Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-12
DOI:10.22034/cmde.2024.61426.2644

Existence results for nonlinear elliptic $\overrightarrow{p}(\cdot)-$equations

Mokhtar Naceri[*]
ENS of Laghouat, Box 4033 Station post avenue of Martyrs, 03000, Laghouat, Algeria.

Abstract

This paper studies the existence of distributional solutions for nonlinear elliptic $\overrightarrow{p}(\cdot)-$equations, focusing on the right-hand side which is a sum of a datum $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$ independent of u, and a compound nonlinearity composed of a given function $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, the solution u and its partial derivatives $\partial_i u$, $i \in \{1, \dots, N\}$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$, $L^{\overrightarrow{p}'(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces.

Keywords. Variable exponent, Nonlinear elliptic equation, Anisotropic Lebesgue-Sobolev space, Distributional solution, Existence, Compound nonlinearity.
2010 Mathematics Subject Classification. 35J60, 35J66, 35J67.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open Lipschitz domain (i.e. with Lipschitz boundary $\partial\Omega$).

Our endeavor here is to prove the existence of distributional solution to the anisotropic nonlinear elliptic problems of the form

$$-\sum_{i=1}^{N} \partial_i\big(|\partial_i u|^{p_i(x)-2}\partial_i u\big) = f(x) + \sum_{i=1}^{N} \big(|g| + |u| + |\partial_i u|\big)^{p_i(x)-1}, \quad \text{in } \Omega. \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

where, $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$ independent of u, and $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$, $L^{\overrightarrow{p}'(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces defined by

$$L^{\overrightarrow{p}(\cdot)}(\Omega) = \bigcap_{i=1}^{N} L^{p_i(\cdot)}(\Omega),\ L^{\overrightarrow{p}'(\cdot)}(\Omega) = \bigcap_{i=1}^{N} L^{p_i'(\cdot)}(\Omega),$$

where $p_i'(\cdot)$ denotes the Hölder congugate of $p_i(\cdot)$, and $\partial_i u = \frac{\partial u}{\partial x_i}$, $i \in \{1, \dots, N\}$.

This paper is concerned with the study of the existence results of distributional solutions concerning a class of $\overrightarrow{p}(x)-$Laplacian problems (i.e. variable exponents anisotropic Laplace operator equations) characterized by a compound nonlinearity, it should also be noted here that this type of operators has many uses in applied sciences (see[3, 9, 18]), and it represents a generalization of $p(x)-$Laplacian (For more similar problems, you can see, but not limited to, the papers [11–17]). The right-hand side of our problem is given in terms of $L^{\overrightarrow{p}'(\cdot)}-$data and nonlinearity $\big(|g| + |u| + |\partial_i u|\big)^{p_i(x)-1}$ with $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$, $L^{\overrightarrow{p}'(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces.

We began our proof in this work by applying Leray-Schauder's fixed point Theorem of existence (For more about fixed point Theorem, can see [19]) in order to prove the existence of a sequence of suitable approximate solutions (u_n).

Received: 29 April 2024 ; Accepted: 23 December 2024.
∗ Corresponding author. Email: nasrimokhtar@gmail.com, m.naceri@ens-lagh.dz .

Then we needed to provide a priori estimates for u_n and its partial derivatives. So, we proved the boundedness of u_n in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and the a.e. convergence in $\overline{\Omega}$ for $\partial_i u_n$, $i \in \{1,\dots,N,\}$, which can be turned into strong L^1-convergence. Then, we pass to the limit strongly in the term $|\partial_i u_n|^{p_i(x)-2}\partial_i u_n$, and in $\big(|g_n| + |u_n| + |\partial_i u_n|\big)^{p_i(x)-1}$. Through this we were able to deduce the convergence of u_n to the solution u of (1.1).

Our paper was staged as follows: Section 2 for some basic concepts about anisotropic variable exponent Lebesgue-Sobolev spaces and some important properties related to them. Our main results with proof are in section 3.

2. Preliminaries and basic concepts

In this section, we will learn about anisotropic Lebesgue-Sobolev spaces with variable exponent and their most important distinctive properties, as explained, for example, in the papers [4, 5, 7].

