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Existence result for the fuzzy form of the equation governing the unsteady motion of solid particles in a fluid medium

Masoumeh Zeinali[1,*] **and Ghiyam Eslami**[2]

[1]Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran.

[2]Department of Mechanical Engineering, Ahar Branch, Islamic Azad University, Ahar, Iran.

Abstract

The unsteady drag force in the equation governing the dynamics of small solid particles in the fluid medium appears as an integral Volterra operator in the equation, which is known as the history force. The history force has a kernel whose exact and general form is not known to date. In this article, the very general form of this equation is considered so that both the kernel of the history force and the fields affecting the particle motion can have a general linear or non-linear form. In the present work, the fuzzy form of this equation is proposed as a new method for uncertainty analysis of the problem. Using the Shoulder's fixed point theorem in the semi-linear Banach space, it is proved that the fuzzy form of this equation has a solution.

Keywords. Implicit integro-differential equation, GH-differentiability, Particle motion, Schauder fixed point theorem.

2010 Mathematics Subject Classification. 45JXX, 94DXX.

1. Introduction

The study of unsteady motion of solid particles in a fluid medium has been a subject of significant research [9, 14, 17] due to its extensive applications in various scientific and engineering fields, including sediment transport [18], dispersion of pollutants [19], heat transfer enhancement [13], and industrial processes [10, 11]. Traditional models often employ deterministic approaches to describe the dynamics of particle-fluid interactions. However, real-world scenarios are frequently characterized by inherent uncertainties and imprecisions, which can arise from measurement errors, variations in material properties, or simplified modeling assumptions [12, 20]. To address these uncertainties, fuzzy logic and fuzzy differential equations offer a powerful framework [21, 24, 25]. By incorporating fuzzy set theory, one can develop models that better capture the vagueness inherent in physical systems.

The fuzzy differential equation under consideration extends the classical formulation by integrating fuzzy parameters and initial conditions, thereby providing a more realistic representation of the system. We will employ advanced mathematical techniques and theorems to establish the existence of solutions for this fuzzy differential equation. Such existence results are crucial as they lay the foundational groundwork for further qualitative analyses and numerical simulations. Our study contributes to the broader field of fluid-particle dynamics by drawing on the theory of fuzzy mathematics. It not only enhances the theoretical understanding of the unsteady motion of particles in fluids but also provides practical insights for engineers and scientists dealing with complex and uncertain systems.

Here, we provide a comprehensive overview of recent developments in the study of linear and nonlinear fuzzy integro-differential equations. There's a growing interest in both theoretical and numerical aspects of these equations, particularly in using fuzzy quadrature rules and power series methods for numerical solutions [2, 3]. Several studies have established existence and uniqueness results using the various fixed point theorems [5, 6, 23], with recent works extending these results to semilinear cases [21] and generalized differentiability of fuzzy functions [4]. The classical

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∗ Corresponding author. Email: zeinali@tabrizu.ac.ir.

Banach fixed point theorem has been a primary tool, and more recent work has generalized the Schauder fixed point theorem in semilinear Banach space [1], offering weaker conditions.

Implicit fuzzy integro-differential equations are the main topic of this work since they are applicable to a wide range of engineering applications, including fluid mechanics and particle motion in viscous media. Although these equations have many applications, the existence of their solution, especially in the fuzzy state, has not been thoroughly studied. By taking into account the kind of gH-differentiabilty of the first- and second-order derivatives of the equation's solution, an equivalency lemma that corresponds the integro-differential equation to four nonlinear integral equations has been proven in reference [22]. In this article, the integro-differential equation is first transformed into the appropriate integral equation using this equivalency lemma. Next, the existence of the solution to the related integral is established using the semilinear Banach spaces Schauder fixed point theorem.

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2. Preliminaries

In this study, the symbols $\mathbb{R}_{\mathcal{F}}$ and $\tilde{A}^{\alpha} = [a_{-}^{\alpha}, a_{+}^{\alpha}]$, respectively, will be used to depict the set of all fuzzy numbers and the so-called α-cut of fuzzy numbers.

Theorem 2.1. *([15]) Let A be a fuzzy number and $\tilde{A}^{\alpha} = [a_{-}^{\alpha}, a_{+}^{\alpha}]$ be its α-cuts. Then, the endpoints of the α-cuts are defined by the functions $a_{-}, a_{+} : [0,1] \to \mathbb{R}$, which fulfill the following requirements:*

- **(i)** *$a_{-}(\alpha) = a_{-}^{\alpha} \in \mathbb{R}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$, and it is right-continuous at 0.*
- **(ii)** *$a_{+}(\alpha) = a_{+}^{\alpha} \in \mathbb{R}$ is a bounded, non-increasing, left-continuous function in $(0,1]$, and it is right-continuous at 0.*
- **(iii)** *$a_{-}^{1} \leq a_{+}^{1}$.*

On the other hand, there exists a unique fuzzy number A with a_{-}^{α} and a_{+}^{α} as the endpoints of its α-cuts if the functions $a_{-}, a_{+} : [0,1] \to \mathbb{R}$ meet the requirements **(i)**-**(iii)**.

