



On the resolvent operator of a singular impulsive Hahn–Dirac system

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Abstract

In this study, we consider impulsive singular Hahn–Dirac systems. Green’s function and a spectral function for these systems are constructed. Finally, an integral representation of the resolvent operator is obtained.

Keywords. Difference equations, Hahn–Dirac system, discontinuous equations, resolvent.

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1. INTRODUCTION

In 1949, W. Hahn gave a definition of a difference operator [13]. With this definition, the author wanted to combine two important operators under a single structure. These operators are the q -difference operator and the forward difference operator. By 2018, Annaby et al.[6] studied the Hahn–Sturm–Liouville operators. In 2020, F. Hira [15] examined this system by taking the Hahn derivative instead of the ordinary derivative in the classical Dirac system. Allahverdiev and Tuna [1] investigated the basic features of the Hahn–Dirac system.

Differential equations with impulsive boundary conditions are one of the important problems studied in the theory of differential equations. Impulsive equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. There is quite substantial literature on such types of problems (see [4, 7–12, 16–19, 26, 28, 29]).

In this paper, the Hahn–Dirac system under impulsive conditions is considered. Using Levitan and Sargsjan’s method [21], the spectral representation of the resolvent operator for such systems will be obtained. In the Hahn–Dirac system (given below), if $\sigma = 0$, the q -Dirac system is obtained, if $q \rightarrow 1$ and $\sigma = 0$, the classical Dirac system is obtained. The spectral representation of the resolvent operator for the classical Dirac system under impulsive conditions was studied by Allahverdiev and Tuna in [2]. A similar problem in the q -Dirac system was considered in [3]. With this study, a more general version of the classical Dirac system will be analyzed under impulsive conditions. Recently, solutions utilizing the expansion method for different types of equations and the characteristics of the solutions have been the subject of study in [22–25, 30].

2. PRELIMINARIES

In this context, we provide a concise overview of the Hahn calculus [5, 6, 13, 14]. Let $q \in (0, 1)$, $\sigma_0 := \sigma / (1 - q)$, $\sigma > 0$, and let $\Psi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $\sigma_0 \in J$.

Definition 2.1 ([13, 14]). The Hahn derivative $D_{\sigma,q}\Psi$ is defined by

$$D_{\sigma,q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\sigma+q\zeta)-\Psi(\zeta)}{\sigma+(q-1)\zeta}, & \zeta \neq \sigma_0, \\ \Psi'(\sigma_0), & \zeta = \sigma_0. \end{cases}$$

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Definition 2.2 ([5]). Let $a, b, \sigma_0 \in J$. The Hahn integral (σ, q -integral) is defined by

$$\int_a^b \Psi(\zeta) d_{\sigma, q} \zeta := \int_{\sigma_0}^b \Psi(\zeta) d_{\sigma, q} \zeta - \int_{\sigma_0}^a \Psi(\zeta) d_{\sigma, q} \zeta,$$

where

$$\int_{\sigma_0}^{\zeta} \Psi(t) d_{\sigma, q} t := ((1-q)\zeta - \sigma) \sum_{n=0}^{\infty} q^n \Psi\left(\sigma \frac{1-q^n}{1-q} + \zeta q^n\right), \quad \zeta \in J,$$

provided that the series converges at $\zeta = a$ and $\zeta = b$.

3. MAIN RESULTS

Consider the impulsive Hahn–Dirac problem

$$\begin{cases} -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2 + p(\zeta) y_1 = \lambda y_1, & \zeta \in (\sigma_0, d) \cup (d, \frac{1}{q^n}), \\ D_{\sigma, q} y_1 + r(\zeta) y_2 = \lambda y_2, \end{cases} \quad (3.1)$$

$$y_2(\sigma_0, \lambda) \cos \beta + y_1(\sigma_0, \lambda) \sin \beta = 0, \quad (3.2)$$

$$y_1(d-) - k_1 y_1(d+) = 0, \quad (3.3)$$

$$y_2(\mu^{-1}(d-)) - k_2 y_2(\mu^{-1}(d+)) = 0, \quad (3.4)$$

$$y_2\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right) \cos \alpha + y_1\left(\frac{1}{q^n}, \lambda\right) \sin \alpha = 0, \quad (3.5)$$

where $k_1, k_2, \alpha, \beta \in \mathbb{R}$, $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, and λ is a complex eigenvalue parameter.

