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Compact finite difference scheme for numerical solution of caputo-fabrizio fractional riccati differential equations

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Abstract

Riccati differential equation (RDE) is a kind of non-linear differential equation that has been used in many fields, like Newtonian dynamics, quantum mechanics, stochastic processes, propagation, reactor engineering, and optimal control. In this work, we consider the fractional RDE (FRDE) with the Caputo-Fabrizio derivative and use the compact finite difference scheme to solve it numerically. To solve this equation, we initially approximate the first-order derivative appearing in the definition of the Caputo-Fabrizio derivative through the compact finite difference method. By substituting the obtained approximation formula into the original equation, we derive a system of algebraic equations containing unknown values of the solution of the Riccati equation corresponding to specific discrete points in the domain. Solving this system of non-linear equations yields the solution of the Riccati differential equation at the discrete points. We provide some examples to examine the efficiency and accuracy of the suggested method.

Keywords. Fractional Riccati differential equations, Caputo-Fabrizio derivative, Non-linear equations, Compact finite difference method.
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1. Introduction

Fractional calculus is an attractive branch of applied mathematics, physics, quantum mechanics, and engineering that extends the foundations and principles of classical calculus; and deals with integrals and derivatives of arbitrary order. After the initial presentation of the concept of fractional derivatives, for a long time, applied scientists were unaware that they could mathematically employ fractional calculus in modeling real-world phenomena. However, in recent decades, researchers have realized that derivatives and integrals of fractional order are useful for describing many phenomena and properties of various materials in fluid mechanics, chemistry, physics, and finance [22]. Also, scientists have shown a keen interest in developing fractional derivative operators and applying them in modeling natural phenomena through fractional differential equations. This trend has led to the appearance of various definitions of fractional derivatives, such as Riemann-Liouville, Grunwald-Letnikov, Caputo, Marchaud, Erdélyi-Kober, Feller, Sonine-Leontiev, and Hadamard [22]. Some of these fractional derivatives, such as classical Riemann-Liouville and Caputo operators, have defects, especially their singular kernels, affecting the quality of memory and, transferring the genetic features of various materials. This has led to the emersion of new non-singular fractional derivatives like the Caputo-Fabrizio (C-F), and the Atangana-Baleanu [2, 8]. Fractional derivatives with non-singular kernels become more valuable since some phenomena cannot be accurately modeled by fractional derivatives with singular kernels [12]. Thus, many researchers consider fractional differential equations with non-singular fractional derivatives especially Caputo-Fabrizio derivative [1, 5, 13, 15–17, 21, 31, 34, 36, 37, 39, 42, 43].

The Riccati differential equation is a well-known nonlinear differential equation named in honor of the Italian Nobel laureate Count Jacopo Francesco Riccati. This equation is also among the fractional equations that derive

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from modeling many physical phenomena, stochastic processes, optimal control problems, propagation, and financial mathematics [6, 23, 26, 27, 33].

In this paper, we study the FRDE with the Caputo-Fabrizio derivative (C-FFRDE) given by [40, 42]:

$$\begin{aligned} &{}_{a}^{CF}D_t^{\eta} f(t) = a(t) + b(t) f(t) + c(t) f^2(t), \quad 0 < \eta < 1, \qquad a < t \ \leq b, \\ &f(a) = k \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $a(t), b(t)$, and $c(t)$ are known continuous functions on the interval $[a,b]$, and $f(t)$ is an unknown function. Additionally, ${}_{a}^{CF}D_t^{\eta}$ represents the C-F fractional derivative of order η.

So far, many analytical and numerical methods have been used to solve the FRDE [3, 11, 14, 24, 25, 28–30, 32, 38]. Compact Finite Difference (CFD) methods are also among the techniques extensively used for approximating solutions to differential equations [10, 18, 35, 45]. In this method, the derivatives in the equation at a specific point are approximated as a linear combination of the function values at the neighboring point. The key feature of compact schemes is their higher order of convergence compared to classical finite difference schemes.

This paper aim is to apply and investigate the CFD method for the numerical solution of C-FFRDE. The paper is structured as follows: In section 2, the C-F fractional derivative and its properties are introduced. In section 3, a brief overview of the compact finite difference scheme for approximating first-order derivatives is presented; then, it is employed to solve the fractional Riccati differential equation with the C-F derivative. In section 4, the results of CFD method are provided, and a comparison with the results of the other existing methods for solving the RDE is conducted. The conclusion and discussion regarding the used method are given in section 5.

2. The Caputo-Fabrizio Fractional Derivative

The Caputo fractional derivative, due to its strong compatibility with the classical derivative (for example, like the integer-order derivative, the Caputo fractional derivative of a constant function is also zero), has found extensive applications in modeling natural phenomena.

However, one of the drawbacks of the Caputo derivative is the singularity of the integral kernel in its definition. In 2015, Caputo and Fabrizio introduced the Caputo-Fabrizio fractional derivative to solve this issue [8]. Here, we first propose the definition of the Caputo fractional derivative. Subsequently, the C-F fractional derivative, obtained by applying some modifications to the Caputo fractional derivative, is introduced. More details in this regard can be found in [7, 8, 20].

