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Numerical solutions for a Branch Crack in a Half-Plane

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Abstract

Branch crack subjected to a remote stress in a half-plane elasticity is modeled into a singular integral equations (SIE) using the distributed dislocation and complex potential method. Numerical solution to the obtained SIE is discovered using the appropriate quadrature formulas. Numerical works exhibit the nature of stress intensity factors (SIF) for each branch are displayed.

Keywords. Singular integral equation, Branch crack problem, Complex potential method, Stress intensity factors, Half-plane elasticity.
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1. Introduction

The appearance of branch cracks, either due to earthquakes, or collisions, or aging structures, may weaken the strength and life cycles of structures or materials. The analysis of cracks may assists the engineers and scientists in assessing the structural integrity, design and maintenance the mechanical parts. This helps prevent accidents and failures that could cause injury or damage. Hence, many researchers have focused their works on investigating the various cracks problems. Isida and Nishino [14] solved the bent crack problem by proposing the body forced method where a force doublet is positioned at the crack center. Chatterjee [1] formulated the branch crack problem of kinked or bent into integral equations by utilizing the Kolosov - Muskhelishvili stress function. Lo [18] used the Kolosov–Muskhelishvili representation and Green's function to formulate the two-dimensional crack problems into a complex SIE, and solved the obtained equation numerically for symmetrical, asymmetrical and doubly symmetrical branched cracks by applying different integration schemes to different branches. Kitigawa and Yuuki [16] and Hasebe et al. [12] used a series approximation and rational approximation of mapping function for solving a kinked crack and branched crack, respectively. Theocaris [29] expanded the works of Kitigawa and Yuuki [16], and added some dislocation distributions to solve the branch crack problem. Ghorbanpoor et al. [9] formulated an unequal branches of branch crack in an infinite plane into SIEs. They proceeded further to investigate the stability and convergence of the method used. Chen and Cheung [2] used the Chebyshev integration rule to solve an integral equation with logarithmic kernel for cracks in a half-plane. Mogilevskaya [20] formulated the piecewise homogeneous cracks in a half-plane into the complex hypersingular integral equations. Complex Lagrange polynomials were used to approximate the unknown functions, and straight segments and circular arcs for the boundaries. Elfakhakhre et al. [6–8] obtained SIEs for curve cracks in a half-plane employing a complex potential technique and distributed dislocation density function. Lin and Keep [17] investigated two- and three-dimensional multiple edges crack using the distributed dislocation method and

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the weight function. The induced Cauchy integral is assessed by recursion relations. Jin and Noda [15] formulated the SIE for an edge crack problem in a non-homogeneous half-plane subjected to steady heat flux. The Fourier transform method was adopted for solving the obtained equation. Seed [27] applied the distributed dislocation method onto a half-pane surface-breaking crack subjected to different loadings and crack configurations. Monfared et al. [21, 22] solved the crack problem with non-uniform edge dislocation in a functionally graded orthotropic half-plane subjected to mixed loading using a system of SIEs.

Hamzah et al. [11] investigated the problem of an unequal length of slanted cracks subjected to various stresses in bonded planes using hypersingular integral equations. The curved length method is taken into account for solving the obtained equations. Hamzah et al. [10] solved numerically the thermally insulated cracks problem in bonded materials, in which the problem is formulated into hypersingular integral equations; the solutions can be archieved straitforwardly. Nourazar et al. [25] considered the mixed mode cracked piezoelectric plane with a general in-plane thermal load. The problem is formulated into singular integral equations, and the Fourier transform method was used to determine the unknown density of the distributed thermo-mechanical dislocations.

In solving the obtained equations for the crack problem, many researchers (eq. Chen and Cheung [2],Elfakhakhre et al. [6]) used the curved length method to transform the obtained singular integral equations (Elfakhakhre et al. [7, 8]) or hypersingular integral equations (Nik Long and Eshkuvatov [24] and Husin et al. [13]) for the curved or inclined crack problems into the singular integral equations or hypersingular integral equations for the straight line. This transformation process ensure less collocation points used in the numerical computation and hence faster convergence is archived.

In this work, we consider the branch crack problem in an elastic half-plane subjected to shear stress. In section 2, some fundamental equations are provided. In section 3, we formulate the problem of branch crack in an elastic half-plane. In section 4, we present a strategy to numerically solve the obtained SIE. In section 5 we discuss some numerical examples, and lastly in section 6 we give the conclusion.

2. Complex variable function method

The fundamentals of complex variable function method are briefly introduced. The stresses $(\delta_x, \delta_y, \delta_{xy})$, the resultant forces function (X, Y), and the displacements vector (u, v) are given as

$$\delta_x + \delta_y = 4Re\Phi(z), \tag{2.1}$$

$$\delta_y - \delta_x + 2i\delta_{xy} = 2[\bar{z}\acute{\Phi}(z) + \acute{\Psi}(z)], \tag{2.2}$$

$$F = -Y + iX = \varphi(z) + z\overline{\Phi(z)} + \overline{\Psi(z)} + c, \tag{2.3}$$

$$2G(u + iv) = \Lambda\varphi(z) - z\overline{\Phi(z)} - \overline{\Psi(z)}, \tag{2.4}$$

where $\varphi(z)$ and $\psi(z)$ are complex potentials, $\Phi(z) = \acute{\varphi}(z)$, $\Psi(z) = \acute{\psi}(z)$, G is the shear modulus for plane elasticity, $\Lambda = \dfrac{3-\varrho}{1+\varrho}$ and $\Lambda = 3 - 4\varrho$, in the plane stress and in the plane strain problems, respectively, ϱ is the Poisson's ratio, z is a complex variable, and $\overline{X}$ conjugates of X [23].