First, we denote by

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, \qquad 1 < p^- \leq p^+ < \infty\},$$

where, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset,

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- = \min_{x\in\overline{\Omega}} p(x).$$

- Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. Then, $\forall \xi, \xi' \in \mathbb{R}$ and $\forall \varepsilon > 0$ the following inequalities are true :
  - $(*)$ Young's inequality :
  $$|\xi\xi'| \leq \varepsilon|\xi|^{p(x)} + c(\varepsilon)|\xi'|^{p'(x)}, \tag{2.1}$$
  where, $p'(\cdot)$ denotes the Hölder conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$).
  - $(*)$ In addition :
  $$|\xi + \xi'|^{p(x)} \leq 2^{p^+-1}(|\xi|^{p(x)} + |\xi'|^{p(x)}).$$
  - $(*)$ If $(\xi, \xi') \neq (0,0)$,
  $$(|\xi|^{p(x)-2}\xi - |\xi'|^{p(x)-2}\xi')(\xi - \xi') \geq \begin{cases} 2^{2-p^+}|\xi - \xi'|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{2.2}$$
- Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ defined by
$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\},$$

where the function

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx, \text{ is called the convex modular.}$$

It is a Banach and reflexive space when equipped with the Luxemburg norm given by:

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{s > 0 :\ \rho_{p(\cdot)}(\frac{u}{s}) \leq 1\right\},$$

- The following Hölder type inequality holds :
$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$
- Next results(see [4, 5]) we need to use them later. Let (u_n), $u \in L^{p(\cdot)}(\Omega)$, then:

$$\min\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right), \tag{2.3}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.4}$$

- We will now define the main spaces in our paper are anisotropic Sobolev spaces with variable exponents $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1, +\infty)\big),\ i \in \{1, \ldots, N\}$, and $\forall x \in \overline{\Omega}$ we set that

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{1\le i\le N} p_i(x),\, p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}, \quad \overline{p}^\star(x) = \frac{N\overline{p}(x)}{N-\overline{p}(x)}, \text{if } \overline{p}(x) < N.$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \Big\{u \in L^{p_+(\cdot)}(\Omega) \text{ and } \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i \in \{1, \ldots, N\}\Big\},$$

equipped with the following norm :

$$u \mapsto \|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}\,. \tag{2.5}$$

The Banach space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (Our results are based on it) is defined as follow

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega),$$

under the norm (2.5).

• The following important results (see [6, 7]) are needed during the proof steps.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

Lemma 2.1. *If we have $r \in \mathcal{C}_+(\overline{\Omega})$ such that $r(\cdot) < \max(p_+(\cdot),\ \overline{p}^\star(\cdot))$ in $\overline{\Omega}$. Then*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact embedding.} \tag{2.6}$$

Lemma 2.2. *If we have the following condition*

$$p_+(\cdot) < \overline{p}^\star(\cdot) \text{ in } \overline{\Omega}. \tag{2.7}$$

Then, there exists $c >$ independent of u, such that

$$\|u\|_{p_+(\cdot)} \le c\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{2.8}$$

Remark 2.3. If (2.7) holds, then (2.8) implies that

$$u \mapsto \sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)} \text{ is an equivalent norm to (2.5).} \tag{2.9}$$

3. Statement of results and proofs

Definition 3.1. u is a distributional solution of the problem (1.1) if and only if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^\infty(\Omega)$,

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i(x)-2}\partial_i u \partial_i\varphi\, dx = \sum_{i=1}^{N}\int_\Omega (|g| + |u| + |\partial_i u|)^{p_i(x)-1}\varphi\, dx + \int_\Omega f(x)\varphi\, dx.$$

Our main result is that :

Theorem 3.2. *Let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega}),\ i \in \{1, \ldots, N\}$ such that $\overline{p} < N$ and (2.7) holds, and assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then the problem (1.1) has at least one solutions $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the distributional sense.*

Remark 3.3. Condition (2.7) is adopted in our main Theorem in order to consider the norm (2.9) in all steps of our work.

3.1. **Existence of approximate solutions.** Let (f_n) and (g_n) be a two sequences of bounded functions defined in Ω which (f_n), (g_n) converges to f, g in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $L^{\overrightarrow{p}(\cdot)}(\Omega)$ respectively.

Remark 3.4. Since $f_n \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, then from (2.3), we obtain

$$\|f_n\|_{p'_i(\cdot)} \leq 1 + \rho_{p'_i(x)}^{\frac{1}{\overline{p'^{-}_i(\cdot)}}}(f_n) \leq 2 + \rho_{p'_i}^{\frac{1}{\overline{p'^{-}_i}}}(f_n) < \infty.$$

Through this, we conclude that

$$f_n \text{ is bounded in } L^{p'_i(\cdot)}(\Omega),\ i = 1, \ldots, N. \tag{3.1}$$

By following similar arguments to g_n in the space $L^{\overrightarrow{p}(\cdot)}(\Omega)$, we can get that

$$g_n \text{ is bounded in } L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N. \tag{3.2}$$