Definition 2.1. The Caputo fractional derivative of a function $f \in H^1(a,b) = \{f : \ f, f' \in L^2(a,b)\}$ of order $\eta \in [0,1]$ for $a \in [-\infty, t)$ and $b > a$ is denoted by ${}_{a}^{C}D_t^{\eta} f(t)$ and defined as follows [20]:

$$ {}_{a}^{C}D_t^{\eta} f(t) = \frac{1}{\Gamma(1-\eta)} \int_a^t \frac{f'(x)}{(t-x)^{\eta}} dx. \tag{2.1}$$

Definition 2.2. The C-F fractional derivative of a function $f \in H^1(a,b) = \{f : \ f, f' \in L^2(a,b)\}$ of order $\eta \in (0,1)$ for scalar $a \in [-\infty, t)$ and $b > a$ is denoted by ${}_{a}^{CF}D_t^{\eta}\ f(t)$ and defined as follows [20]:

$$ {}_{a}^{CF}D_t^{\eta} f(t) = \frac{M(\eta)}{1-\eta} \int_a^t \exp\left(-\eta \frac{(t-x)}{1-\eta}\right) f'(x)\, dx, \tag{2.2}$$

where, $M(\eta) > 0$ is the normalization function depends η satisfying $M(0) = M(1) = 1$ and $t \geq 0$. But, if $f \notin H^1(a,b)$, then the above derivative can be presented for $f \in L^1(a,b)$ as

$$ {}_{a}^{CF}D_t^{\eta} f(t) = \frac{M(\eta)}{1-\eta} \int_a^t \exp\left(-\frac{\eta(t-x)}{1-\eta}\right) (f(t) - f(x)) dx. \tag{2.3}$$

Essentially, the Caputo-Fabrizio derivative is obtained from the Caputo derivative by converting the integral kernel from $(t-x)^{-\eta}$ to the new kernel, $\exp\left(-\frac{\eta(t-x)}{1-\eta}\right)$, and, by changing the coefficient of integral from $\frac{1}{\Gamma(1-\eta)}$ to $\frac{M(\eta)}{1-\eta}$.

In [20], the Caputo-Fabrizio fractional integral equation was defined and it was suggested that the function M in the Definition 2.2 be considered as $M(\eta)=1$. We also consider $M(\eta)=1$ in our calculations. Some characteristics of the C-F derivative include [8, 20]:

(1) The kernel of the C-F derivative at $x=t$ is non-singular.
(2) The C-F derivative of a constant function is equal to zero.
(3) $\lim\limits_{\eta\to 1} {}_{a}^{CF}D_t^{\eta} f(t) = f'(t)$.
(4) $\lim\limits_{\eta\to 0} {}_{a}^{CF}D_t^{\eta} f(t) = f(t) - f(a)$.

3. Compact Finite Difference Scheme for Solving C-FFRDE

In this paper, our main purpose is to employ the CFD method to numerically solve the C-FFRDE. First, we briefly introduce the CFD scheme for approximating the first-order derivatives. Then, we explain the method of applying the CFD scheme to solve the C-FFRDE.

3.1. CFD Scheme for Approximating the First-Order Derivatives.

We consider the discrete point set $\{t_n\}$ defined by $t_i = a + ih$, $i = 0,1,\dots,N = \dfrac{b-a}{h}$ that they produce equidistant points in the interval $[a,b]$; and the parameter $h = \dfrac{b-a}{N}$ is called step length.
The fourth-order CFD scheme for approximating $f'(t)$ at the nodes t_i is given by [44]:

$$\begin{cases} 4f_1' + f_2' = \dfrac{1}{h}\left(-\dfrac{11}{12}f(t_0) - 4f(t_1) + 6f(t_2) - \frac{4}{3}f(t_3) + \frac{1}{4}f(t_4)\right), \\ f_{i-1}' + 4f_i' + f_{i+1}' = \frac{3}{h}\left(-f(t_{i-1}) + f(t_{i+1})\right), \ i = 1,\dots,N-1, \\ f_{N-2}' + 4f_{N-1}' = \frac{1}{h}\left(-\frac{1}{4}f(t_{N-4}) + \frac{4}{3}f(t_{N-3}) - 6f(t_{N-2}) + 4f(t_{N-1}) + \frac{11}{12}f(t_N)\right), \end{cases} \tag{3.1}$$

where, $f_i' \approx f'(t_i), i = 1,\dots,N-1$.
All the above relations have the accuracy of order $O\left(h^4\right)$. The matrix form for Eq. (3.1) is

$$A_1 \boldsymbol{f}' = \frac{1}{h} B_1 \boldsymbol{f}, \tag{3.2}$$

where

$$A_1 = \begin{bmatrix} 0 & 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \qquad \boldsymbol{f} = [f(t_0), f(t_1), f(t_2), \dots, f(t_N)]^T,$$

$$B_1 = \begin{bmatrix} -\dfrac{11}{12} & -4 & 6 & -\dfrac{4}{3} & \dfrac{1}{4} & 0 & \dots & 0 \\ -3 & 0 & 3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -4 & 0 & 3 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & 0 & 0 & 0 & -3 & 0 & 4 \\ 0 & \dots & 0 & -\dfrac{1}{4} & \dfrac{4}{3} & -6 & 4 & \dfrac{11}{12} \end{bmatrix}_{(N+1)\times(N+1)}, \qquad \boldsymbol{f}' = [f_0', f_1', \dots, f_N']^T.$$