Let $f(z)$ and $g(z)$ be two analytic functions. We define [3]:

$$\frac{d}{dz}\left\{f(z)\overline{g(z)}\right\} = \acute{f}(z)\overline{g(z)} + f(z)\overline{\acute{g}(z)}\frac{d\bar{z}}{dz}. \tag{2.5}$$

It is also obtainable that, Eqs. (2.3) and (2.4) yield

$$\begin{aligned} J_1\left(z, \bar{z}, \frac{d\bar{z}}{dz}\right) &= \frac{d}{dz}\{-Y + iX\} = N + iT \\ &= \Phi(z) + \overline{\Phi(z)} + \frac{d\bar{z}}{dz}(z\overline{\acute{\Phi}(z)} + \overline{\Psi(z)}), \end{aligned} \tag{2.6}$$

$$\begin{aligned} J_2(z, \bar{z}, \frac{d\bar{z}}{dz}) &= (\kappa + 1)\Phi(z) - J_1 = 2G\frac{d}{dz}\{u + iv\} \\ &= \kappa\Phi(z) - \overline{\Phi(z)} - \frac{d\bar{z}}{dz}(z\overline{\acute{\Phi}(z)} + \overline{\Psi(z)}). \end{aligned} \tag{2.7}$$

3. Branch crack in a half-plane elasticity

Consider a branch crack subjected to the remote stress $\sigma_x^\infty = \rho$. Along boundary of a half-plane, the traction free condition is employed, see Figure 3.1(a). In formulating this problem, we proceed as follows: Figure 3.1(a) is a superposition of a remote stress and a half-plane elasticity, Fig. 3.1(b), and an opposite and equivalent loading magnitude imposed on a branch crack, Figure 3.1(c). In addition, the problem shown in Figure 3.1(c) is broke down further into a branch crack in an infinite plane, Figure 3.1(d), and a regular stress field in Figure 3.1(e). The germane complex potentials for the problems in Figures 3.1(d) and 3.1(e) are denoted by $\Phi_p(z)$, $\Psi_p(z)$ and $\Phi_c(z)$ and $\Psi_c(z)$, respectively. Applying the superposition principle, the modified complex potentials are expressed by

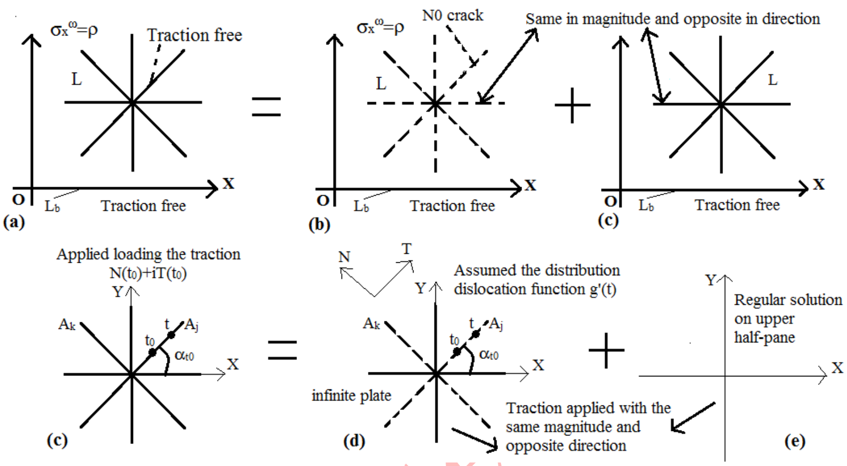

FIGURE 3.1. (a) The original problem, $\sigma_x^\infty = p$, (b) An elastic half-plane, $\sigma_x^\infty = p$, (c) A branch crack with the applied loading, (d) A branch crack in an infinite plate, (e) The upper half-plane's regular solution.

$$\Phi(z) = \Phi_p(z) + \Phi_c(z), \quad \Psi(z) = \Psi_p(z) + \Psi_c(z), \tag{3.1}$$

where p and c denote the principal and the complementary parts, respectively. z is a complex variable. Along the boundary L_b, the traction free condition reads

$$\begin{aligned}(N+iT)_{L_b} &= \Phi_p(z) + \overline{\Phi_p(z)} + z\acute{\Phi}_p(z) + \Psi_p(z) \\ &+ \Phi_c(z) + \overline{\Phi_c(z)} + z\acute{\Phi}_c(z) + \Psi_c(z) = 0,\end{aligned} \tag{3.2}$$

where $\Phi_p(z)$ and $\Psi_p(z)$ are acquired from the complex potential for a crack in an infinite plate.