Definition 2.2. (See [7].) Assume that $\lambda \in \mathbb{R}$ and $\tilde{A}, \tilde{B} \in \mathbb{R}_{\mathcal{F}}$. Level-wise definitions of the sum, H-difference, and scalar product are as follows:

$$(\tilde{A} + \tilde{B})^{\alpha} = \tilde{A}^{\alpha} + \tilde{B}^{\alpha},$$

$$(\tilde{A} \ominus \tilde{B})^{\alpha} = [a_{-}^{\alpha} - b_{-}^{\alpha}, a_{+}^{\alpha} - b_{+}^{\alpha}],$$

$$(\lambda.\tilde{A})^{\alpha} = \lambda \tilde{A}^{\alpha}.$$

The space $\mathbb{R}_{\mathcal{F}}$ is equipped with the metric

$$D(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0,1]} \max\{|a_{-}^{\alpha} - b_{-}^{\alpha}|, |a_{+}^{\alpha} - b_{+}^{\alpha}|\},$$

where $[\tilde{A}]^{\alpha} = [a_{-}^{\alpha}, a_{+}^{\alpha}]$ and $[\tilde{B}]^{\alpha} = [b_{-}^{\alpha}, b_{+}^{\alpha}]$ are α-cuts of $A, B \in \mathbb{R}_{\mathcal{F}}$. The space $\mathbb{R}_{\mathcal{F}}$ with this metric is a complete metric space and

- (i) $D(\tilde{A} + \tilde{B}, \tilde{A} + \tilde{C}) = D(\tilde{B}, \tilde{C})$,
- (ii) description $D(\lambda.\tilde{A}, \lambda.\tilde{B}) = |\lambda| D(\tilde{A}, \tilde{B})$,
- (iii) $D(\tilde{A} + \tilde{B}, \tilde{C} + \tilde{E}) \leq D(\tilde{A}, \tilde{C}) + D(\tilde{B}, \tilde{E})$.

The space $\mathbb{R}^c_{\mathcal{F}}$ is defined as follows:

$$\mathbb{R}^c_{\mathcal{F}} = \left\{ A \in \mathbb{R}_{\mathcal{F}} | \alpha \to [u]^\alpha \quad \text{is continuous} \right\}.$$

Definition 2.3. Let $B \subseteq \mathbb{R}^c_{\mathcal{F}}$. If there exists $M \subseteq \mathbb{R}$ such that for all $x \in B$, we have $[x]^0 \subseteq M$, then it is called compactly supported.

Definition 2.4. Let $B \subseteq \mathbb{R}^c_{\mathcal{F}}$, $\alpha_0 \in [0,1]$, and

$$d_H([x]^\alpha, [x]^{\alpha_0}) = max\{|x^\alpha_- - x^{\alpha_0}_-|, |x^\alpha_+ - x^{\alpha_0}_+|\},$$

be the Hausdorff distance. If

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{s.t } |\alpha - \alpha_0| < \delta \Rightarrow d_H([x]^\alpha, [x]^{\alpha_0}) < \epsilon, \qquad \forall x \in B,$$

then, B is called level-equicontinuous at α_0.

Theorem 2.5. *([1]) Assume that B is a subset of $\mathbb{R}^c_{\mathcal{F}}$ that has compact support. Then, B is level-equicontinuous on $[0,1]$ if and only if it is a relatively compact subset of $(\mathbb{R}^c_{\mathcal{F}}, D)$.*

Definition 2.6. ([1]) The term "semilinear metric space" refers to a semilinear space M that has a metric $d : M \times M \to \mathbb{R}_+$ if

- $d(kA, kB) = kd(A, B)$ for any $A, B \in S$ and $k \geq 0$ (positive homogeneity),
- $d(A + B, C + B) = d(A, C)$ for any $A, B, C \in M$ (translation invariance).

The definition of a norm in this space is $\|x\| = d(x, 0)$. If B is both a complete metric space and a semilinear space at the same time, it is referred to as a semilinear Banach space. Consequently, since the set of real fuzzy numbers is both semilinear and a complete metric space, even though it is not a Banach space, it can be a semilinear Banach space.

Theorem 2.7. *([1]) Assume that $T : S \to S$ is a compact operator and that S is a nonempty, closed, bounded, and convex subset of a semilinear Banach space B with the cancellation property. Then, T has at least one fixed point in S.*

Remark 2.8. Theorem 2.7 may be applied to $\mathbb{R}^c_{\mathcal{F}}$ since it is a semilinear Banach space with the cancellation property.

3. The Maxey-Riley equation of particle motion

The most advanced and complete equation known to date for the prediction of small solid particle dynamics in the fluid medium is the Maxey-Riley equation. This equation is a balance between different forces acting on the particle, including buoyancy force, the force caused by the stress gradient of the fluid, the virtual mass force, steady Stokes drag, and the unsteady Basset force. It is

$$\begin{aligned} m_p \frac{dv_p}{dt} &= (m_p - \rho_f V_p)g + \rho_f V_p \frac{Dv_f}{Dt} - k\rho_f V_p \frac{d}{dt}(v_p - v_f - \frac{1}{10}a^2 \nabla^2 v_f) \\ &- 6\pi\mu a(v_p - v_f - \frac{1}{6}a^2\nabla^2 v_f) - 6\pi\mu a^2 \int_0^t \frac{\frac{d}{d\tau}(v_p(\tau) - v_f(\tau) - \frac{1}{6}a^2\nabla^2 v_f)}{\sqrt{\pi\nu(t-\tau)}} d\tau, \end{aligned} \tag{3.1}$$

where the subscripts p and f stand for the particle and fluid, respectively. $v_p = v_p(t)$ is the Lagrangian velocity of the particle, and $v_f = v_f(r, t)$ is an arbitrary Eulerian velocity of the fluid flow at the particle location. m is mass, ρ is mass density, a is the particle radius, and μ is the fluid dynamic viscosity. In the case of having any other force originating from agents such as electric fields, the corresponding force should be added to the equation.