Lemma 3.1. *The coefficient matrix A_1 is invertible [10].*

According to Lemma 3.1, from Eq. (3.2), we have $\boldsymbol{f}' = \frac{1}{h}A_1^{-1}B_1\boldsymbol{f}$, and by defining $D = A_1^{-1}B_1$, the following relation holds for $\boldsymbol{f}'$:

$$\boldsymbol{f}' = \frac{1}{h}D\boldsymbol{f}. \tag{3.3}$$

We can write Eq. (3.3) in component form as follows:

$$f_i' = \frac{1}{h}\sum_{j=0}^{N} d_{i+1,j+1}f(t_j), \qquad i = 0, 1, \ldots, N, \tag{3.4}$$

where $d_{i+1,j+1}$, $i = 0, 1, \ldots, N, j = 0, 1, \ldots, N$ are the elements of the matrix D.

3.2. **Solving C-FFRDE Using the Compact Finite Difference Scheme.** Here, we describe the CFD scheme for solving the C-FFRDE (Eq. (1.1)). Without loss of generality, we study the Eq. (1.1) over the interval $[0, T]$. Thus, the C-FFRDE takes the following form:

$$\begin{aligned} &{}_0^{CF}D_t^{\eta}f(t) = a(t) + b(t)f(t) + c(t)f^2(t), \qquad 0 < \eta < 1,\ t \in (0, T], \\ &f(0) = k \in \mathbb{R}, \end{aligned} \tag{3.5}$$

where,

$${}_0^{CF}D_t^{\eta}f(t) = \frac{M(\eta)}{1-\eta}\int_0^t \exp\left(-\frac{\eta(t-x)}{1-\eta}\right) f'(x)dx.$$

We consider the points $t_i = ih$, $i = 0, 1, 2, \ldots, N$, which are equidistant points in the interval $[0, T]$ with the step length $h = \frac{T}{N}$. We rewrite Eq. (3.5) at the discrete points t_i, $i = 1, 2, \ldots, N$:

$$\begin{aligned} &{}_0^{CF}D_t^{\eta}f(t_i) = a(t_i) + b(t_i)f(t_i) + c(t_i)f^2(t_i), \qquad i = 1, 2, \ldots, N, \\ &f(t_0) = k, \end{aligned} \tag{3.6}$$

where ${}_0^{CF}D_t^{\eta}f(t_i)$ is defined as follows:

$${}_0^{CF}D_t^{\eta}f(t_i) = \frac{M(\eta)}{1-\eta}\int_0^{t_i} \exp\left(-\eta\frac{t_i-x}{1-\eta}\right) f'(x)dx. \tag{3.7}$$

Equation (3.7) can be rewritten as:

$${}_0^{CF}D_t^{\eta}f(t_i) \approx \frac{M(\eta)}{1-\eta}f_i'\int_0^{t_i} \exp\left(-\eta\frac{t_i-x}{1-\eta}\right) dx = \frac{M(\eta)}{\eta}f_i'\left[1 - \exp\left(-\eta\frac{t_i-t_0}{1-\eta}\right)\right]. \tag{3.8}$$

Substituting $t_i = ih$ into Eq. (3.7), we have:

$${}_0^{CF}D_t^{\eta}f(t_i) \approx \frac{M(\eta)}{1-\eta}f_i'\left[1 - \exp\left(\frac{-\eta}{1-\eta}(ih)\right)\right]. \tag{3.9}$$

Substituting Eq. (3.9) into Eq. (3.6), we have:

$$\frac{M(\eta)}{\eta}f_i'\left[1 - \exp\left(\frac{-\eta}{1-\eta}(ih)\right)\right] = a(t_i) + b(t_i)f(t_i) + c(t_i)f^2(t_i), \qquad i = 1, 2, \ldots, N. \tag{3.10}$$

Now, by substituting the CFD scheme, $f_i' = \frac{1}{h}\sum_{j=0}^{N} d_{i+1,j+1}f(t_j)$, into Eq. (3.10), assuming that f_i is an approximation of $f(t_i)$, and $M(\eta) = 1$ we obtain the following relation:

$$\frac{1}{\eta h}\sum_{j=0}^{N} d_{i+1,j+1}f_j\left[1 - \exp\left(\frac{-\eta}{1-\eta}(ih)\right)\right] = a(t_i) + b(t_i)f_i + c(t_i)f_i^2, \qquad i = 1, 2, \ldots, N. \tag{3.11}$$