Lemma 3.5. *Let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$, $i \in \{1, \ldots, N\}$ such that $\overline{p} < N$ and (2.7) holds, and assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then, there exists at least one solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the weak sense to the approximated problems*

$$\begin{aligned} &-\sum_{i=1}^{N} \partial_i\left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right) = f_n(x) + \sum_{i=1}^{N} \left(|g_n| + |u_n| + |\partial_i u_n|\right)^{p_i(x)-1},\ \textit{in } \Omega, \\ &u_n = 0,\ \textit{on } \partial\Omega, \end{aligned} \tag{3.3}$$

in the sense that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2}\partial_i u_n \partial_i \varphi\, dx = \sum_{i=1}^{N} \int_\Omega \left(|g_n| + |u_n| + |\partial_i u_n|\right)^{p_i(x)-1} \varphi\, dx + \int_\Omega f_n(x)\varphi\, dx, \tag{3.4}$$

for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. For $n \geq 1$ fixed in $\mathbb{N}$ and $\forall (v, \xi) \in X \times [0,1]$ which $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right) = \xi\left(f_n + \sum_{i=1}^{N}\left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1}\right), & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Let be the operator: $T : X \times [0,1] \longrightarrow X$ such that :

$$\forall (v, \xi) \in X \times [0,1] :\ u = T(v, \xi) \Leftrightarrow$$

u is the only weak solution of the problem (3.5), verify :

$$\forall \varphi \in X :\ \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2}\partial_i u \partial_i \varphi\, dx = \xi\left(\int_\Omega f_n \varphi\, dx + \sum_{i=1}^{N} \int_\Omega \left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1} \varphi\, dx\right). \tag{3.6}$$

Since it is easy to verify that,

$$\forall (v, \xi) \in X \times [0,1] : \left(f_n + \sum_{i=1}^{N}\left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1}\right) \in L^{\overrightarrow{p}'(\cdot)}(\Omega),$$

then the main Theorem on monotone operators (see [1, 2, 10, 20]) guarantees us the existence of a weak solution u to the problem (3.5) in X, and its uniqueness results directly from the uniqueness of the solution to the problem $(= 0)$, which results from the assumption that there are two weak solutions to (3.5) with taking into account the above assumption that f is independent of u.

Now we will estimate the solution u. choosing $\varphi = u$ in (3.6), and using (2.7), Hölder inequality, and (2.3), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \leq\ & 2\big\|f_n\big\|_{p_i'(\cdot)}\big\|u\big\|_{p_i(\cdot)} + 2\big\|(|g_n| + |v| + |\partial_i v|)^{p_i(x)-1}\big\|_{p_i'(\cdot)}\big\|u\big\|_{p_i(\cdot)} \\ \leq & 2\big\|f_n\big\|_{p_i'(\cdot)}\big\|u\big\|_{p_i(\cdot)} + c\left(1 + \sum_{i=1}^{N}\int_{\Omega}(|g_n| + |v| + |\partial_i v|)^{p_i(x)}\,dx\right)^{\frac{1}{{p'}^{-}_{-}}}\big\|u\big\|_{p_i(\cdot)} \\ \leq & C\big\|u\big\|_{\overrightarrow{p}(\cdot)} + C'\left(1 + \sum_{i=1}^{N}\int_{\Omega}\left(|g_n|^{p_i(x)} + |v|^{p_i(x)} + |\partial_i v|^{p_i(x)}\right)\,dx\right)\big\|u\big\|_{\overrightarrow{p}(\cdot)}. \end{aligned} \tag{3.7}$$

By (2.7), Lemma 2.1, and (2.4), we get

$$\begin{aligned}\int_{\Omega}|v|^{p_i(x)}\,dx &\leq 1 + \|v\|_{p_i(\cdot)}^{p_i^+(x)} \\ &\leq 2 + \|v\|_{p_i(\cdot)}^{p_+^+} \\ &\leq 2 + c\|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}. \end{aligned} \tag{3.8}$$

By (2.7), (2.8), and (2.4), we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_i v|^{p_i(x)}\,dx &\leq N + \sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}^{p_i^+(x)} \\ &\leq 2N + \sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}^{p_+^+} \\ &\leq 2N + \left(\sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}\right)^{p_+^+} \\ &= 2N + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}. \end{aligned} \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we find that

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \leq C\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)\|u\|_{\overrightarrow{p}(\cdot)}. \tag{3.10}$$

From another side, by using (2.4), we can obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \geq \sum_{i=1}^{N}\min\{\big\|\partial_i u\big\|_{p_i(x)}^{p_i^-}, \big\|\partial_i u\big\|_{p_i(x)}^{p_i^+}\}.$$