Our basic assumptions throughout the paper are the following:

(A₁) Let $q \in (0, 1)$, $\sigma_0 := \sigma / (1 - q)$, $\sigma > 0$, $\mu(\zeta) = q\zeta + \sigma$, $I_1 := [\sigma_0, d]$, $I_2 := (d, \frac{1}{q^n}]$, $\sigma_0 < d < \frac{1}{q^n}$ ($n \in \mathbb{N}$), $I := I_1 \cup I_2$ and $k_1 k_2 = \delta > 0$.

(A₂) $p, r : I \rightarrow \mathbb{R}$ are continuous functions on I and have finite limits $p(d\pm)$, $r(d\pm)$.

Now, we introduce the Hilbert space $H_1 = L^2_{\sigma, q}(I_1) \dot{+} L^2_{\sigma, q}(I_2)$ with its inner product

$$\langle u, \omega \rangle_{H_1} := \int_{\sigma_0}^d (u, \omega)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} (u, \omega)_{\mathbb{C}^2} d_{\sigma, q} \zeta,$$

where

$$u(\zeta) = \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix}, \quad u_1(\zeta) = \begin{cases} u_{11}(\zeta), & \zeta \in I_1, \\ u_{12}(\zeta), & \zeta \in I_2, \end{cases}$$

$$u_2(\zeta) = \begin{cases} u_{21}(\zeta), & \zeta \in I_1, \\ u_{22}(\zeta), & \zeta \in I_2, \end{cases}$$

and

$$\omega(\zeta) = \begin{pmatrix} \omega_1(\zeta) \\ \omega_2(\zeta) \end{pmatrix}, \quad \omega_1(\zeta) = \begin{cases} \omega_{11}(\zeta), & \zeta \in I_1, \\ \omega_{12}(\zeta), & \zeta \in I_2, \end{cases}$$

$$\omega_2(\zeta) = \begin{cases} \omega_{21}(\zeta), & \zeta \in I_1, \\ \omega_{22}(\zeta), & \zeta \in I_2. \end{cases}$$

Set

$$D_{\max} = \left\{ y \in H_1 : \begin{array}{l} y_1 \text{ and } y_2 \text{ are continuous at } \sigma_0, \\ \text{one-sided limits } y_1(d\pm) \text{ and } y_2(\mu^{-1}(d\pm)) \\ \text{exist and finite, } y_1(d-) - k_1 y_1(d+) = 0, \\ y_2(\mu^{-1}(d-)) - k_2 y_2(\mu^{-1}(d+)) = 0 \text{ and } \tau y \in H_1 \end{array} \right\},$$



where

$$\tau y := \begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2 + p(\zeta) y_1, \\ D_{\sigma, q} y_1 + r(\zeta) y_2, \end{cases} \quad \text{and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then the maximal operator \mathcal{L}_{\max} on D_{\max} is defined by $\mathcal{L}_{\max} y = \tau y, y \in D_{\max}$.

Green's formula is given by

$$\int_{\sigma_0}^{\frac{1}{q^n}} [(\tau u, v)_{\mathbb{C}^2} - (u, \tau v)_{\mathbb{C}^2}] d_{\sigma, q} \zeta = W_{\sigma, q}(u, \bar{v}) \left(\frac{1}{q^n} \right) - W_{\sigma, q}(u, \bar{v})(d+) + W_{\sigma, q}(u, \bar{v})(d-) - W_{\sigma, q}(u, \bar{v})(\sigma_0), \tag{3.6}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, u, v \in D_{\max},$$

and

$$W_{\sigma, q}(u, \bar{v})(\zeta) := u_1(\zeta) \overline{v_2(\mu^{-1}(\zeta))} - u_2(\mu^{-1}(\zeta)) \overline{v_1(\zeta)}.$$

Let

$$\varphi(\zeta, \lambda) = \begin{pmatrix} \varphi_1(\zeta, \lambda) \\ \varphi_2(\zeta, \lambda) \end{pmatrix}, \tag{3.7}$$

$$\varphi_1(\zeta, \lambda) = \begin{cases} \varphi_1^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \varphi_1^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases} \quad \varphi_2(\zeta, \lambda) = \begin{cases} \varphi_2^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \varphi_2^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases}$$

and

$$\theta(\zeta, \lambda) = \begin{pmatrix} \theta_1(\zeta, \lambda) \\ \theta_2(\zeta, \lambda) \end{pmatrix},$$

$$\theta_1(\zeta, \lambda) = \begin{cases} \theta_1^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \theta_1^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases} \quad \theta_2(\zeta, \lambda) = \begin{cases} \theta_2^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \theta_2^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases}$$

be two solutions of Eq. (3.1) satisfying the initial conditions

$$\begin{aligned} \varphi_1^{(1)}(\sigma_0, \lambda) &= \sin \beta, \quad \varphi_2^{(1)}(\sigma_0, \lambda) = -\cos \beta, \\ \theta_1^{(1)}(\sigma_0, \lambda) &= \cos \beta, \quad \theta_2^{(1)}(\sigma_0, \lambda) = \sin \beta. \end{aligned} \tag{3.8}$$

and impulsive conditions (3.3) and (3.4).