Now, considering the variations in i in Eq. (3.11) and using the notations $a_i = a(t_i)$, $b_i = b(t_i)$, and $c_i = c(t_i)$, we obtain the following relations:

$$
\begin{aligned}
&i = 1: \ \frac{1}{\eta h}\left(1 - \exp\left(\frac{-\eta}{1-\eta}(h)\right)\right)[d_{2,1}f_0 + d_{2,2}f_1 + \ldots + d_{2,N+1}f_N] = a_1 + b_1 f_1 + c_1 f_1^2,\\
&i = 2: \ \frac{1}{\eta h}\left(1 - \exp\left(\frac{-\eta}{1-\eta}(2h)\right)\right)[d_{3,1}f_0 + d_{3,2}f_1 + \ldots + d_{3,N+1}f_N] = a_2 + b_2 f_2 + c_2 f_2^2,\\
&\vdots\\
&i = N-1: \ \frac{1}{\eta h}\left(1 - \exp\left(\frac{-\eta}{1-\eta}((N-1)h)\right)\right)[d_{N,1}f_0 + d_{N,2}f_1 + \ldots + d_{N,N+1}f_N]\\
&\qquad\qquad = a_{N-1} + b_{N-1}f_{N-1} + c_{N-1}f_{N-1}^2,\\
&i = N: \ \frac{1}{\eta h}\left(1 - \exp\left(\frac{-\eta}{1-\eta}(Nh)\right)\right)[d_{N+1,1}f_0 + d_{N+1,2}f_1 + \ldots + d_{N+1,N+1}f_N] = a_N + b_N f_N + c_N f_N^2.
\end{aligned}
$$

By solving this system of algebraic equations using the Mathematica software, the approximate solution values at the points t_i, $i = 1, 2, \ldots, N$ are obtained.

4. Numerical results

In this section, to investigate the efficiency and, accuracy of the proposed method, we employ it for the numerical solution of three test problems of the C-FFRDE. The algorithm is implemented using Mathematica 12 software. The simulations are run by an office laptop with Ubuntu 22.04.3 LTS equipped Intel® Core™ i7-3740QM CPU @ 2.70GHz × 8 processor and 8.00GB of RAM on a 64-bit operating system. To measure the accuracy of the numerical solutions, we consider the absolute error at grid points, denoted as $|f(t_i) - f_i|$, and use the maximum absolute error (MAE) as a criterion for evaluation,

$$\text{MAE} = \max_{1 \le i \le N} |f(t_i) - f_i|,$$

where $h = \dfrac{T}{N}$, $t_i = 1, 2, \ldots, N$, also, $f(t_i)$, and f_i represent the exact solution and the approximate solution at the point t_i respectively. To confirm the theoretical convergence of the proposed approach, we compute the order of convergence (ROC) numerically by the following formula:

$$\text{ROC} = \log_2 \frac{\text{MAE}(2h)}{\text{MAE}(h)}.$$

Also, we report the Central Processing Unit (CPU) time for running the algorithm of the proposed method for different values of N.
Additionally, we compare the accuracy of our method with the used methods in [4, 9, 19, 38, 40, 41].

Example 4.1. Consider the following C-FFRDE

$$ {}_0^{CF}D_t^{\eta} f(t) = t^5 + 1 - 2t^4 f(t) - t^3 f^2(t), \qquad 0 < \eta < 1, \ 0 < t \le 1, \tag{4.1}$$

The exact solution for $\eta = 1$ (integer order RDE) is $f(t) = t$ [4, 38].
Table 1, presents the absolute error values of approximate solution at points t_i, for $\eta = 1$ and $N = 10$. Table 2, provides the approximate solution values of Eq. (4.1) at the grid points t_i for $N = 10$ and $\eta = 0.9, 0.8.0.7$. Table 3 shows the MAE of the solution of Eq. (4.1) and CPU time for $\eta = 1$, and different values of N. Also, Table 4 compares the accuracy of the solution the Example 4.1, for $\eta = 1$ using the proposed method with the accuracy of the used methods used in [38], and [4]. The results indicate the high accuracy of the obtained approximate solution. Figure 1 illustrates the exact and approximate solutions of Eq. (4.1) with $\eta = 1$ and $N = 10$; and in Figure 2, the behavior of the curve for the approximate solution of Example 4.1, can be seen for $N = 40$ and different values of η.

TABLE 1. The error of the approximate solution for Eq. (4.1) at the grid points t_i, $i = 1, 2, \ldots, N$ for $N = 10$ and $\eta = 1$.

t_i	$\vert f(t_i) - f_i \vert$
0.1	$4.5796699766 \times 10^{-16}$
0.2	$3.3306690739 \times 10^{-16}$
0.3	$3.8857805862 \times 10^{-16}$
0.4	$3.8857805862 \times 10^{-16}$
0.5	$5.5511151231 \times 10^{-16}$
0.6	$3.3306690739 \times 10^{-16}$
0.7	$3.3306690739 \times 10^{-16}$
0.8	$4.4408920985 \times 10^{-16}$
0.9	$3.3306690739 \times 10^{-16}$
1	$4.4408920985 \times 10^{-16}$

TABLE 2. Values of the approximate solution of Eq. (4.1) at some grid points t_i for $N = 40$, and $\eta = 0.9,\ 0.8,\ 0.7$.

t_i	approximate solution for $\eta = 0.7$	approximate solution for $\eta = 0.8$	approximate solution for $\eta = 0.9$
0.1	0.915035	0.627548	0.341454
0.2	1.16075	0.81039	0.46561
0.3	1.32331	0.938842	0.566838
0.4	1.45524	1.04733	0.661311
0.5	1.57444	1.14705	0.753523
0.6	1.69043	1.24339	0.84513
0.7	1.81015	1.33968	0.936843
0.8	1.94032	1.43845	1.02902
0.9	2.08928	1.54211	1.12188
1	2.26973	1.65343	1.21559

TABLE 3. The MAE of the solution of the Eq. (4.1) and CPU time for $\eta = 1$ and $N = 5,\ 10,\ 20$.