We put for every $i \in \{1,\dots,N\}$, $\quad \eta_i = \begin{cases} p_+^+, & \text{if } \|\partial_i u\|_{p_i(\cdot)} < 1 \\ p_-^-, & \text{if } \|\partial_i u\|_{p_i(\cdot)} \geq 1 \end{cases}$, we get

$$\begin{aligned}\sum_{i=1}^{N} \min\{\|\partial_i u\|_{p_i(\cdot)}^{p_i^-}, \|\partial_i u\|_{p_i(\cdot)}^{p_i^+}\} &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{\eta_i} \\ &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i,\xi_i = p_+^+\}} \left(\|\partial_i u\|_{p_i(\cdot)}^{p_-^-} - \|\partial_i u\|_{p_i(\cdot)}^{p_+^+}\right) \\ &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i:\eta_i = p_+^+\}} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} \geq \Big(\frac{1}{N}\sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}\Big)^{p_-^-} - N.\end{aligned}$$

Then, we get

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)}\, dx \geq \left(\frac{1}{N}\|u\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} - N. \tag{3.11}$$

From (3.10) and (3.11), we conclude

$$\|u\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq C' \left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right) \|u\|_{\overrightarrow{p}(\cdot)}. \tag{3.12}$$

Then, there exists $c > 0$ such that

$$\|u\|_{\overrightarrow{p}(\cdot)} \leq c\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)^{\frac{1}{p_-^- - 1}}. \tag{3.13}$$

• Prove the continuity of the operator T :
Let $n \geq 1$ fixed in $\mathbb{N}$, and let $(v_k, \xi_k)_{k\geq 1} \subset X \times [0,1]$ be a sequence converges to $(v,\xi) \in X \times [0,1]$. Then, we have

$$v_k \longrightarrow v, \quad \text{Strongly}, \tag{3.14}$$

$$\xi_k \longrightarrow \xi, \quad \text{Strongly}. \tag{3.15}$$

We consider the sequence $(u_k)_{\{k\in\mathbb{N}^*\}}$ where $u_k = T(v_k,\xi_k)$. Then, we get $\forall \varphi \in X$;

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_k|^{p_i(x)-2} \partial_i u_k \partial_i \varphi\, dx = \xi_k \left(\int_{\Omega} f_n \varphi\, dx + \sum_{i=1}^{N} \int_{\Omega} (|g_n| + |v_k| + |\partial_i v_k|)^{p_i(x)-1} \varphi\, dx \right). \tag{3.16}$$

By (3.13) and the fact that $\|v_k\|_{\overrightarrow{p}(\cdot)} < +\infty$ (due (3.14)):

$$\|u_k\|_{\overrightarrow{p}(\cdot)} = \|T(v_k,\xi_k)\|_{\overrightarrow{p}(\cdot)} \leq c\left(1 + \|v_k\|_{\overrightarrow{p}(x)}^{p_+^+}\right)^{\frac{1}{p_-^- - 1}} \leq \delta, \tag{3.17}$$

with $\delta > 0$ independent of k.

From (3.17) we conclude the boundedness of (u_k) in X.
So, there exists a subsequence (still denoted by (u_k)) and $u \in X$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } X. \tag{3.18}$$

First of all, let us show that,

$$\lim_{k\longrightarrow+\infty} \Phi_{i,k} = 0, \tag{3.19}$$

where

$$\Phi_{i,k} = \int_{\Omega} \left(|\partial_i u_k|^{p_i(x)-2} \partial_i u_k - |\partial_i u|^{p_i(x)-2} \partial_i w \right) (\partial_i u_k - \partial_i u)\, dx, \quad i \in \{1,\dots,N,\}.$$

After choosing $\varphi = u_k - u$ in (3.16), we can obtain that

$$\sum_{i=1}^{N} \Phi_{i,k} = \xi_k \int_\Omega f_n(u_k - u) + \sum_{i=1}^{N} \int_\Omega (|g_n| + |v_k| + |\partial_i v_k|)^{p_i(x)-1}(u_k - u)\, dx$$
$$+ \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2}\partial_i u(\partial_i u_k - \partial_i u)\, dx. \tag{3.20}$$

Since $\|f_n\|_{p_i'(\cdot)} < +\infty$ and $\|(|g_n| + |v_k| + |\partial_i v_k|)^{p_i(x)-1}\|_{p_i'(\cdot)} < +\infty$, $u_k \longrightarrow u$ strongly in $L^{r(\cdot)}(\Omega)$ where $r(\cdot)$ mentioned in Lemma 2.1, the fact that $\||\partial_i u|^{p_i(x)-2}\partial_i u\|_{p_i'(\cdot)} < +\infty$, and (3.18), we conclude that the right side of (3.20) goes to 0 when $k \longrightarrow +\infty$, with this we get (3.19).
Now we put

$$\Omega_i^{(1)} = \{x \in \Omega, p_i(x) \geq 2\}, \text{ and } \Omega_i^{(2)} = \{x \in \Omega, 1 < p_i(x) < 2\}.$$