The Maxey-Riley equation can predict the time evolution of the particle trajectory and velocity inside an arbitrary moving fluid provided that the particle Reynolds number remains smaller than unity ($Re << 1$). For fairly large

Reynolds numbers, it is essential to modify the last term on the right-hand side of the equation, known as the history force. Generally, the history force is

$$F_h = 6\pi\mu a^2 \int_0^t K(t-\tau)\frac{dv_p(\tau)}{d\tau}d\tau, \tag{3.2}$$

where $K(t-\tau)$ denotes the history force kernel. No general form has been found for this kernel up to now. The Basset kernel [14, 16]

$$K_{\text{Basset}}(t-\tau) = \left[\frac{4\pi\nu(t-\tau)}{d_p^2}\right]^{-1/2},$$

and Mei and Adrian's kernel [16]

$$K(t-\tau) = \left(\left[\frac{4\pi\nu(t-\tau)}{d_p^2}\right]^{1/2c_1} + \left[\frac{\pi(t-\tau)^2}{f_H\tau_d}Re_p^3\right]^{1/c_1}\right)^{-c_1}, \tag{3.3}$$

are two common ones used for $Re << 1$ and $Re < 170$, respectively (for details see ref. [16]). Mathematically speaking, Eq. (3.1) with a general form of the history force, Eq. (3.2) reduces to

$$x''(t) = f(t, x(t), x'(t)) + \int_0^t k(t, \tau, x(\tau), x'(\tau), x''(\tau))d\tau.$$

Because in practice the values related to the properties of the fluid and the immersed particle are determined experimentally and are uncertain, the dynamic behavior of the particle will also be uncertain. To predict the effect of the uncertainty related to the parameters involved in the problem on the dynamics of the particle, these parameters and the initial conditions of the particle can be considered as fuzzy numbers. In this case, we are faced with the fuzzy form of the governing equation. In the first step, it should be checked whether the fuzzy form of this equation has a solution in principle or not. This work is dedicated to studying the fuzzy form of this equation as described in the next section.

4. Problem description

In the above-discussed subject of particle dynamics in fluid media, we want to demonstrate the existence of an implicit form of the fuzzy integro-differential equation, Eq. (4.1), with fuzzy initial conditions (4.2) and (4.3):

$$x''(t) = f(t, x(t), x'(t)) + \int_0^t k(t, \tau, x(\tau), x'(\tau), x''(\tau))d\tau, \quad t \in I = [0, b], \tag{4.1}$$

$$x(0) = x_0, \tag{4.2}$$

$$x'(0) = v_0, \tag{4.3}$$

in which $f \in C([0, b] \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}}, \mathbb{R}^c_{\mathcal{F}})$, $k \in C(G \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}}, \mathbb{R}^c_{\mathcal{F}})$, $x_0, v_0 \in \mathbb{R}^c_{\mathcal{F}}$, and

$$G = \left\{(t, s) | t \in [0, b], s \in [0, t]\right\}.$$

Let us represent the space of fuzzy functions with continuous second derivative (in the sense of gH-derivative) as $C^2([0, b], \mathbb{R}_{\mathcal{F}})$. The fuzzy initial value problem (4.1)-(4.3) shall be referred to as FIVP in the rest of this work. Depending on the kind of gH-differentiability for x and x', we present four distinct forms of the FIVP solution in the following definition.

Definition 4.1. A fuzzy function $x \in C^2([0, b], \mathbb{R}^c_{\mathcal{F}})$ that satisfies the Equations (4.1)-(4.3) is a solution for the FIVP. Regarding the distinct type of gH-differentiability, the following solutions can be taken into consideration for FIVP (4.1)-(4.3):

- (i)-(i)-solution: For a solution $x \in C^2([0, b], \mathbb{R}^c_{\mathcal{F}})$, if x and x' are (i)-gH-differentiable, then x is referred to as a (i)-(i)-solution.

- (i)-(ii)-solution: For a solution $x \in C^2([0,b], \mathbb{R}^c_{\mathcal{F}})$, if x is (i)-gH-differentiable and x' is (ii)-gH-differentiable, then x is referred to as a (i)-(ii)-solution.
- (ii)-(i)-solution: For a solution $x \in C^2([0,b], \mathbb{R}^c_{\mathcal{F}})$, if x is (ii)-gH-differentiable and x' is (i)-gH-differentiable, then x is referred to as a (ii)-(i)-solution.
- (ii)-(ii)-solution: For a solution $x \in C^2([0,b], \mathbb{R}^c_{\mathcal{F}})$, if x and x' are (ii)-gH-differentiable, then x is referred to as a (ii)-(ii)-solution.

In the next lemma, FIVP is reduced to an equivalent integral equation (EIE). Therefore, it is possible to analyze the existence and uniqueness of the corresponding integral equation rather than the existence and uniqueness of the solution of the problem under study.

Lemma 4.2. *(The Equivalence Lemma [22]) The equivalent integral equation to FIVP is*