The suitable complex potentials for the focus distribution functions $\varphi_{p1}(z), \psi_{p1}(z)$ with intensity $H = H_1 + iH_2$ are [5]

$$\varphi_{p1}(z) = \frac{H}{2\pi}\log z, \quad \psi_{p1}(z) = \frac{\bar{H}}{2\pi}\log z. \tag{3.3}$$

Whereas for the distributed dislocation functions $\varphi_{p2}(z), \psi_{p2}(z)$ with density $-\acute{g}(t)$ $(0 < t < a)$, enforce at each arm of the branch crack reads

$$\begin{aligned}\varphi_{p2}(z) &= \frac{-1}{2\pi}e^{i\alpha}\int_0^a \acute{g}(t)\log(z_* - t)dt, \\ \psi_{p2}(z) &= \frac{-1}{2\pi}e^{-i\alpha}\left\{\int_0^a \overline{\acute{g}(t)}\log(z_* - t)dt - \int_0^a \frac{t\acute{g}(t)dt}{t - z_*}\right\},\end{aligned} \tag{3.4}$$

where $g^{\prime}(t)$ is the distributed dislocation function defined as

$$g^{\prime}(t) = \frac{-2Gi}{\kappa+1}\frac{d\left((u(t)+iv(t))^{+}-(u(t)+iv(t))^{-}\right)}{dt}, \tag{3.5}$$

$(u(t)+iv(t))^{+}$ and $(u(t)+iv(t))^{-}$ denote the displacements of the upper and lower faces of crack L, respectively. In (3.4), $z_{*} = ze^{-i\alpha}$. Applying the essential rule of superposition to Eqs. (3.3) and (3.4), the complex potential for the problem in Figure 3.1(d) is obtainable as

$$\begin{aligned}
\varphi_{p}(z) &= \varphi_{p1}(z)+\varphi_{p2}(z) = \frac{H}{2\pi}\log z - \frac{1}{2\pi}\sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j} g_j^{\prime}(t)\log(z_j-t)dt,\\
\psi_{p}(z) &= \psi_{p1}(z)+\psi_{p2}(z) = \frac{\overline{H}}{2\pi}\log z - \frac{1}{2\pi}\sum_{j=1}^{N} e^{-i\alpha_j}\left\{\int_0^{a_j}\overline{g_j^{\prime}(t)}\log(z_j-t)dt\right\}\\
&- \frac{1}{2\pi}\sum_{j=1}^{N} e^{-i\alpha_j}\left\{\int_0^{a_j}\frac{tg_j^{\prime}(t)dt}{t-z_j}\right\}, \qquad (z_j = ze^{-i\alpha_j}).
\end{aligned} \tag{3.6}$$

Differentiating Eqs. (3.6) with respect to z yields

$$\begin{aligned}
\Phi_{p}(z) &= \frac{H}{2\pi z} + \frac{1}{2\pi}\sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j}\frac{g_j^{\prime}(t)dt}{t-z_j},\\
\Psi_{p}(z) &= \frac{\overline{H}}{2\pi z} + \frac{1}{2\pi}\sum_{j=1}^{N} e^{-i\alpha_j}\left\{\int_0^{a_j}\frac{\overline{g_j^{\prime}(t)}dt}{t-z_j} - \int_0^{a_j}\frac{tg_j^{\prime}(t)dt}{(t-z_j)^2}\right\},
\end{aligned} \tag{3.7}$$

From Eq. (3.2), we obtain $\Phi_c(z)$ and $\Psi_c(z)$ as

$$\Phi_c(z) = -\overline{\Phi_p}(z) - z\overline{\Phi_p^{\prime}}(z) - \overline{\Psi_p}(z), \tag{3.8}$$

$$\Psi_c(z) = 3z\overline{\Phi_p^{\prime}}(z) + z^2\overline{\Phi_p^{\prime}}(z) + \overline{\Psi_p}(z) + z\overline{\Psi_p^{\prime}}(z). \tag{3.9}$$

In Eqs. (3.8) and (3.9), $\bar{\Phi}_c(z)$ is analytic, $\overline{\Phi_p}(z) = \overline{\Phi_p(\bar{z})}$, and $\overline{\Psi_p}(z) = \overline{\Psi_p(\bar{z})}$. Consequently, $\phi_c(z)$ and $\psi_c(z)$ are derivable, and perform the differentiation with respect to z, we have

$$\begin{aligned}
\Phi_c(z) &= \frac{-H}{2\pi z} - \frac{1}{2\pi}\sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j}\frac{g_j^{\prime}(t)dt}{\bar{t}-z_j} + \frac{1}{2\pi}\sum_{j=1}^{N} e^{-i\alpha_j}\int_0^{a_j}\frac{(t-\bar{t})\overline{g_j^{\prime}(t)dt}}{(\bar{t}-z_j)^2},\\
\Psi_c(z) &= \frac{\overline{-H}}{2\pi z} + \frac{1}{2\pi}\sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j}\left(\frac{\bar{t}}{(\bar{t}-z_j)^2}\right)g_j^{\prime}(t)dt\\
&+ \frac{1}{2\pi}\sum_{j=1}^{N} e^{-i\alpha_j}\int_0^{a_j}\left\{\frac{3t_0-t}{(\bar{t}-z_j)^2} + \frac{2t_0(t_0-t)}{(\bar{t}-z_j)^3}\right\}\overline{g_j^{\prime}(t)}\overline{dt} \qquad (z_j = ze^{-i\alpha_j}).
\end{aligned} \tag{3.10}$$

The $\phi_p(z)$ and $\psi_p(z)$ are the complex potentials for the crack problem in the infinite region exterior to the crack faces. They may possess a singular point along the arms of branch crack, and may generate some tractions on the boundary of a half-plane, whereas $\phi_c(z)$ and $\psi_c(z)$ define the complex potential functions for a crack problem in a half-plane, and eliminate tractions along the boundary of half-plane produced by $\phi_p(z)$ and $\psi_p(z)$.