We will denote by $\theta(\zeta, \lambda) + m_{\frac{1}{q^n}}(\lambda) \varphi(\zeta, \lambda)$ the solution of Eq. (3.1) which satisfies (3.5). Then, we find

$$m_{\frac{1}{q^n}}(\lambda) = -\frac{\theta_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) \cot \alpha + \theta_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}{\varphi_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) \cot \alpha + \varphi_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}. \tag{3.9}$$

The function (3.9) is a meromorphic function of λ due to θ and φ are entire functions of λ . This function is called the *Titchmarsh–Weyl function* of Problem (3.1)-(3.5). By (3.9), we see that

$$m_{\frac{1}{q^n}}(\lambda, z) = -\frac{\theta_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) z + \theta_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}{\varphi_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) z + \varphi_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}, \tag{3.10}$$

where $z = \cot \alpha$. From the properties of the mapping (3.10) the real axis of the z -plane has as its image a circle in the m -plane.



Let $H = L^2_{\sigma,q}(I_1) + L^2_{\sigma,q}(I_3)$ be a Hilbert space endowed with the following inner product

$$\langle u, \omega \rangle_H := \int_{\sigma_0}^d (u, \omega)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\infty} (u, \omega)_{\mathbb{C}^2} d_{\sigma,q}\zeta,$$

where $I_3 := (d, \infty)$,

$$u(\zeta) = \begin{pmatrix} u_1(\zeta, \lambda) \\ u_2(\zeta, \lambda) \end{pmatrix},$$

$$u_1(\zeta, \lambda) = \begin{cases} u_{11}(\zeta, \lambda), & \zeta \in I_1, \\ u_{12}(\zeta, \lambda), & \zeta \in I_3, \end{cases} \quad u_2(\zeta, \lambda) = \begin{cases} u_{21}(\zeta, \lambda), & \zeta \in I_1, \\ u_{22}(\zeta, \lambda), & \zeta \in I_3, \end{cases}$$

and

$$\omega(\zeta) = \begin{pmatrix} \omega_1(\zeta, \lambda) \\ \omega_2(\zeta, \lambda) \end{pmatrix},$$

$$\omega_1(\zeta, \lambda) = \begin{cases} \omega_{11}(\zeta, \lambda), & \zeta \in I_1, \\ \omega_{12}(\zeta, \lambda), & \zeta \in I_3, \end{cases} \quad \omega_2(\zeta, \lambda) = \begin{cases} \omega_{21}(\zeta, \lambda), & \zeta \in I_1, \\ \omega_{22}(\zeta, \lambda), & \zeta \in I_3. \end{cases}$$

Lemma 3.1. *Let*

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) := \theta(\zeta, \lambda) + m_{\frac{1}{q^n}}(\lambda) \varphi(\zeta, \lambda), \quad \zeta \in I. \quad (3.11)$$

For every nonreal λ , we have

$$\begin{aligned} \chi_{\frac{1}{q^n}}(\zeta, \lambda) &\rightarrow \chi(\zeta, \lambda), \quad \frac{1}{q^n} \rightarrow \infty, \\ \int_{\sigma_0}^d \left\| \chi_{\frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \chi_{\frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \\ &\rightarrow \int_{\sigma_0}^d \left\| \chi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\infty} \left\| \chi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta, \quad \frac{1}{q^n} \rightarrow \infty. \end{aligned}$$

Proof. It is obvious that

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) = \chi(\zeta, \lambda) + \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi(\zeta, \lambda),$$

where $\chi(\zeta, \lambda) \in H$ and $m_{\frac{1}{q^n}}(\lambda)$ is a point of the circle. From [21], we infer that

$$\left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right| \leq 2r_{\frac{1}{q^n}}(\lambda) = \left[|v| \left(\begin{array}{c} \int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \\ + \delta \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \end{array} \right) \right]^{-1},$$

where $r_{\frac{1}{q^n}}$ denotes the radius of the circle and $\text{Im } \lambda = v \neq 0$. Letting $\frac{1}{q^n} \rightarrow \infty$ yields