(a)

$$\begin{aligned} y(t) = f\Big[t, x_0 + v_0 t + \int_0^t (t-s)y(s)ds, v_0 + \int_0^t y(s)ds\Big] \\ + \int_0^t k\Big[t, \tau, x_0 + v_0\tau + \int_0^\tau (\tau - s)y(s)ds, v_0 + \int_0^\tau y(s)ds, y(\tau)\Big]d\tau, \end{aligned} \tag{4.4}$$

if FIVP has (i)-(i)-solution x. The following is the relationship between the FIVP solutions and the equivalent integral equation:

$$x(t) = x_0 + v_0 t + \int_0^t (t-s)y(s)ds.$$

(b)

$$\begin{aligned} y(t) = f\Big[t, x_0 + v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds, v_0 \ominus (-1)\int_0^t y(s)ds\Big] \\ + \int_0^t k\Big[t, \tau, x_0 + v_0\tau \ominus (-1)\int_0^\tau (\tau - s)y(s)ds, v_0 \ominus (-1)\int_0^\tau y(s)ds, y(\tau)\Big]d\tau, \end{aligned} \tag{4.5}$$

if FIVP has (i)-(ii)-solution x. The following is the relationship between the FIVP solutions and the equivalent integral equation:

$$x(t) = x_0 + v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds.$$

(c)

$$\begin{aligned} y(t) = f\Big[t, x_0 \ominus (-1)v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds, v_0 + \int_0^t y(s)ds\Big] \\ + \int_0^t k\Big[t, \tau, x_0 \ominus (-1)v_0\tau \ominus (-1)\int_0^\tau (\tau - s)y(s)ds, v_0 + \int_0^\tau y(s)ds, y(\tau)\Big]d\tau, \end{aligned} \tag{4.6}$$

if FIVP has (ii)-(i)-solution x. The following is the relationship between the FIVP solutions and the equivalent integral equation

$$x(t) = x_0 \ominus (-1)v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds.$$

(d)

$$\begin{aligned} y(t) = f\Big[t, x_0 \ominus (-1)v_0 t + \int_0^t (t-s)y(s)ds, v_0 \ominus (-1)\int_0^t y(s)ds\Big] \\ + \int_0^t k\Big[t, \tau, x_0 \ominus (-1)v_0\tau + \int_0^t (t - s)y(s)ds], v_0 \ominus (-1)\int_0^t y(s)ds, y(\tau)\Big]d\tau, \end{aligned} \tag{4.7}$$

if FIVP has (ii)-(ii)-solution x. The following is the relationship between the FIVP solutions and the equivalent integral equation

$$x(t) = x_0 \ominus (-1)v_0 t + \int_0^t (t-s)y(s)ds.$$

Using the equivalence lemma, FIVP has a unique solution if and only if the equivalent fuzzy integral equation (FIE) has a unique continuous solution. In the following theorem, we use the Schauder fixed point theorem in semilinear Banach spaces (introduced in section 2) to show the existence of the solution for the pertaining FIEs, Equations (4.4)-(4.7).

Theorem 4.3. *Suppose that*

(i) $k \in C(G \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}}, \mathbb{R}^c_{\mathcal{F}})$, *which yields*

$$\exists M_k > 0; \|k\| \leq M_k,$$

(ii) $f \in C([0,b] \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}}, \mathbb{R}^c_{\mathcal{F}})$, *which yields*

$$\exists M_f > 0; \|f\| \leq M_f,$$

(iii) $M_f + bM_k \leq R$,

(iv) *The functions f and k are compact,*

(v) *The functions v_0^- and v_0^+ (defined as in Theorem 2.1) are strictly increasing differentiable and strictly decreasing differentiable on $[0,1]$, respectively. There exist the constants $\underline{c_1}, \overline{c_1} > 0$, and $\underline{c_2}, \overline{c_2} < 0$, such that*

$$0 < \underline{c_1} < ([v_0]^{\alpha}_{-})' < \overline{c_1}, \qquad \underline{c_2} < ([v_0]^{\alpha}_{+})' < \overline{c_2} < 0,$$

for all $\alpha \in [0,1]$ and $len[v_0]^1 = 0$,

(vi) *The functions x_0^- and x_0^+ are strictly increasing differentiable and strictly decreasing differentiable on $[0,1]$, respectively, and there exist the constants $\underline{d_1}, \overline{d_1} > 0$, and $\underline{d_2}, \overline{d_2} < 0$, such that*

$$0 < \underline{d_1} < ([x_0]^{\alpha}_{-})' < \overline{d_1}, \qquad \underline{d_2} < ([x_0]^{\alpha}_{+})' < \overline{d_2} < 0,$$

for all $\alpha \in [0,1]$ and $len[x_0]^1 = 0$.

Then,

A. *Assuming* **(i)**-**(iv)**, *there exists a continuous global solution for FIE (4.4).*
B. *Assuming* **(i)**-**(v)**, *there exists a continuous global solution for FIE (4.5).*
C. *Assuming* **(i)**-**(vi)**, *there exists a continuous global solution for FIE (4.6).*
D. *Assuming* **(i)**-**(v)**, *there exists a continuous global solution for FIE (4.7).*

Proof. **Case A**. Consider the operator $T : \Lambda \to \Lambda$ defined as

$$Ty(t) = f\left[t, x_0 + v_0 t + \int_0^t (t-s)y(s)ds, v_0 + \int_0^t y(s)ds\right] + \int_0^t k\left[t, \tau, x_0 + v_0\tau + \int_0^\tau (\tau-s)y(s)ds, v_0 + \int_0^\tau y(s)ds, y(\tau)\right]d\tau,$$

where

$$\Lambda := \left\{ y \in C([0,b], \mathbb{R}^c_{\mathcal{F}}); d(y, \hat{0}) \leq R \right\}.$$

Obviously, the fixed point of the operator T is the solution of Eq. (4.4). So, to establish the existence of the solution of Eq. (4.4), it is sufficient to prove the operator T has a fixed point.