The single valuedness condition is

$$\sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j} g_j(t)dt = H \quad (H = H_1 + iH_2). \tag{3.11}$$

Substituting Eq. (3.7) into Eq. (2.6), let z approaches $t_0^{\pm}$, $(t_0^{\pm} \in L^{\pm})$, changing $\frac{d\bar{z}}{dz}$ by $\frac{d\bar{t_0}}{dt_0}$, and the formulas stated in Appendix A, the traction $(N+iT)_p$ on the kth arm of branch crack reads:

$$\frac{1}{\pi} \quad \int_0^{a_k} \frac{\acute{g}_k(t)dt}{t-t_{k0}} + \frac{1}{\pi}\sum_{j=1}^{\acute{N}} \int_0^{a_j} \left[K_{jk}(t_j,t_{k0})\acute{g}_j(t_j) + L_{jk}(t_j,t_{k0})\overline{\acute{g}_j(t_j)}\right] dt_j + \frac{H}{\pi t_{k0}} e^{-i\alpha_k}$$
$$= (N_k(t_{k0}) + iT_k(t_{k0}))_p, \qquad 0 < t_{k0} < a_k \ (k = 1, 2, \ldots, N), \tag{3.12}$$

where $t_{k0} \in L_k$ and L_k is kth arm of branch crack. The first integral is singular. The kernels for the regular integrals are given by

$$K_{jk}(t_j,t_{k0}) \quad = \quad \frac{1}{2}\left(\frac{1}{t_j - t_{k0}e^{i(\alpha_k-\alpha_j)}} + e^{[2i(\alpha_j-\alpha_k)]}\frac{1}{t_j - \overline{t_{k0}e^{i(\alpha_k-\alpha_j)}}}\right), \tag{3.13}$$

and

$$L_{jk}(t_j,t_{k0}) \quad = \quad \frac{1}{2}\left(\frac{1}{t_j - \overline{t_{k0}e^{i(\alpha_k-\alpha_j)}}} - e^{[2i(\alpha_j-\alpha_k)]}\frac{t_j - t_{k0}e^{i(\alpha_k-\alpha_j)}}{\left(t_j - \overline{\left\{t_{k0}e^{i(\alpha_k-\alpha_j)}\right\}}\right)^2}\right). \tag{3.14}$$

Note that the kernels $K_{jk}(t_j,t_{k0})$ and $L_{jk}(t_j,t_{k0})$ depend on t and t_0, and also on α_j and α_k. They are the "two points−two directions" functions [4].

Similarly, substituting Eq. (3.10) into Eq. (2.6), we obtain the traction $(N+iT)_c$ influence along the kth arm of branch crack

$$\frac{1}{2\pi}\sum_{j=1}^{\acute{N}} \int_0^{a_j} D_1(t_j,t_{k0})\acute{g}_j(t_j)dt_j + \frac{1}{2\pi}\sum_{j=1}^{\acute{N}} \int_0^{a_j} D_2(t_j,t_{k0})\acute{g}_j(t_j)d\bar{t}_j$$
$$+\frac{1}{2\pi}\sum_{j=1}^{\acute{N}} \int_0^{a_j} D_3(t_j,t_{k0})\overline{\acute{g}_j(t_j)}dt_j + \frac{1}{2\pi}\sum_{j=1}^{\acute{N}} \int_0^{a_j} D_4(t_j,t_{k0})\overline{\acute{g}_j(t_j)dt_j} - \frac{H}{\pi t_{k0}} e^{-i\alpha_k}$$
$$= (N_k(t_{k0}) + iT_k(t_{k0}))_c, \qquad 0 < t_{k0} < a_k \ (k = 1, 2, \ldots, N), \tag{3.15}$$

where

$$\begin{aligned}
D_1(t_j,t_{k0}) &= \frac{-1}{\bar{t_j} - t_{k0}} + \frac{\bar{t_j} - t_j}{(\bar{t_j} - t_{k0})^2},\\
D_2(t_j,t_{k0}) &= \frac{-1}{t_j - \bar{t_{k0}}} + \frac{t_j - \bar{t_j}}{(\bar{t_j} - t_{k0})^2},\\
D_3(t_j,t_{k0}) &= \frac{d\bar{t_{k0}}}{dt_{k0}}\left(\frac{2t_{k0}(\bar{t_j} - t_j)}{(t_j - \bar{t_{k0}})^3} + \frac{(3\bar{t_{k0}} - \bar{t_j})}{(t_j - \bar{t_{k0}})^2} + \frac{2\bar{t_{k0}}(\bar{t_{k0}} - \bar{t_j})}{(t_j - \bar{t_{k0}})^3}\right),\\
D_4(t_j,t_{k0}) &= \frac{d\bar{t_{k0}}}{dt_{k0}}\frac{t_j - t_{k0}}{(t_j - \bar{t_{k0}})^2}.
\end{aligned} \tag{3.16}$$

For the curved crack, Figure 3.1(c), the boundary condition reads:

$$N(t_{k0}) + iT(t_{k0}) \quad = \quad \tilde{N}(t_{k0}) + i\tilde{T}(t_{k0}), \qquad (t_{k0} \in L_k), \tag{3.17}$$

where the right-hand term is obtained as

$$\tilde{N}(t_{k0}) + i\tilde{T}(t_{k0}) \quad = \quad -p(\sin^2\gamma + i\sin\gamma\cos\gamma), \qquad \text{with } \sigma_x^{\infty} = p, \tag{3.18}$$

and γ is the inclined angle.