From (2.2), we get

$$2^{2-p_i^+} \int_{\Omega_i^{(1)}} |\partial_i(u_k - u)|^{p_i(x)}\, dx$$
$$\leq \int_{\Omega_i^{(1)}} \left[|\partial_i u_k|^{p_i(x)-2}\partial_i u_k - |\partial_i u|^{p_i(x)-2}\partial_i u\right]\partial_i(u_k - u)\, dx \leq \Phi_{i,k}. \tag{3.21}$$

From another side, we have

$$\int_{\Omega_i^{(2)}} |\partial_i(u_k - u)|^{p_i(x)}\, dx$$
$$\leq \int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^{p_i(x)}}{(|\partial_i u_k| + |\partial_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}}} (|\partial_i u_k| + |\partial_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}}\, dx$$
$$\leq 2\max\left\{\left(\int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^2}{(|\partial_i u_k| + |\partial_i u|)^{2-p_i(x)}}\, dx\right)^{\frac{p_i^-}{2}}, \left(\int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^2}{(|\partial_i u_k| + |\partial_i u|)^{2-p_i(x)}}\, dx\right)^{\frac{p_i^+}{2}}\right\}$$
$$\times \max\left\{\left(\int_\Omega (|\partial_i u_k| + |\partial_i u|)^{p_i(x)}\, dx\right)^{\frac{2-p_i^+}{2}}, \left(\int_\Omega (|\partial_i u_k| + |\partial_i u|)^{p_i(x)}\, dx\right)^{\frac{2-p_i^-}{2}}\right\}$$
$$\leq 2c\max\left\{\left(\Phi_{i,k}\right)^{\frac{p_i^-}{2}}, \left(\Phi_{i,k}\right)^{\frac{p_i^+}{2}}\right\} \times \left(1 + (\rho_{p_i}(|\partial_i u_k| + |\partial_i u|))^{\frac{2-p_-^-}{2}}\right). \tag{3.22}$$

Since u_k, $u \in X$, and (3.19), after letting $k \longrightarrow +\infty$ in (3.21) and in (3.22), we obtain that

$$\lim_{k\to+\infty} \int_\Omega |\partial_i u_k - \partial_i u|^{p_i(x)} = 0, \quad i \in \{1, \ldots, N, \}. \tag{3.23}$$

By using (2.7) and (2.3), we obtain that

$$\|u_k - u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} \|\partial_i(u_k - u)\|_{p_i(\cdot)}$$
$$\leq \sum_{i=1}^{N} \max\left(\rho_{p_i(\cdot)}^{\frac{1}{p_i^+}}(\partial_i u_k - \partial_i u), \rho_{p_i(\cdot)}^{\frac{1}{p_i^-}}(\partial_i u_k - \partial_i u)\right), \tag{3.24}$$

where,

$$\rho_{p_i(\cdot)}(\partial_i u_k - \partial_i u) = \int_\Omega |\partial_i(u_k - u)|^{p_i(x)}\, dx, \ i \in \{1, \ldots, N, \}.$$

By combining (3.23) and (3.24), we conclude that

$$\lim_{k \longrightarrow +\infty} \left\| u_k - u \right\|_{\overrightarrow{p}(\cdot)} = 0. \tag{3.25}$$

Then, (3.25) implies that

$$u_k \longrightarrow u, \text{ Strongly in } X. \tag{3.26}$$

Since the continuity of the function $w \mapsto \sum_{i=1}^{N} (|g_n| + |w| + |\partial_i w|)^{p_i(x)-1}$ on X, we can pass to the limit in (3.16) when $k \longrightarrow +\infty$, and (3.26), we obtain $\forall \varphi \in X$,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx = \xi \left(\int_\Omega f_n \varphi \, dx + \sum_{i=1}^{N} \int_\Omega (|g_n| + |v| + |\partial_i v|)^{p_i(x)-1} \varphi \, dx \right). \tag{3.27}$$

This means that, $u = T(v, \xi)$.
The uniqueness of the weak solution of (3.5) gives us

$$T(v_k, \xi_k) = u_k \longrightarrow u = T(v, \xi), \quad \text{Strongly in } X. \tag{3.28}$$

Then, (3.28) implies the continuity of T.