First of all, we have to prove T maps Λ to itself. To do this, let $y \in \Lambda$ and $t_1, t_2 \in [0,b]$ $(t_1 < t_2)$, so that

$$\begin{aligned}
D(Ty(t_1), Ty(t_2)) \leq D\Bigg(& f\left[t_1, x_0 + v_0 t_1 + \int_0^{t_1} (t_1 - s)y(s)ds, v_0 + \int_0^{t_1} y(s)ds\right] \\
& f\left[t_2, x_0 + v_0 t_2 + \int_0^{t_2} (t_2 - s)y(s)ds, v_0 + \int_0^{t_2} y(s)ds\right]\Bigg) \\
& + \int_0^{t_1} D\Bigg(k\left[t_1, \tau, x_0 + v_0\tau + \int_0^{\tau} (\tau - s)y(s)ds, v_0 + \int_0^{\tau} y(s)ds, y(\tau)\right] \\
& k\left[t_2, \tau, x_0 + v_0\tau + \int_0^{\tau} (\tau - s)y(s)ds, v_0 + \int_0^{\tau} y(s)ds, y(\tau)\right]\Bigg) d\tau \\
& + \int_{t_1}^{t_2} D\Bigg(\hat{0}, k\left[t_2, \tau, x_0 + v_0\tau + \int_0^{\tau} (\tau - s)y(s)ds, v_0 + \int_0^{\tau} y(s)ds, y(\tau)\right]\Bigg) d\tau.
\end{aligned}$$

Taking into account the continuity of f and k, we can conclude that $D(Ty(t_1), Ty(t_2)) \to 0$ when $t_1 \to t_2$. Thus, $Ty \in C([0,b], \mathbb{R}^c_{\mathcal{F}})$, and to prove that $Ty \in \Lambda$, it is sufficient to show $d(Ty, \hat{0}) \leq R$. By conditions (i), (ii), and (iii), we obtain

$$\begin{aligned}
D(Ty(t), \hat{0}) \leq D\Bigg(& f\left[t, x_0 + v_0 t + \int_0^{t} (t - s)y(s)ds, v_0 + \int_0^{t} y(s)ds\right], \hat{0}\Bigg) \\
& + \int_0^{t} D\Bigg(k\left[t, \tau, x_0 + v_0\tau + \int_0^{\tau} (\tau - s)y(s)ds, v_0 + \int_0^{\tau} y(s)ds, y(\tau)\right], \hat{0}\Bigg) d\tau \\
& \leq M_f + \int_0^{t} M_k d\tau \leq M_f + bM_k \leq R,
\end{aligned}$$

and so

$$d(Ty, \hat{0}) = \sup_{t \in [0,b]} D(Ty(t), \hat{0}) \leq R.$$

Therefore, T maps Λ to itself. We now show the compactness of T. As per the compact operator definition, we need to demonstrate that $T(\Lambda)$ is relatively compact. The Arzela-Ascoli theorem can be used to demonstrate
(i) $T(\Lambda)$ is an equicontinuous subset of $C([0,b], \mathbb{R}^c_{\mathcal{F}})$,
(ii) $T(\Lambda)(t)$ is relatively compact in $\mathbb{R}^c_{\mathcal{F}}$ for each $t \in [0,b]$.
For the case (i), let $y \in \Lambda$ and ϵ be given. Using the continuity of f, there is some $\delta_1 > 0$, such that for (t_1, ν_1, ω_1), $(t_2, \nu_2, \omega_2) \in [0,b] \times \mathbb{R}^c_{\mathcal{F}} \times \mathbb{R}^c_{\mathcal{F}}$:

$$\max\{|t_1 - t_2|, D(\nu_1, \nu_2), D(\omega_1, \omega_2)\} \leq \delta_1 \quad \to \quad D\Big(f(t_1, \nu_1, \omega_1), f(t_2, \nu_2, \omega_2)\Big) \leq \frac{\epsilon}{3}. \tag{4.8}$$

By a simple calculation, it can be proved that if $|t_1 - t_2| \leq \delta_1/R$, then

$$D\left(v_0 + \int_0^{t_1} y(s)ds, v_0 + \int_0^{t_2} y(s)ds\right) \leq \delta_1, \tag{4.9}$$

and if $|t_1 - t_2| \leq \frac{\delta_1}{\|v_0\| + 2bR}$, then

$$D\left(x_0 + v_0 t_1 + \int_0^{t_1} (t_1 - s)y(s)ds, x_0 + v_0 t_2 + \int_0^{t_2} (t_2 - s)y(s)ds\right) \leq \delta_1. \tag{4.10}$$

By combining (4.8), (4.9), and (4.10), if $|t_1 - t_2| \leq \delta_2$ in which $\delta_2 = \min\{\delta_1, \delta_1/R, \frac{\delta_1}{\|v_0\| + 2bR}\}$, then

$$\begin{aligned}
D\Bigg[& f[t_1, x_0 + v_0 t_1 + \int_0^{t_1} (t_1 - s)y(s)ds, v_0 + \int_0^{t_1} y(s)ds], \\
& f([t_2, x_0 + v_0 t_2 + \int_0^{t_2} (t_2 - s)y(s)ds, v_0 + \int_0^{t_2} y(s)ds]\Bigg) \leq \frac{\epsilon}{3}.
\end{aligned} \tag{4.11}$$

On the other hand, since k is continuous, there exists some $\delta_3 > 0$ such that $|t_1 - t_2| \leq \delta_3$ implies

$$D\bigg(k[t_1,\tau,x_0+v_0\tau+\int_0^\tau(\tau-s)y(s)ds, v_0+\int_0^\tau y(s)ds, y(\tau)],$$
$$k[t_2,\tau,x_0+v_0\tau+\int_0^\tau(\tau-s)y(s)ds, v_0+\int_0^\tau y(s)ds, y(\tau)]\bigg) \leq \frac{\epsilon}{3b}. \tag{4.12}$$