N	MAE(h)	CPU time(s)
5	1.04383×10^{-12}	0.919481
10	5.55112×10^{-16}	1.087944
20	3.40006×10^{-16}	3.306098

Example 4.2. Consider the following C-FFRDE:

$$\begin{aligned} &{}_0^{CF}D_t^{\eta} f(t) = f^2(t) + 1, \qquad 0 < \eta < 1,\ 0 < t \le 1, \\ &f(0) = 0. \end{aligned} \tag{4.2}$$

The exact solution of Eq. (4.2) for $\eta = 1$ (integer order RDE) is $f(t) = \tan t$ [19, 40, 41].

We solve Eq. (4.2) using the compact finite difference method for various values of η and N. The results are presented in Tables 5, 6, 7, and 8, as well as Figures 3 and 4. In Table 5, the exact and approximate solutions of Eq. (4.2) at the grid points t_i along with the absolute error for $\eta = 1$ and $N = 40$, are provided. In Table 6, the approximate solution of Eq. (4.2) at the grid points t_i are provided for $N = 10$ and $\eta = 0.95, 0.85, 0.75$. Table 7 presents the MAE of the solution of Eq. (4.2), ROC, and CPU time for $\eta = 1$ and $N = 5, 10, 20, 40$. Table (3.4) compares the absolute error of the solution obtained from solving Example 4.2 for $\eta = 1$, and $N = 40$ using the

TABLE 4. The error of solution of Eq. (4.1) for $\eta = 1$ and $N = 10$ and comparison with the methods of [38] and [4].

t_i	absolute error of our proposed method	absolute error of method [38]	absolute error of method [4]
0.1	$4.5796699766 \times 10^{-16}$	7.45×10^{-7}	1.98×10^{-8}
0.2	$3.3306690739 \times 10^{-16}$	8.51×10^{-7}	1.03×10^{-6}
0.3	$3.8857805862 \times 10^{-16}$	9.30×10^{-7}	8.85×10^{-6}
0.4	$3.8857805862 \times 10^{-16}$	1.08×10^{-6}	3.33×10^{-5}
0.5	$5.5511151231 \times 10^{-16}$	1.14×10^{-6}	7.26×10^{-5}
0.6	$3.3306690739 \times 10^{-16}$	1.14×10^{-6}	9.98×10^{-5}
0.7	$3.3306690739 \times 10^{-16}$	1.21×10^{-6}	8.84×10^{-5}
0.8	$4.4408920985 \times 10^{-16}$	1.04×10^{-6}	1.54×10^{-5}
0.9	$3.3306690739 \times 10^{-16}$	1.13×10^{-6}	4.99×10^{-4}
1	$4.4408920985 \times 10^{-16}$	4.84×10^{-7}	3.47×10^{-3}

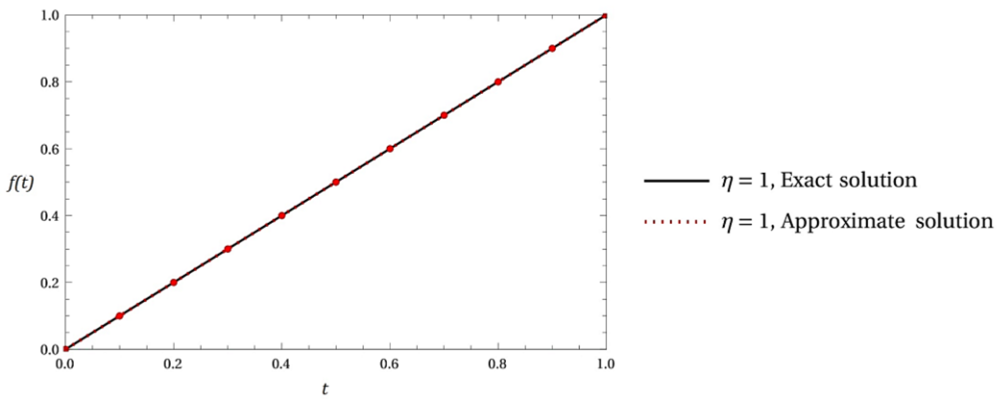

FIGURE 1. Exact solution and approximate solution of Eq. (4.1) for $\eta = 1$ and $N = 10$.

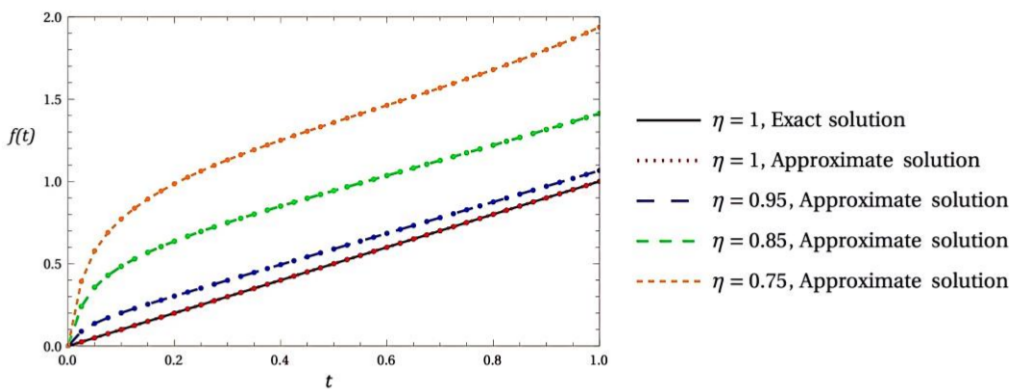

FIGURE 2. Exact solution of Eq. (4.1) for $\eta = 1$ and approximate solutions for $\eta = 1, 0.9, 0.8, 0.7$, $N = 40$.