• Prove the compactness of the operator T : Let $\widehat{B}$ be a bounded of $X \times [0,1]$. Thus $\widehat{B}$ is contained in a product of the type $B \times [0,1]$ with B a bounded of X, which can be assumed to be a ball of center O and of radius $r > 0$.
For $u \in T(\widehat{B})$, we have, thanks to (3.13):

$$\left\| u \right\|_{\overrightarrow{p}(\cdot)} \leq c \left(1 + r^{p_+^+} \right)^{\frac{1}{p_-^- - 1}} = \rho.$$

For $u = T(v, \xi)$ with $(v, \xi) \in B \times [0,1]$ ($\left\| v \right\|_{\overrightarrow{p}(\cdot)} \leq r$).
This proves that T applies $\widehat{B}$ in the closed ball of center O and radius ρ (ρ depend on r) in X.
Let $(u_k) \subset T(\widehat{B})$ be a sequence, so $u_k = T(v_k, \xi_k)$ where $(v_k, \xi_k) \in \widehat{B}$.
Since u_k remains in a bounded of X, it is possible to extract a subsequence (still denoted (u_k)) which converges weakly to an element u of X, and like (3.28) we can get that

$$T(v_k, \xi_k) = u_k \longrightarrow u = T(v, \xi), \quad \text{Strongly in } X.$$

This implies that $\overline{T(\widehat{B})}^{X}$ is compact. Thus, we have proven the compactness of T.
• Let's prove now that $\exists C > 0$, such that

$$\forall (v, \xi) \in X \times [0,1] : v = T(v, \xi) \Rightarrow \left\| v \right\|_{\overrightarrow{P}(\cdot)} \leq C.$$

We have for $v \in X$ such that $v = T(v, \xi)$ meaning that

$$\text{for all } \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) :$$
$$\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)-1} \partial_i v \partial_i \varphi \, dx = \xi \left(\int_\Omega f_n \varphi \, dx + \sum_{i=1}^{N} \int_\Omega (|g| + |v| + |\partial_i v|)^{p_i(x)-1} \varphi \, dx \right). \tag{3.29}$$

Choosing $\varphi = v$ in (3.29), and using Young's inequality, Hölder inequality, Lemma 2.1, and (2.7), we get for all $\varepsilon > 0$:

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i v|^{p_i(x)}\, dx \le 2\|f_n\|_{p_i'(\cdot)}\|v\|_{p_i(\cdot)}$$
$$+ \varepsilon \sum_{i=1}^{N} \int_{\Omega} (|g_n|^{p_i(x)} + |v|^{p_i(x)} + |\partial_i v|^{p_i(x)})\, dx + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega} |v|^{p_i(x)}\, dx$$
$$\le c\|f_n\|_{p_i'(\cdot)}\|v\|_{\overrightarrow{p}(\cdot)} + \varepsilon \sum_{i=1}^{N} \int_{\Omega} (|g_n|^{p_i(x)} + |v|^{p_i(x)} + |\partial_i v|^{p_i(x)})\, dx$$
$$+ C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega} |v|^{p_i(x)}\, dx. \tag{3.30}$$

By choosing $\varepsilon = \frac{1}{2}$, and the use of (3.1), (3.2), and the fact that $v \in L^{p_+(\cdot)}(\Omega)$, we obtain that

$$\|v\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \le c\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}\right). \tag{3.31}$$

By using separation of cases ($\|v\|_{\overrightarrow{p}(\cdot)} > 1$ and $\|v\|_{\overrightarrow{p}(\cdot)} \le 1$), we can easily get from (3.31) that, $\exists C > 0$ independent of n such that

$$\|v\|_{\overrightarrow{p}(\cdot)} \le C. \tag{3.32}$$

Since it is clear that $T(v, 0) = 0$.
Then, the conditions for Leray-Schauder's fixed point Theorem were met. So, the operator $T_0 : X \longrightarrow X$ such that $T_0(u) = T(u, 1)$ accepts a fixed point. Therefore, the proof of Lemma 3.5 was completed. □

3.1.1. *A priori estimates.*

Lemma 3.6. *Let $\{u_n\}$ be the sequence of approximating solutions of (3.4) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Assume f, g, p_i, $i \in \{1, \dots, N\}$ be restricted as in Theorem 3.2. Then, there exists $C > 0$ such that*

$$\|u_n\|_{\overrightarrow{p}(\cdot)} \le C. \tag{3.33}$$

Moreover,

$$\partial_i u_n \longrightarrow \partial_i u \quad a.e.\ in\ \overline{\Omega}, \quad i \in \{1, \dots, N\}. \tag{3.34}$$

Proof. By choosing $\varphi = u_n$ in (3.4), we get that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i(x)}\, dx = \int_{\Omega} f_n u_n\, dx + \sum_{i=1}^{N} \int_{\Omega} (|g| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} u_n\, dx.$$

By using the same way as proof (3.32), we easily get (3.33).