Let $\delta = \min\{\delta_2, \delta_3, \frac{\epsilon}{3M_k}\}$, $|t_1 - t_2| \leq \delta$, and $t_2 > t_1$. Then, (4.11) and (4.12) yield

$$D(Ty(t_1), Ty(t_2)) \leq \epsilon,$$

which means $T(\Lambda)$ is equicontinuous. Now, for the case (ii), based on Theorem 2.5, it is enough to prove $T(\Lambda)(t)$ is level-equicontinuous and compactly supported. Define the sets Λ_1 and Λ_2 as

$$\Lambda_1 := \Big\{w \in C^1([0,b], \mathbb{R}^c_{\mathcal{F}}) \mid w(t) = v_0 + \int_0^t y(s)ds \ \&\ y \in \Lambda\Big\}, \tag{4.13}$$

$$\Lambda_2 := \Big\{w \in C^2([0,b], \mathbb{R}^c_{\mathcal{F}}) \mid w(t) = x_0 + v_0 t + \int_0^t (t-s)y(s)ds \ \&\ y \in \Lambda\Big\}, \tag{4.14}$$

clearly, Λ_1 and Λ_2 are bounded.

Using the compactness of f, it can be concluded that $f([0,b], \Lambda_2, \Lambda_1)$ is relatively compact. As a result, it is level-equicontinuous according to Theorem 2.5. Similarly, one can conclude that $k(G, \Lambda_2, \Lambda_1, \Lambda)$ is level-equicontinuous. Thus, for a given $\epsilon > 0$, there exists $\delta > 0$, such that if $|\alpha - \beta| < \delta$, then

$$d_H\bigg([f(t, x_0+v_0t+\int_0^t(t-s)y(s)ds, v_0+\int_0^t y(s)ds)]^\alpha$$
$$[f(t, x_0+v_0t+\int_0^t(t-s)y(s)ds, v_0+\int_0^t y(s)ds)]^\beta\bigg) \leq \frac{\epsilon}{2}, \quad \forall t \in [0,b], y \in \Lambda,$$
$$d_H\bigg([k(t,\tau, x_0+v_0t+\int_0^t(t-s)y(s)ds, v_0+\int_0^t y(s)ds, y(\tau))]^\alpha$$
$$[k(t,\tau, x_0+v_0t+\int_0^t(t-s)y(s)ds, v_0+\int_0^t y(s)ds, y(\tau))]^\beta\bigg) \leq \frac{\epsilon}{2b}, \quad \forall (t,\tau) \in G, y \in \Lambda.$$

Hence, direct calculation implies

$$d_H\bigg([T(\Lambda)(t)]^\alpha, [T(\Lambda)(t)]^\beta\bigg) \leq \epsilon.$$

Therefore, $T(\Lambda)(t)$ is level-equicontinuous in $\mathbb{R}^c_{\mathcal{F}}$. Finally, we have to prove $T(\Lambda)(t)$ is compactly supported. Let $y \in \Lambda$, then

$$[T(y)(t)]^0 = f\bigg(t, [x_0]^0 + t[v_0]^0 + \int_0^t (t-s)[y(s)]^0 ds, [v_0]^0 + \int_0^t [y(s)]^0 ds\bigg)$$
$$+ \int_0^t k\bigg(t, \tau, [x_0]^0 + \tau[v_0]^0 + \int_0^\tau (\tau-s)[y(s)]^0 ds, [v_0]^0 + \int_0^\tau [y(s)]^0 ds, [y(\tau)]^0\bigg) d\tau$$
$$\subseteq f([0,b], \Lambda_2, \Lambda_1) + \int_0^t k(G, \Lambda_2, \Lambda_1, \Lambda) d\tau.$$

Since f and k are compact, $f([0,b], \Lambda_2, \Lambda_1)$ and $k(G, \Lambda_2, \Lambda_1, \Lambda)$ will be relatively compact, and consequently, there exist compact sets k_1 and k_2, such that $f([0,b], \Lambda_2, \Lambda_1) \subseteq k_1$ and $k(G, \Lambda_2, \Lambda_1, \Lambda) \subseteq k_2$. Hence, $T(\Lambda)(t)$ is compactly supported. Now, Theorem 2.7 shows that Equation (4.4) has at least one solution y.

For the cases **B-D**, the proving procedure is very similar to the case **A**, i.e., we consider operators $T_B, T_C, T_D : \Lambda' \to \Lambda'$ defined by

$$T_B(y) = f\Big[t, x_0 + v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds, v_0 \ominus (-1)\int_0^t y(s)ds\Big]$$
$$+ \int_0^t k\Big[t, \tau, x_0 + v_0\tau \ominus (-1)\int_0^\tau (\tau-s)y(s)ds, v_0 \ominus (-1)\int_0^\tau y(s)ds, y(\tau)\Big]d\tau,$$

$$T_C(y) = f\Big[t, x_0 \ominus (-1)v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds, v_0 + \int_0^t y(s)ds\Big]$$
$$+ \int_0^t k\Big[t, \tau, x_0 \ominus (-1)v_0\tau \ominus (-1)\int_0^\tau (\tau-s)y(s)ds, v_0 + \int_0^\tau y(s)ds, y(\tau)\Big]d\tau,$$

$$T_D(y) = f\Big[t, x_0 \ominus (-1)v_0 t + \int_0^t (t-s)y(s)ds, v_0 \ominus (-1)\int_0^t y(s)ds\Big]$$
$$+ \int_0^t k\Big[t, \tau, x_0 \ominus (-1)v_0\tau + \int_0^t (t-s)y(s)ds], v_0 \ominus (-1)\int_0^t y(s)ds, y(\tau)\Big]d\tau,$$

for the case **B**, **C**, and **D** respectively where

$$\Lambda' := \Big\{y \in C([0,b], \mathbb{R}^c_{\mathcal{F}}); d(y, \hat{0}) \leq R \ \& \ |\frac{\partial y^\alpha_-(t)}{\partial \alpha}|, |\frac{\partial y^\alpha_+(t)}{\partial \alpha}| \leq M \ \forall t \in [0,b], \forall \alpha \in [0,1] \ \& \ len([y]^1) = 0\Big\}.$$