TABLE 5. The absolute error of the approximate solution for Eq. (4.2) using the proposed method, for $N = 40$, and $\eta = 1$.

t_i	exact solutions	approximate solutions	absolute error
0.1	0.10033467209	0.10032768441	$6.9876785563 \times 10^{-6}$
0.2	0.20271003551	0.20270281345	$7.2220567901 \times 10^{-6}$
0.3	0.30933624961	0.30932861416	$7.6354497600 \times 10^{-6}$
0.4	0.42279321874	0.42628285418	$8.2628285418 \times 10^{-6}$
0.5	0.54630248984	0.54629332770	$9.1621426559 \times 10^{-6}$
0.6	0.68413680834	0.68412638197	$1.0426370521 \times 10^{-6}$
0.7	0.84228838046	0.84227617515	$1.2205318022 \times 10^{-6}$
0.8	1.0296385571	1.0296238114	$14745603556 \times 10^{-6}$
0.9	1.2601582176	1.2601397528	$1.8464770642 \times 10^{-6}$
1	1.5574077247	1.5573836422	$2.4082466655 \times 10^{-6}$

TABLE 6. Values of the approximate solution of Eq. (4.2) at the grid points t_i for $N = 10$, and $\eta = 0.95, 0.85, 0.75$.

t_i	approximate solution for $\eta = 0.75$	approximate solution for $\eta = 0.85$	approximate solution for $\eta = 0.95$
0.1	0.20295	0.274999	0.129639
0.2	0.020003	0.450523	0.235989
0.3	0.0890875	0.593397	0.339721
0.4	0.0278064	0.737066	0.450043
0.5	0.0511954	0.889313	0.569499
0.6	0.00192346	1.06399	0.703231
0.7	0.051855	1.26856	0.855597
0.8	0.0882871	1.52298	1.03558
0.9	0.113735	1.85031	1.25445
1	1.28679	2.30151	1.53332

TABLE 7. MAE of the solution of Eq. (4.2), ROC, and CPU time for $\eta = 1$, and $N = 5, 10, 20, 40$.

N	MAE(h)	ROC	CPU time (s)
5	1.0374×10^{-1}	$-$	1.123224
10	7.96028×10^{-3}	3.704009	1.407011
20	5.18273×10^{-4}	3.941035	3.408236
40	2.40825×10^{-5}	4.427655	21.569029

proposed method with the absolute error obtained from the methods in [40], [41], and [19]. Additionally, Figure 3, displays a graph comparing the exact and approximate solutions of the Eq. (4.2) obtained using the compact finite difference method for $\eta = 1$ and $N = 10$. Figure 4 displays a graph showing the behavior of the approximate solution of Example 4.2 for $N = 10$, and various values of η ($\eta = 1, 0.95, 0.85, 0.75$).

Example 4.3. Consider the following C-FFRDE as the third example:

$$
\begin{aligned}
&{}_{0}^{CF}D_t^{\eta} f(t) = -f^2(t) + 1, \qquad 0 < \eta < 1,\ 0 < t \leq 1, \\
&f(0) = 0.
\end{aligned} \tag{4.3}
$$

TABLE 8. Comparison of the exact and approximate solutions of Eq. (4.2) for $\eta = 1$, and $N = 40$ with the methods in [40], [41], and [19].

t_i	exact solution	approximate solution of our proposed method	approximate solution of method [40]	approximate solution of method [41]	approximate solution of method [19]
0.1	0.10033467209	0.10032768441	0.100330	0.100335	0.100342
0.2	0.20271003551	0.20270281345	0.202708	0.20271	0.202726
0.3	0.30933624961	0.30932861416	0.309334	0.309336	0.309372
0.4	0.42279321874	0.42628285418	0.422791	0.422793	0.422832
0.5	0.54630248984	0.54629332770	0.546300	0.546302	0.546363
0.6	0.68413680834	0.68412638197	0.684134	0.684131	0.684251
0.7	0.84228838046	0.84227617515	0.842285	0.842245	0.842411
0.8	1.0296385571	1.0296238114	1.029635	1.02937	1.029849
0.9	1.2601582176	1.2601397528	1.260154	1.2588	1.260573
1	1.5574077247	1.5573836422	1.557402	1.55137	1.557938