Decomposing $N+iT$ into its principle and complementary parts, Eq. (3.17) gives

$$(N_k(t_{k0})+iT_k(t_{k0}))_p+(N_k(t_{k0})+iT_k(t_{k0}))_c \quad = \quad \tilde{N}_k(t_{k0})+i\tilde{T}_k(t_{k0}). \tag{3.19}$$

Due to Eqs. (3.12), (3.15), and Eq. (3.19) yields

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{a_k}\frac{\acute{g}_k(t)dt}{t-t_{k0}}+\frac{1}{\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}K_{jk}(t_j,t_{k0})\acute{g}_j(t_j)dt_j\\
&+\frac{1}{\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}L_{jk}(t_j,t_{k0})\overline{\acute{g}_j(t_j)}dt_j+\frac{1}{2\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}D_1(t_j,t_{k0})\acute{g}_j(t_j)dt_j\\
&+\frac{1}{2\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}D_2(t_j,t_{k0})\acute{g}_j(t_j)\overline{dt_j}+\frac{1}{2\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}D_3(t_j,t_{k0})\overline{\acute{g}_j(t_j)}dt_j\\
&+\frac{1}{2\pi}\acute{\sum}_{j=1}^{N}\int_0^{a_j}D_4(t_j,t_{k0})\overline{\acute{g}_j(t_j)dt_j}=\tilde{N}(t_{k0})+i\tilde{T}(t_{k0}), \qquad (3.20)\\
&0<t_{k0}<a_k \ (k=1,2,\ldots,N),
\end{aligned}$$

where $\acute{\Sigma}$ means the terms corresponding to $j=k$ are omitted. The three leading integrals in Eq. (3.20) means the aftermath of dislocation on the crack k itself, whilst the remaining integrals represent the aftermath on the crack j, for $j=1,2,3,\ldots,n, j\neq k$. The theoretical analysis of the solution of Eq. (3.20), which includes the stability and convergence of solution can be found in the work of Ghorbanpoor et al. [9].

4. Curved length coordinate method

For solving Eq. (3.20) subjected to Eq. (3.11), each of branch cracks is mapped into a real axis s, and s_k is in the interval $[0,a_k], k=1,2,\ldots,n$. The mapping functions are

$$\begin{aligned}
&\acute{g}_k(t_k)_{t_k=t_k(s_k)}=\sqrt{\frac{s_k}{a_k-s_k}}H_k\left(s_k\right), \qquad 0<s_k<a_k,\\
&H_k\left(s_k\right)=H_{k1}\left(s_k\right)+iH_{k2}\left(s_k\right).
\end{aligned} \tag{4.1}$$

Using (4.1) into Eqs. (3.20) and (3.11) result, respectively,

$$\begin{aligned}
&I_{k1}(s_{k0})+I_{k2}(s_{k0})+I_{k3}(s_{k0})+\acute{\Sigma}_{j=1}^{N}\left\{I_{k4}(s_{k0})+I_{k5}(s_{k0})+I_{k6}(s_{k0})+I_{k7}(s_{k0})\right\}\\
&=N_k(s_{k0})+iT_k(s_{k0}), \qquad 0<s_{k0}<a_k, \qquad (4.2)\\
&I_{k8}(s,s_0)-H=0, \qquad \left(H=H_1+iH_2\right), \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
I_{k1}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_k}\sqrt{\frac{s_k}{a_k-s_k}}\frac{1}{s_k-s_{k0}}A_k\left(s_k,s_{k0}\right)ds_k,\\
I_{k2}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}B_k\left(s_l,s_{k0}\right)ds_l,\\
I_{k3}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}C_k\left(s_l,s_{k0}\right)ds_l,\\
I_{k4}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}E_k\left(s_l,s_{k0}\right)ds_l,\\
I_{k5}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}G_k\left(s_l,s_{k0}\right)ds_l,\\
I_{k6}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}M_k\left(s_l,s_{k0}\right)ds_l,\\
I_{k7}(s_{k0}) &= \frac{1}{\pi}\int_0^{a_l}\sqrt{\frac{s_l}{a_l-s_l}}N_k\left(s_l,s_{k0}\right)ds_l,
\end{aligned}$$

$$I_{k8}(s,s_{k0}) = \sum_{j=1}^{N} e^{i\alpha_j}\int_0^{a_j}\sqrt{\frac{s_k}{a_k-s_k}}Y_k(s,s_{k0})\,ds - H = 0,$$

and

$$\begin{aligned}
A_k(s_k,s_{k0}) &= \frac{s_k-s_{0k}}{t_k-t_{0k}}\frac{dt_k}{ds_k}H_k(s_k),\\
B_k(s_l,s_{k0}) &= K_{lk}(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
C_k(s_l,s_{k0}) &= L_{lk}(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
E_k(s_l,s_{k0}) &= D_1(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
G_k(s_l,s_{k0}) &= D_2(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
M_k(s_l,s_{k0}) &= D_3(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
N_k(s_l,s_{k0}) &= D_4(t_l,t_k)\frac{dt_l}{ds_l}H_l(s_l),\\
Y_k(s,s_0) &= H_j(s)\frac{dt}{ds}.
\end{aligned}$$