The big challenge of these cases is the existence of H-differences that appeared in the above operators. The remaining part of the proof is very similar to the case **A**. So, what we are about to show is the existence of the H-differences involved in the operators T_B, T_C, and T_D. In general, for $u, v \in \mathbb{R}_{\mathcal{F}}$, the H-difference $u \ominus v$ exists if and only if $[u^\alpha_- - v^\alpha_-, u^\alpha_+ - v^\alpha_+]$ defines the α-cuts of a fuzzy number (see [8]). According to Theorem 2.1, we define the operators $w^-, w^+ : [0,1] \to \mathbb{R}$ as

$$w^-(\alpha) = u^\alpha_- - v^\alpha_-,$$
$$w^+(\alpha) = u^\alpha_+ - v^\alpha_+.$$

It is sufficient to prove that the conditions **(i)-(iii)** of Theorem 2.1 hold true for w^- and w^+. Since u and v are fuzzy numbers, the continuity conditions clearly hold true. So, we have to prove w^- is nondecreasing, w^+ is nonincreasing, and $w^-(1) \leq w^+(1)$. Below, we prove the existence of the H-differences for the cases **B-D** separately.

Case B. In definition of the operator T_B the following H-differences appeared

$$v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds, \tag{4.15}$$

$$v_0 \ominus (-1)\int_0^t y(s)ds. \tag{4.16}$$

The existence of H-difference (4.16) can be concluded from [8] Lemma 2.2. In order to prove the existence of (4.15), we have to prove

$$[v_0 t]^\alpha_- - [(-1)\int_0^t (t-s)y(s)ds]^\alpha_- \ is \ nondecreasing,$$
$$[v_0 t]^\alpha_+ - [(-1)\int_0^t (t-s)y(s)ds]^\alpha_+ \ is \ nonincreasing,$$
$$[v_0 t]^1_- - [(-1)\int_0^t (t-s)y(s)ds]^1_- \leq [v_0 t]^1_+ - [(-1)\int_0^t (t-s)y(s)ds]^1_+,$$

or equivalently

$$t[v_0]^\alpha_- + \int_0^t (t-s)y^\alpha_+(s)ds \text{ is nondecreasing}, \tag{4.17}$$

$$t[v_0]^\alpha_+ + \int_0^t (t-s)y^\alpha_-(s)ds \text{ is nonincreasing}, \tag{4.18}$$

$$\int_0^t \frac{t-s}{t} len([y(s)]^1)ds \leq len([v_0]^1). \tag{4.19}$$

Obviously (4.19) holds true because $len([v_0]^1) = 0$ and $len([y(s)]^1) = 0$ for all $s \in [0, b]$. To prove (4.17) and (4.18), it is sufficient to show

$$t([v_0]^\alpha_-)' + \int_0^t (t-s)\frac{\partial y^\alpha_+(s)}{\partial \alpha}ds > 0,$$

$$t([v_0]^\alpha_+)' + \int_0^t (t-s)\frac{\partial y^\alpha_-(s)}{\partial \alpha}ds < 0.$$

Since $([v_0]^\alpha_-)' > \underline{c_1}$ and $\frac{\partial y^\alpha_+(s)}{\partial \alpha} \geq -M$, we have

$$t([v_0]^\alpha_-)' + \int_0^t (t-s)\frac{\partial y^\alpha_+(s)}{\partial \alpha}ds > t\underline{c_1} - \frac{Mt^2}{2} = t(\underline{c_1} - \frac{Mt}{2}) \geq 0,$$

for all $t \in [0, 2\frac{\underline{c_1}}{M}]$, which implies (4.17).

Since $([v_0]^\alpha_+)' < \overline{c_2}$ and $\frac{\partial y^\alpha_+(s)}{\partial \alpha} \leq M$, we have

$$t([v_0]^\alpha_+)' + \int_0^t (t-s)\frac{\partial y^\alpha_-(s)}{\partial \alpha}ds < t\overline{c_2} + \frac{Mt^2}{2} = t(c_2 + \frac{Mt}{2}) \leq 0,$$

for all $t \in [0, -2\frac{\overline{c_2}}{M}]$, which implies (4.18).

Finally, the H-difference (4.15) exists for all $t \in [0, h]$, where $h = \min\{2\frac{\underline{c_1}}{M}, -2\frac{\overline{c_2}}{M}\}$.