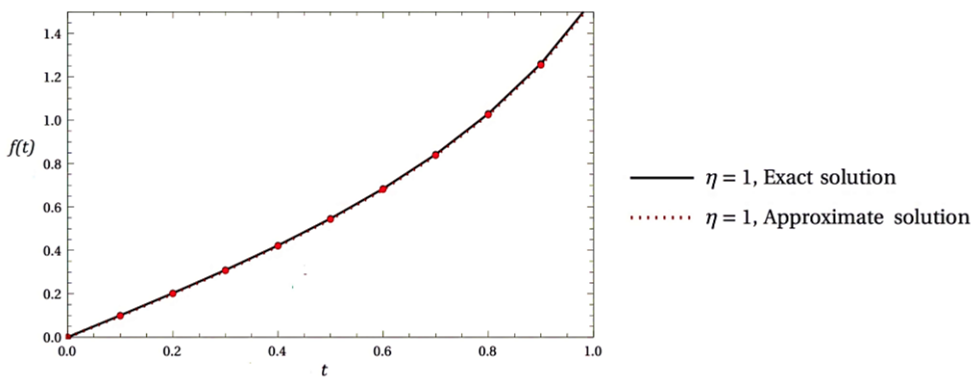

FIGURE 3. The exact solution and approximate solution of Eq. (4.2) for $\eta = 1$ and $N = 10$.

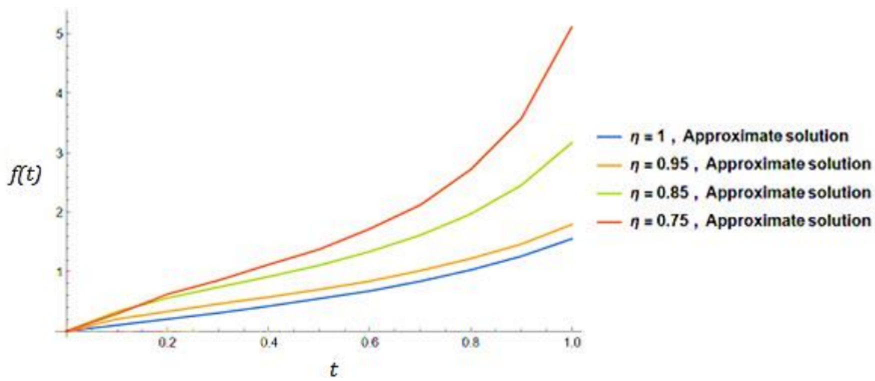

FIGURE 4. Exact solution of Eq. (4.2) for $\eta = 1$ and approximate solutions for $\eta = 1, 0.95, 0.85, 0.75$, and $N = 10$.

TABLE 9. The error of the approximate solution of Eq. (4.3), for $N = 40$, and $\eta = 1$.

t_i	exact solution	approximate solution	absolute error
0.1	0.099667994625	0.099667975251	$1.9373759208 \times 10^{-8}$
0.2	0.19737532022	0.19737530417	$1.6057372432 \times 10^{-8}$
0.3	0.29131261245	0.29131259893	$1.3518546893 \times 10^{-8}$
0.4	0.37994896226	0.37994895027	$1.1981791670 \times 10^{-8}$
0.5	0.46211715726	0.46211714589	$1.1372007946 \times 10^{-8}$
0.6	0.53704956700	0.53704955559	$1.1408848866 \times 10^{-8}$
0.7	0.60436777712	0.60436776538	$1.1739751615 \times 10^{-8}$
0.8	0.66403677027	0.66403675821	$1.2053518517 \times 10^{-8}$
0.9	0.71629787020	0.71629785806	$1.2143552830 \times 10^{-8}$
1	0.76159415596	0.76159414404	$1.1919934373 \times 10^{-8}$

TABLE 10. Values of the approximate solution of Example 4.3 at the grid points t_i for $N = 20$, and $\eta = 0.9, 0.8, 0.7$.

t_i	approximate solution for $\eta = 0.7$	approximate solution for $\eta = 0.8$	approximate solution for $\eta = 0.9$
0.1	0.59683	0.448651	0.264848
0.2	0.732413	0.582324	0.376157
0.3	0.798332	0.66047	0.459516
0.4	0.839661	0.716407	0.530533
0.5	0.868755	0.760013	0.593169
0.6	0.89063	0.795527	0.64879
0.7	0.907775	0.825161	0.698044
0.8	0.921594	0.850235	0.741401
0.9	0.932954	0.87162	0.779317
1	0.942425	0.889937	0.812263

TABLE 11. MAE of the solution of Eq. (4.3), ROC, and CPU time for $\eta = 1$, and $N = 5, 10, 20, 40$.

N	MAE(h)	ROC	CPU time (s)
5	6.56854×10^{-4}	–	0.809669
10	1.12236×10^{-5}	5.87096	1.294869
20	5.91402×10^{-7}	4.24625	3.464271
40	2.11699×10^{-8}	4.80405	21.072132

The exact solution for $\eta = 1$ (integer order RDE) is $f(t) = \dfrac{e^{2t} - 1}{e^{2t} + 1}$ [9, 40].