Now, (3.33) implies that, there exists a subsequence (still denoted by (u_n)) and $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega. \tag{3.35}$$

We put

$$\Delta_n = \sum_{i=1}^{N} \Delta_{n,i},$$

where,

$$\Delta_{n,i} = \int_{\Omega} \left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n - |\partial_i u|^{p_i(x)-2}\partial_i u\right)(\partial_i u_n - \partial_i u)\, dx.$$

Let us first prove that,

$$\lim_{n \to +\infty} \Delta_n = 0. \tag{3.36}$$

We have

$$\Delta_n = \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx$$
$$- \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx.$$

By choosing $\varphi = u_n - u$ in (3.4), and using (3.35), the facts that $\|f_n\|_{p_i'(\cdot)} < \infty$, $\|(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\|_{p_i'(\cdot)} < \infty$, we obtain that

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx = 0. \tag{3.37}$$

By (3.35) and the fact that $\||\partial_i u|^{p_i(x)-2}\partial_i u\|_{p_i'(\cdot)} < \infty$, we obtain

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx = 0. \tag{3.38}$$

From (3.37) and (3.38) we obtain (3.36).

From (2.2) we get that

$$\Delta_{n,i} > 0,\ i \in \{1, \ldots, N\}. \tag{3.39}$$

Then, by (3.39) and (3.36) we obtain

$$\Delta_{n,i} \to 0,\ \text{strongly in } L^1(\Omega),\ i \in \{1, \ldots, N\}. \tag{3.40}$$

By extracting a subsequence (still denoted by (u_n)), we conclude that

$$\Delta_{n,i} \longrightarrow 0 \quad \text{a.e. in } \Omega,\ i \in \{1, \ldots, N\}. \tag{3.41}$$

So there is a subset $\Omega' \subset \Omega$ where $|\Omega'| = 0$, and $\forall x \in \Omega - \Omega'$

$$|\partial_i u(x)| < \infty,\ \text{and } \Delta_{n,i} \to 0$$

By (3.41), we get

$$\Delta_{n,i}(x) \le \phi(x), \tag{3.42}$$

for some functions ϕ.

Let's prove the existence of a function ψ such that

$$\big|\partial_i u_n(x)\big| \le \psi(x). \tag{3.43}$$

By (3.42) and (2.2), we obtain

$$\begin{cases} \phi(x) \ge c\left((|\partial_i u_n| - |\partial_i u|)^{p_-^-} - 1\right), & \text{if } p_i(x) \ge 2 \\ \phi(x) \ge c'\left(\frac{|\partial_i u_n| - |\partial_i u|}{|\partial_i u_n| + |\partial_i u| + 1}\right)^2, & \text{if } 1 < p_i(x) < 2. \end{cases} \tag{3.44}$$

Then, (3.44) implies (3.43).

We proceed by contradiction to prove that

$$\partial_i u_n(x) \longrightarrow \partial_i u(x) \text{ in } \Omega - \Omega'. \tag{3.45}$$

Suppose there exists $a \in \Omega - \Omega'$ such that $\lim_{n\to+\infty} \partial_i u_n(a) \ne \partial_i u(a)$.

So, Bolzano Weierstrass Theorem implies that

$$\partial_i u_n(a) \longrightarrow \eta \in \mathbb{R}. \tag{3.46}$$

By passing to the limit in $\Delta_{n,i}(a)$ when $n \longrightarrow +\infty$ and using (3.46), we obtain

$$\left(|\eta|^{p_i(a)-2}\eta - |\partial_i u(a)|^{p_i(a)-2}\partial_i u(a)\right)(\eta - \partial_i u(a)) = 0, \tag{3.47}$$

By (3.47) and (2.2), we conclude that $\eta = \partial_i u(a)$. This gives us (3.34). □

3.2. **Proof of the Theorem 3.2 :** By (3.34) and (3.35), we get that

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \longrightarrow |\partial_i u|^{p_i(x)-2}\partial_i u \text{ a.e. in } \Omega,\ i \in \{1,\dots,N\}. \tag{3.48}$$

By (3.33) we obtain that

$$\int_\Omega \left||\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right|^{p_i'(x)} dx = \int_\Omega |\partial_i u_n|^{p_i(x)} dx \le c,\ i \in \{1,\dots,N\}. \tag{3.49}$$

Then, (3.49) and (2.3) implies that

$$\left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right) \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega),\ i \in \{1,\dots,N\}. \tag{3.50}$$

From Young's inequality and that $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we get $\forall \varepsilon > 0$

$$\begin{aligned}\int_\Omega \left||\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right| dx &= \int_\Omega |\partial_i u_n|^{p_i(x)-1} dx \\ &\le C(\varepsilon) + \varepsilon \int_\Omega |\partial_i u_n|^{p_i(x)} dx \\ &\le C(\varepsilon) + \varepsilon c = C'(\varepsilon).\end{aligned} \tag{3.51}$$

So, we conclude that

$$\left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right) \in L^1(\Omega),\ i \in \{1,\dots,N\}. \tag{3.52}$$