In evaluating the integrals in Eqs. (4.2) and (4.2), the following formulas are adopted [19]

$$\begin{aligned}
\int_0^a \frac{Q(t)}{t-t_0}\sqrt{\frac{t}{a-t}}dt &= \sum_{r=1}^{R}\frac{A_rQ(t_r)}{t_r-t_0},\\
\int_0^a P(t)\sqrt{\frac{t}{a-t}}dt &= \sum_{r=1}^{R}A_rP(t_r),
\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}
A_r &= \frac{\pi a}{R}\sin^2\frac{r\pi}{2R}\quad (r=1,2,\ldots,R-1),\qquad A_R=\frac{\pi a}{2R},\\
t_r &= a\sin^2\frac{r\pi}{2R}\quad (r=1,2,\ldots,R),\\
t_{r0} &= a\sin^2\frac{(r-0.5)\pi}{2R}\quad (r=1,2,\ldots,R).
\end{aligned}$$

The stress intensity factors at each of crack tips is computed as

$$\begin{aligned}
(K_1-iK_2)_j &= -\sqrt{2\pi}\lim_{s_j\to a_j}\sqrt{a_j-s_j}\acute{g}_j(s_j)\\
&= -\sqrt{2\pi a_j}H_j(a_j)\quad (j=1,2,\ldots,N).
\end{aligned}\tag{4.5}$$

5. Numerical examples

In order to algorithmically approach this problem, we formally assume the traction free condition subjected to $\sigma_x = p$. To do so, we establish complex initial functions with the principal and the complementary parts which are denoted as Eq. (3.1). The process of the used algorithm is the following:

Step 1: Assume $KCT = 1$ where KCT represents remote loading condition, and $MT = 4$ which MT is the number arms of branch crack.

Step 2: Consider $MP(J)$ as the number of terms used in integration, $AP(J)$ as the half length of branch crack and $HQ(J)$ as the inclined angle in terms of degree for J-crack when $J = 1$ to MT.

Step 3: Convert the angles of arms in terms of radian using $HP(J) = HQ(J) * PAI/180.D0$.
Step 4: Define MUA as the sum of MPs for all arms of branch crack and calculate $MUB = 2*MUA$, $MUC = MUB+1$, $MUD = MUB + 2$, $MUE = MUB + 3$.
Step 5: Consider matrix DA with MUE*MUE arrays.
Step 6: Consider k constant and j variable. For each k and j, compute the mutual influence matrix from integral equation which is derived from the kernel K_{jk} and L_{jk} of equations after Eq. (3.15) and call it ST.
Step 7: Consider k constant, for each k, applying influence of H_1 and $H_2 (H = H_1 + iH_2)$ is stored in matrix KL which obtained from the part $\exp(-i\alpha_k)H/\pi s_k$ of equation Eq. (3.15).
Step 8: For each j which varied from $J = 1 : MT$, single-valuedness condition of displacement (CSD) is calculated from Eq. (3.9) and is formed in MN.
Step 9: Assemble matrices ST, KL, and MN into their appropriate locations within DA. The following two sentences also belong to CSD. $DA(MUC, MUC) = -1.D0$, $DA(MUD, MUD) = -1.D0$
Step 10: Based on the value of KCT, $KCT(= 1, 2, 3, or 4)$, the right hand term of algebraic equation is calculated using the hand right of Eq. (3.15) and assembled in DA.
Step 11: Finally, the matrix DA is solved by GAUS elimination process, and solution is stored in the MUE-column (last column) of DA.
Step 12: Consider $TRA = 2.D0 * 0.5D0$ and then $S1 = -TRA * DA(K1G, MUE)$, $S2 = TRA * DA(K1H, MUE)$ which K1G and K1H are the even and odd rows respectively, so we have $SIF(K, 1) = S1$, $SIF(K, 2) = S2$.
Using this algorithm we can solve the problems step by step.

Example 5.1. To validate our results, we first look at a perpendicular crack problem in a half-plane with traction free subjected to $\sigma_x = p$ (see Figure 5.1)). Table 1 indicates our approache converges faster, and comply well with Chen and Cheung[2] and Tada et al.[28].

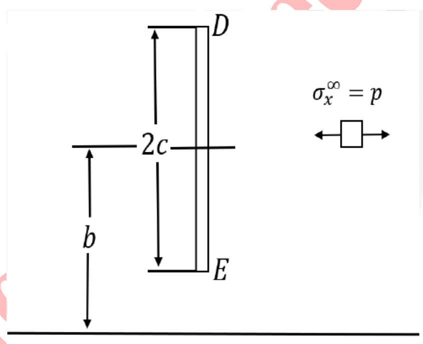

FIGURE 5.1. perpendicular crack in a half-plane subjected to $\sigma_x = \rho$.

Example 5.2. Consider a branch crack problem subjected to $\sigma_x^\infty = p$ in a half-plane elasticity (see Figure 5.2). Table 2 shows F_{1A} increases as c/b increases, which means the plane becomes more vulnerable as the size of crack grows. Table 3 displays that, for $c/b = 0.7$, F_{1A} and F_{2A} decrease as the number of arms increases. We note that $F_{1A} = F_{1B} = F_{1C} = F_{1D} = F_{1E} = F_{1G} = F_{1H} = F_{1L}$.