The relationship between the exact solution and obtained approximate solution of the Eq. (4.3) using the compact finite difference method at the nodes t_i, was investigated for different values of N, and $\eta = 1, 0.9, 0.8, 0.7$. Table 9, presents the exact and approximate solutions of Example 4.3 at the nodes t_i, along with the absolute error for $\eta = 1$ and $N = 40$. In Table 10, the approximate solution of Eq. (4.3) at the nodes t_i are provided for $N = 40$ and $\eta = 0.9, 0.8, 0.7$. Additionally, Table 11 presents the MAE of Example 4.3, ROC, and CPU time concerning $\eta = 1$ and $N = 5, 10, 20, 40$. In Table 12, our approximate solutions of Example 4.3 for $\eta = 1$ are compared with the results of [40] and [9]. Figure 5 shows the plot of the exact and approximate solutions of Example 4.3 for $\eta = 1$ and $N = 10$; and in Figure 6, the behavior of the approximate solution of Eq. (4.3) can be seen for various values of η and $N = 20$.

TABLE 12. Comparison of the exact solution of Eq. (4.3) for $\eta = 1$ and $N = 40$ with the approximate solution obtained from our method and the methods proposed in [40] and [9].

t_i	exact solution	approximate solution of our proposed method	approximate solution of method [40]	approximate solution of method [9]
0.1	0.099667994625	0.099667975251	–	0.099668
0.2	0.19737532022	0.19737530417	0.197374	0.197375
0.3	0.29131261245	0.29131259893	–	0.291312
0.4	0.37994896226	0.37994895027	0.379949	0.379949
0.5	0.46211715726	0.46211714589	–	0.462117
0.6	0.53704956700	0.53704955559	0.537049	0.537050
0.7	0.60436777712	0.60436776538	–	0.604368
0.8	0.66403677027	0.66403675821	0.664037	0.664037
0.9	0.71629787020	0.71629785806	–	0.716298
1	0.76159415596	0.76159414404	0.761594	0.761594

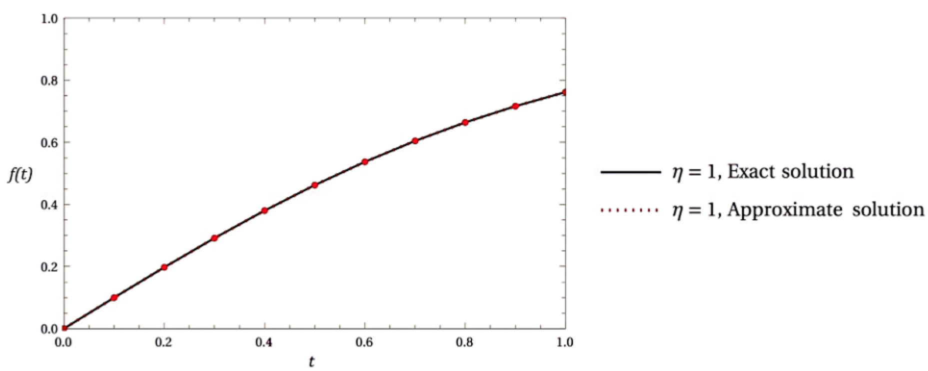

FIGURE 5. Exact solution and approximate solution of Eq. (4.3) for $\eta = 1$ and $h = 0.1$.

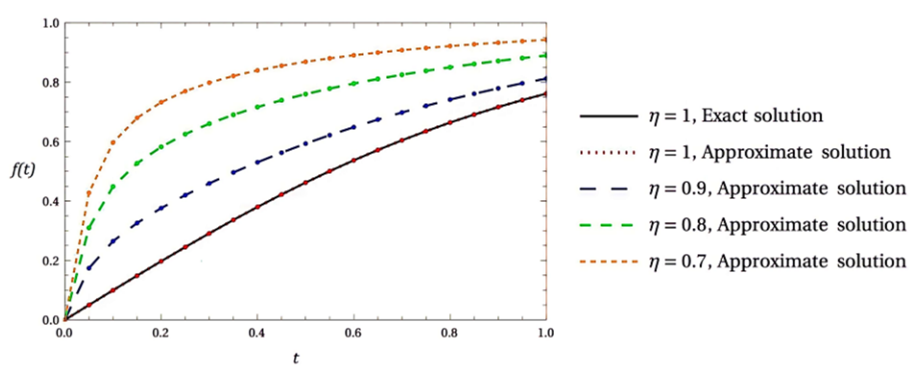

FIGURE 6. Exact solution of Eq. (4.3) for $\eta = 1$ and approximate solutions for $\eta = 1, 0.9, 0.8, 0.7$ and $N = 20$.

5. Conclusion

As mentioned, the Caputo-Fabrizio derivative, due to its non-singular kernel, has a greater capability to describe the natural and physical phenomena compared to classical fractional derivatives and derivatives with a singular kernel. In this work, the non-linear fractional Riccati equation with the Caputo-Fabrizio derivative was considered, and a compact finite difference scheme was proposed for its numerical solution. The proposed scheme has a straightforward algorithm that is computationally efficient, making it applicable for solving the various fractional differential equations with the Caputo-Fabrizio derivative. Our results indicate acceptable accuracy without the need for an excessively small step length. The accuracy increases as the step length decreases. Also, the numerical rate of convergence confirms the theoretical ROC, that it means the used approach has a high order of convergence. Additionally, the runtime of the algorithm is low. The behavior of curves for different values of η preserves the property $\lim_{\eta\to 1} {}_{a}^{CF}D_t^{\eta} f(t) = f'(t)$ of the Caputo-Fabrizio derivative. Also, we compare our results with some other existing methods to solve the mentioned equation; this comparison shows the acceptable accuracy of our method.

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