Case C. In the definition of T_C, the only H-difference is

$$x_0 \ominus (-1)v_0 t \ominus (-1)\int_0^t (t-s)y(s)ds. \tag{4.20}$$

In a similar way to the case B, in order to prove the existence of this H-difference, we have to prove

$$[x_0]^\alpha_- + t[v_0]^\alpha_+ + \int_0^t (t-s)[y(s)]^\alpha_+ ds \text{ is nondecreasing},$$

$$[x_0]^\alpha_+ + t[v_0]^\alpha_- + \int_0^t (t-s)[y(s)]^\alpha_- ds \text{ is nonincreasing},$$

$$[x_0]^1_- + t[v_0]^1_+ + \int_0^t (t-s)[y(s)]^1_+ ds \leq [x_0]^1_+ + t[v_0]^1_- + \int_0^t (t-s)[y(s)]^1_- ds,$$

or equivalently

$$([x_0]^\alpha_-)' + t([v_0]^\alpha_+)' + \int_0^t (t-s)\frac{\partial [y(s)]^\alpha_+}{\partial \alpha}ds > 0, \tag{4.21}$$

$$([x_0]^\alpha_+)' + t([v_0]^\alpha_-)' + \int_0^t (t-s)\frac{\partial [y(s)]^\alpha_-}{\partial \alpha}ds < 0, \tag{4.22}$$

$$len([v_0]^1)t + \int_0^t (t-s)len([y(s)]^1)ds \leq len([x_0]^1). \tag{4.23}$$

Obviously, (4.23) holds true because $len([v_0]^1) = 0$, $len([x_0]^1) = 0$, and $len([y(s)]^1) = 0$ for all $s \in [0, b]$. Since

$$([x_0]^\alpha_-)' > \underline{d_1}, \qquad ([v_0]^\alpha_+)' > \underline{c_2}, \qquad \frac{\partial [y(s)]^\alpha_+}{\partial \alpha} \geq -M,$$
$$([x_0]^\alpha_+)' < \overline{d_2}, \qquad ([v_0]^\alpha_-)' < \overline{c_1}, \qquad \frac{\partial [y(s)]^\alpha_-}{\partial \alpha} \leq M,$$

hence,

$$([x_0]^\alpha_-)' + t([v_0]^\alpha_+)' + \int_0^t (t-s)\frac{\partial [y(s)]^\alpha_+}{\partial \alpha} ds > \underline{d_1} + t\underline{c_2} - M\frac{t^2}{2}, \tag{4.24}$$
$$([x_0]^\alpha_+)' + t([v_0]^\alpha_-)' + \int_0^t (t-s)\frac{\partial [y(s)]^\alpha_-}{\partial \alpha} ds < \overline{d_2} + t\overline{c_1} + M\frac{t^2}{2}. \tag{4.25}$$

By elementary calculus, it can be easily verified that for $t \in \left[0, \frac{\underline{c_2} + \sqrt{\underline{c_2}^2 + 4M\underline{d_1}}}{M}\right]$, we have

$$\underline{d_1} + t\underline{c_2} - M\frac{t^2}{2} > 0,$$

and for $t \in \left[0, \frac{-\overline{c_1} + \sqrt{\overline{c_1}^2 - 2M\overline{d_2}}}{M}\right]$, we have

$$\overline{d_2} + t\overline{c_1} + M\frac{t^2}{2} < 0.$$

So for $t \in [0, h]$, where $h = \min\{\frac{\underline{c_2} + \sqrt{\underline{c_2}^2 + 4M\underline{d_1}}}{M}, \frac{-\overline{c_1} + \sqrt{\overline{c_1}^2 - 2M\overline{d_2}}}{M}\}$, the following inequalities hold true:

$$\underline{d_1} + t\underline{c_2} - M\frac{t^2}{2} > 0, \quad \overline{d_2} + t\overline{c_1} + M\frac{t^2}{2} < 0.$$

These, along with the inequalities (4.24) and (4.25), imply (4.21) and (4.22). Accordingly, for $t \in [0, h]$, the H-difference (4.20) exists.

Case D. In the definition of T_D, the following H-differences appeared:

$$x_0 \ominus (-1)v_0 t, \qquad v_0 \ominus (-1)\int_0^t y(s)ds.$$

The existence of the second H-differences can be concluded from [8] Lemma 2.2. In order to prove the existence of $x_0 \ominus (-1)v_0 t$ in a similar reasoning to previous cases, it is enough to prove

$$([x_0]^\alpha_-)' + ([v_0]^\alpha_+)' t > 0, \tag{4.26}$$
$$([x_0]^\alpha_+)' + ([v_0]^\alpha_-)' t < 0, \tag{4.27}$$
$$[x_0]^1_- + [v_0]^1_+ t \leq [x_0]^1_+ + [v_0]^1_- t. \tag{4.28}$$

Obviously, (4.28) holds true because $len([x_0]^1) = 0$ and $len([v_0]^1) = 0$. Since

$$([x_0]^\alpha_-)' > \underline{d_1}, \qquad ([v_0]^\alpha_+)' > \underline{c_2},$$
$$([x_0]^\alpha_+)' < \overline{d_2}, \qquad ([v_0]^\alpha_-)' < \overline{c_1},$$

we have

$$([x_0]^\alpha_-)' + ([v_0]^\alpha_+)' t > \underline{d_1} + t\underline{c_2} > 0, \tag{4.29}$$
$$([x_0]^\alpha_+)' + ([v_0]^\alpha_-)' t < \overline{d_2} + t\overline{c_1} < 0, \tag{4.30}$$

for any $t \in [0, h]$, where $h = \min\{-\frac{\underline{d_1}}{\underline{c_2}}, -\frac{\overline{d_2}}{\overline{c_1}}\}$. Therefore, for any $t \in [0, h]$, H-difference $x_0 \ominus (-1)v_0 t$ exists. □

5. Conclusion

The fuzzy form of the equation governing the accelerated motion of particles in a fluid medium was investigated by including the force related to unsteady drag (the history force) with an arbitrary kernel. Using the Schauder fixed point theorem in semilinear Banach spaces, it was proved that under certain conditions the aforementioned equation has a solution. Due to the arbitrary nature of the history force kernel, there is no Stokes flow limitation in this problem.

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