Example 5.3. Consider a branch crack problem as shown in Figure 5.3. $OA = OB = OC = OD = ON = OP = OQ = a$ and $OE = OG = OH = OL = OM = c$. In the evaluation, we have used

$$M_A = M_B = M_C = M_D = M_N = M_P = M_Q = M_{\max} = 29,$$
$$M_E = M_G = M_H = M_L = M_M = M_A * \sqrt{(c/a)}.$$

At the end tips of $A, B, \ldots, Q$, the SIFs are given by

$$K_{1D} = F_{1D}(n, c/a)\, p\sqrt{\pi a},$$

TABLE 1. Non-dimensional SIF, $F_{1E}(c/b)$ and $F_{1D}(c/b)$ for a perpendicular crack problem in an elastic half-plane (Figure 5.1).

M	c/b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F_{1E} values$										
6	*	1.0026	1.0112	1.0272	1.0528	1.0914	1.1495	1.2401	1.3970	1.7492
10	*	1.0026	1.0113	1.0272	1.0528	1.0913	1.1490	1.2379	1.3881	1.7161
13	*	1.0027	1.0112	1.0272	1.0528	1.0913	1.1490	1.2379	1.3876	1.7095
17	*	1.0026	1.0112	1.0272	1.0528	1.0913	1.1490	1.2379	1.3875	1.7079
25	*	1.0026	1.0112	1.0272	1.0528	1.0913	1.1490	1.2379	1.3875	1.7079
17	**	1.0153	1.0240	1.0403	1.0664	1.1058	1.1649	1.2563	1.4108	1.7470
——	***	——	——	1.033	1.052	1.094	1.148	1.243	1.385	1.688
$F_{1D} values$										
6	*	1.0024	1.0092	1.0201	1.0349	1.0539	1.0775	1.1071	1.1459	1.2056
10	*	1.0024	1.0092	1.0201	1.0349	1.0539	1.0776	1.1074	1.1463	1.2034
13	*	1.0024	1.0092	1.0201	1.0349	1.0539	1.0776	1.1074	1.1464	1.2039
17	*	1.0024	1.0092	1.0201	1.0349	1.0539	1.0776	1.1074	1.1464	1.2038
25	*	1.0024	1.0092	1.0201	1.0349	1.0539	1.0776	1.1074	1.1464	1.2038
17	**	1.0150	1.0219	1.0328	1.0477	1.0667	1.0904	1.1201	1.1588	1.2157
——	***	——	——	1.018	1.031	1.050	1.072	1.106	1.135	1.194
CPU time	–	–	–	–	–	–	–	0.646521	–	–

* Present study ** Chen and Cheung[2] *** Tada et.al[28]

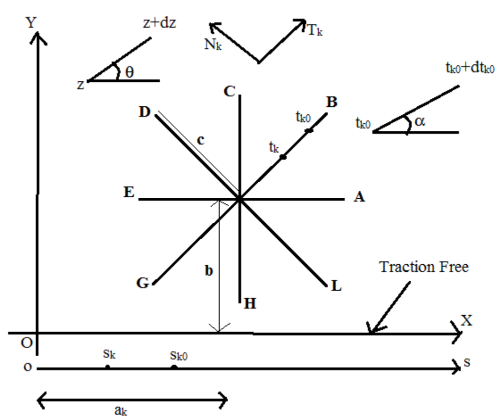

FIGURE 5.2. A branch crack problem in a half-plane elasticity.

$$
\begin{aligned}
K_{1H} &= F_{1H}\left(n, c/a\right) p\sqrt{\pi a},\\
K_{1B} &= K_{1Q} = F_{1B}\left(n, c/a\right) p\sqrt{\pi a}, -K_{2B} = K_{2Q} = -F_{2B}\left(n, c/a\right) p\sqrt{\pi a},\\
K_{1C} &= K_{1P} = F_{1C}\left(n, c/a\right) p\sqrt{\pi a}, -K_{2C} = K_{2P} = -F_{2C}\left(n, c/a\right) p\sqrt{\pi a},\\
K_{1A} &= K_{1N} = F_{1A}\left(n, c/a\right) p\sqrt{\pi a}, -K_{2A} = K_{2N} = -F_{2A}\left(n, c/a\right) p\sqrt{\pi a},
\end{aligned}
$$

Table 2. Non-dimensional SIF, $F_{1A}(c/b)$ for a branch crack problem in a half-plane, Figure (5.2).

M	c/b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F_{1A}values$										
7	*	0.3308	0.3342	0.3392	0.3453	0.3521	0.3594	0.3671	0.3751	0.3835
13	*	0.3308	0.3342	0.3392	0.3453	0.3521	0.3594	0.3671	0.3751	0.3832
17	*	0.3308	0.3342	0.3392	0.3453	0.3521	0.3594	0.3671	0.3751	0.3832
CPU time	–	–	–	–	–	–	–	16.325454	–	–

Table 3. Non-dimensional SIF, $F_1(c/b)$ for a various number of arms , at $c/b = 0.7$.

