A study of fractional Lotka–Volterra population model using Haar wavelet and Adams–Bashforth–Moulton methods*, *Math. Meth. Appl. Sci.*, 43(8) (2020), 5564–5578.
- [14] T. R. Malthus, *Population: The First Essay*, Ann Arbor, MI: University of Michigan Press, 1959.
- [15] P. Manohar, L. Chanchlani, V. Kumar, S. D. Purohit, and D. L. Suthar, *Fractional Lotka–Volterra equations by fractional reduced differential transform method*, *Part. Differ. Eq. Appl. Math.*, 11 (2024), 100816.
- [16] T. Modis, *Technological forecasting at the stock market*, *Technol. Forecast. Soc.*, 62(3) (1999), 173–202.
- [17] M. Ortigueira and G. Bengochea, *A new look at the fractionalization of the logistic equation*, *Physica A*, 467 (2017), 554–561.
- [18] N. Pareek, A. Gupta, D. L. Suthar, G. Agarwal, and K. S. Nisar, *Homotopy analysis approach to study the dynamics of fractional deterministic Lotka–Volterra model*, *Arab J. Basic Appl. Sci.*, 29(1) (2022), 121–128.
- [19] I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
- [20] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Yverdon: Gordon and Breach Science Publishers, 1993.
- [21] D. Stutson and A. S. Vatsala, *Generalized monotone method for Caputo fractional differential systems via coupled lower and upper solutions*, *Dynam. Syst. Appl.*, 20(4) (2011), 495.
- [22] P. F. Verhulst, *Notice sur la loi que la population suit dans son accroissement*, *Corr. math. phys.*, 10 (1838), 113–121.
- [23] B. J. West, *Exact solution to fractional logistic equation*, *Physica A*, 429 (2015), 103–108.
- [24] J. C. Yu and Y. Q. Feng, *Lie symmetry analysis and exact solutions of space-time fractional cubic Schrödinger equation*, *Int. J. Geom. Methods M.*, 19 (2022), 2250077.



- [25] J. C. Yu, *Lie symmetry analysis of time fractional Burgers equation, Korteweg-de Vries equation and generalized reaction-diffusion equation with delays*, *Int. J. Geom. Methods M.*, 19 (2022), 2250219.
- [26] J. C. Yu and Y. Q. Feng, *Group classification of time fractional Black-Scholes equation with time-dependent coefficients*, *Fract. Calc. Appl. Anal.*, 27 (2024), 2335–2358.
- [27] J. C. Yu and Y. Q. Feng, *Lie symmetry, exact solutions and conservation laws of some fractional partial differential equations*, *J. Appl. Anal. Comput.*, 13 (2023), 1872-1889.
- [28] J. C. Yu and Y. Q. Feng, *On the generalized time fractional reaction–diffusion equation: Lie symmetries, exact solutions and conservation laws*, *Chaos Solitons Fractals*, 182 (2024), 114855.
- [29] Z. Y. Zhang and G. F. Li, *Lie symmetry analysis and exact solutions of the time-fractional biological population model*, *Phys. A*, 540 (2020), 123134.