So, through (3.48), (3.52), and (3.50), and Vitali's Theorem [Lemma 3.4. in [8]], we derive, $\forall i \in \{1,\dots,N\}$

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \longrightarrow |\partial_i u|^{p_i(x)-2}\partial_i u\ ,\ \text{Strongly in } L^1(\Omega). \tag{3.53}$$

Now, through (3.34) and (3.35), we obtain that

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \longrightarrow (|g| + |u| + |\partial_i u|)^{p_i(x)-1} \quad \text{a.e. in } \Omega. \tag{3.54}$$

From another side, since $g_n,\ u_n,\ \partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we obtain, $\forall i \in \{1,\dots,N\}$

$$\begin{aligned}\int_\Omega \left|(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\right|^{p_i'(x)} dx &= \int_\Omega (|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)} dx \\ \le c \int_\Omega &\left(|g_n|^{p_i(x)} + |u_n|^{p_i(x)} + |\partial_i u_n|^{p_i(x)}\right) dx \le C.\end{aligned} \tag{3.55}$$

Then, (3.55) and (2.3) implies that, $\forall i \in \{1,\dots,N\}$

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.56}$$

Like the proof of (3.52) with the note that $g_n,\ u_n,\ \partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we can obtain, $\forall i \in \{1,\dots,N\}$

$$\left((|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\right) \in L^1(\Omega). \tag{3.57}$$

Then, from (3.57), (3.54), and (3.56), and Vitali's Theorem, we obtain that, $\forall i \in \{1,\dots,N\}$

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \longrightarrow (|g| + |u| + |\partial_i u|)^{p_i(x)-1} \text{ Strongly in } L^1(\Omega). \tag{3.58}$$

So through this, we can easily pass to the limit in (3.4). Thus Theorem 3.2 has been proven.

Acknowledgment

The author would like to thank the referees for their comments and suggestions.

There is no conflict of interest, nor is there any funding.

References

[1] H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier., *18* (1968), 115–175

[2] F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc., *69* (1963), 862–874.

[3] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., *66* (2006),1383–1406.

[4] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky, and J. Wirth, *Variable Lebesgue Spaces and Hyperbolic Systems*, Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, Basel, 2014.

[5] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics. Springer, New York, *2017* (2011).

[6] X. Fan, *Anisotropic variable exponent Sobolev spaces and* $\overrightarrow{p}(x)$*-Laplacian equations*, Complex Var. Elliptic Equ., *56* (2011), 623–642.

[7] X. Fan and D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$, J. Math. Anal. Appl., *263* (2001), 424–446.

[8] K. H. Karlsen (CMA), *Notes on weak convergence*, (MAT4380 - Spring 2006), 2006.

[9] M. Mihailescu and V. Radulescu, *A multiplicity result for a nonlinear degenrate problem arising in the theory of eletrorheological fluids*, Proc. R. Soc. A., *462* (2006), 2625–2641.

[10] G. J. Minty, *On a "monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A., *50*(1963), 1038–1041.

[11] M. Naceri,*Anisotropic nonlinear elliptic equations with variable exponents and two weighted first order terms*, Filomat, *38*(3) (2024), 1043–1054.

[12] M. Naceri, *Anisotropic nonlinear elliptic systems with variable exponents, degenerate coercivity and* $L^{q}(\cdot)$ *data,* Ann. Acad. Rom. Sci. Ser. Math. Appl., *14*(1-2) (2022), 107–140.

[13] M. Naceri, *Anisotropic nonlinear weighted elliptic equations with variable exponents,* Georgian Math. J., vol., *30*(2) (2023), pp. 277-285.

[14] M. Naceri, *Existence results for anisotropic nonlinear weighted elliptic equations with variable exponents and* L^1 *data*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci., *23*(4) (2022), 337–346.

[15] M. Naceri, *Singular Anisotropic Elliptic Problems with Variable Exponents*, Mem. Differ. Equ. Math. Phys., *85*(2022) (2022), 119–132.

[16] M. Naceri and M. B. Benboubker, *Distributional solutions of anisotropic nonlinear elliptic systems with variable exponents: existence and regularity*, Adv. Oper. Theory., *7*(2) (2022), 1–34.

[17] M. Naceri, F. Mokhtari, *Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity*, Appl. Anal., *100*(11) (2021), 2347–2367.

[18] M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Springer, Berlin. Lecture Notes in Mathematics, *1748*, 2000.

[19] E. Zeidler, *Nonlinear functional analysis and its applications,* Volume I : Fixed-point theorems, Springer-Verlag, New York, 1986.

[20] E. Zeidler, *Nonlinear functional analysis and its applications.* II, B. Nonlinear monotone operators. Translated from the German by the author and Leo F, Boron Springer-Verlag, New York, 1990.