Arm	2	3	4	5	6	7	8	9	10	11
F_{1A}	0	0.7244	0.9083	0.7589	0.5981	0.4669	0.3671	0.2928	0.2369	0.1944
F_{2A}	0	0.4656	0	-0.3684	-0.5873	-0.7285	-0.8318	-0.9406	-1.1157	-1.4617

$$K_{1E} = K_{1M} = F_{1E}(n, c/a)\, p\sqrt{\pi a}, -K_{2E} = K_{2M} = -F_{2E}(n, c/a)\, p\sqrt{\pi a},$$
$$K_{1G} = K_{1L} = F_{1G}(n, c/a)\, p\sqrt{\pi a}, \ -K_{2G} = K_{2L} = -F_{2G}(n, c/a)\, p\sqrt{\pi a}.$$

Figures 5.4(a) and 5.4(b) exhibit F_1 and F_2 versus c/b, respectively. It is detect that, for the symmetric arms, $F_{1A} = F_{1N}, F_{1B} = F_{1Q}, F_{1C} = F_{1P}, F_{1E} = F_{1M}, F_{1G} = F_{1L}$ whereas, F_2 has the inverse value, $F_{2A} = -F_{2N}, F_{2B} = -F_{2Q}, F_{2C} = -F_{2P}, -F_{2E} = F_{2M}, -F_{2G} = F_{2L}$. For the arm OD, F_{1D} ascends smoothly, and for the arm OH, F_{1H} descends drastically and then oscillates with smaller magnitude. Similar behavior can be seen for F_{1G} and F_{1L}, and $F_{2D} = F_{2H} = 0$. It is also observed that F_1 for a longer arm is higher than a short arm. The total run time for this example is 0.765622.

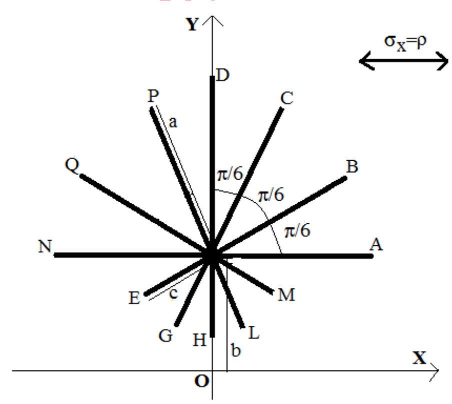

Figure 5.3. A long and short arms branch crack subjected to $\sigma_x^\infty = \wp$ in a half-plane.

6. Conclusion

In this paper, we have formulated a problem of a branch crack subjected to a remote stress in a half-plane elasticity into a singular integral equations. The length coordinate method and semi-open Gauss quadrature rules are employed to secure the numerical outcome. For a perpendicular crack in a half-plane, our results comply well with the previous

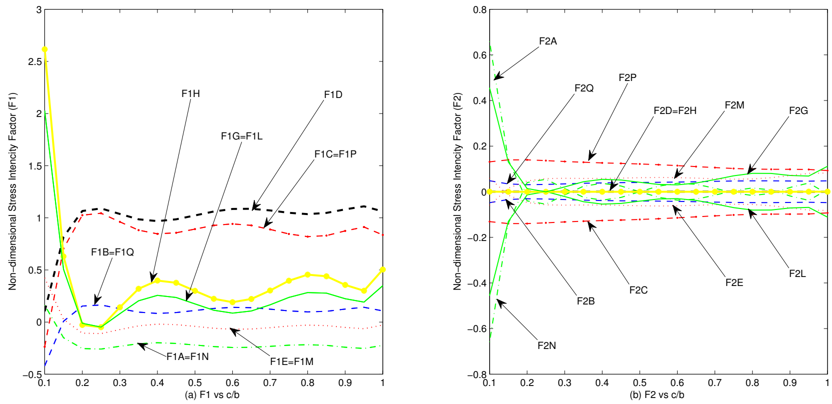

FIGURE 5.4. Non-dimensional SIFs for a branch crack with long and short arms in a half-plane (Figure 5.3).

reported results. The increment of SIF lean on the size of a crack, the distance between center of a branch crack and the boundary of a half-plane. The SIF decreases as the number of arms of the branch crack increases.

APPENDIX A. LIMIT VALUES OF SOME FUNCTIONS

For obtaining the integral equation for the branch crack problem, the following functions are necessary:

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z}, \\ G(z) &= \frac{1}{2\pi i}\int \frac{f(t)\overline{dt}}{t-z}, \\ H(z,\bar{z}) &= \frac{1}{2\pi i}\int \frac{\bar{t}-\bar{z}}{(t-z)^2}h(t)dt. \end{aligned} \tag{A.1}$$

We suppose that the functions $F(z), G(z)$, and $H(z)$ satisfy the Holder condition [23]. Obviously, the first two integrals are analytic and the third is not analytic.
The functions $F(z), G(z)$, and $H(z,\bar{z})$ are defined in the region exterior to the branch in Figure 3.1. Generally speaking, the integrals in Eqs. (A.1) take the different values when $z \to t_0^+$ or $z \to t_0^-$. The limit value of functions (A.1) when we are on the upper side (+) or lower side (-), reads as follows (see [26]):

$$\begin{aligned} F^{\pm}(t_0) &= \pm\frac{f(t_0)}{2} + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-t_0}, \\ G^{\pm}(t_0) &= \pm\frac{g(t_0)}{2}\frac{d\bar{t_0}}{dt_0} + \frac{1}{2\pi i}\int_L \frac{g(t)d\bar{t}}{t-t_0}, \\ H^{\pm}(t,\bar{t_0}) &= \pm\frac{h(t_0)}{2}\frac{d\bar{t_0}}{dt_0} + \frac{1}{2\pi i}\int \frac{\bar{t}-\bar{t_0}}{(t-t_0)^2}h(t)dt. \end{aligned}$$

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