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) \rightarrow \chi(\zeta, \lambda),$$

due to $r_{\frac{1}{q^n}}(\lambda) \rightarrow 0$. Moreover, we deduce that

$$\begin{aligned} &\int_{\sigma_0}^d \left\| \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \\ &= \left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right|^2 \int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right|^2 \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \\ &\leq \left(|v|^2 \left[\int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \right] \right)^{-1}. \end{aligned}$$



Letting $\frac{1}{q^n} \rightarrow \infty$, we obtain

$$\int_{\sigma_0}^d \left\| \chi_{\frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \chi_{\frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \rightarrow \int_{\sigma_0}^d \left\| \chi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\infty} \left\| \chi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta.$$

□

Set

$$G_{\frac{1}{q^n}}(\zeta, t, \lambda) = \begin{cases} \chi_{\frac{1}{q^n}}(\zeta, \lambda)\varphi^T(t, \lambda), & t \leq \zeta, \zeta \neq d, t \neq d, \\ \varphi(\zeta, \lambda)\chi_{\frac{1}{q^n}}^T(t, \lambda), & t > \zeta, \zeta \neq d, t \neq d, \end{cases} \tag{3.12}$$

where $\chi_{\frac{1}{q^n}}$ and φ are defined above with the formulas (3.11) and (3.7). Let $R_{\frac{1}{q^n}, \lambda} : H_1 \rightarrow H_1$ and

$$(R_{\frac{1}{q^n}, \lambda}g)(\zeta) = y(\zeta, \lambda) = \int_{\sigma_0}^d G_{\frac{1}{q^n}}(\zeta, t, \lambda)g(t)d_{\sigma,q}t + \delta \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, \lambda)g(t)d_{\sigma,q}t, \lambda \in \mathbb{C}, \tag{3.13}$$

where

$$g(\zeta) = \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix}, g \in H_1.$$

Our next objective is to prove that (3.13) satisfies $\tau y = \lambda y + g$, where $\zeta \in I$, $g \in H_1$ and conditions (3.2)-(3.5). From (3.13), we get

$$\begin{aligned} y_1(\zeta, \lambda) &= \chi_{\frac{1}{q^n}1}^{(1)}(\zeta, \lambda)q \int_{\sigma_0}^{\zeta} \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t \\ &+ \varphi_1^{(1)}(\zeta, \lambda)q \int_{\zeta}^d \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t \\ &+ \varphi_1^{(1)}(\zeta, \lambda)\delta q \int_d^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t, \zeta \in I_1, \end{aligned} \tag{3.14}$$

$$\begin{aligned} y_1(\zeta, \lambda) &= \chi_{\frac{1}{q^n}1}^{(2)}(\zeta, \lambda)q \int_{\sigma_0}^d \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t \\ &+ \chi_{\frac{1}{q^n}1}^{(2)}(\zeta, \lambda)\delta q \int_d^{\zeta} \begin{pmatrix} \varphi_1^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t \\ &+ \varphi_1^{(2)}(\zeta, \lambda)\delta q \int_{\zeta}^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t, \zeta \in I_2, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} y_2(\zeta, \lambda) &= \chi_{\frac{1}{q^n}2}^{(1)}(\zeta, \lambda)q \int_{\sigma_0}^{\zeta} \left(\varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) + \varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \right) d_{\sigma,q}t \\ &+ \varphi_2^{(1)}(\zeta, \lambda)q \int_{\zeta}^d \left(\chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda)g_1(\mu(t)) + \chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \right) d_{\sigma,q}t \\ &+ \varphi_2^{(1)}(\zeta, \lambda)\delta q \int_d^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{pmatrix} d_{\sigma,q}t, \zeta \in I_1. \end{aligned} \tag{3.16}$$



$$\begin{aligned}
y_2(\zeta, \lambda) &= \chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda)q \int_{\sigma_0}^d \left(\begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ \chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda)\delta q \int_d^\zeta \left(\begin{array}{c} \varphi_1^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ \varphi_2^{(2)}(\zeta, \lambda)\delta q \int_\zeta^{\frac{1}{q^n}} \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt}, \quad \zeta \in I_2,
\end{aligned} \tag{3.17}$$

By (3.15), we conclude that

$$\begin{aligned}
D_{\sigma,q}y_1(\zeta, \lambda) &= D_{\sigma,q}\chi_{\frac{1}{q^n}1}^{(1)}(\zeta, \lambda)q \int_{\sigma_0}^\zeta \left(\begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ D_{\sigma,q}\varphi_1^{(1)}(\zeta, \lambda)q \int_\zeta^d \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ D_{\sigma,q}\varphi_1^{(1)}(\zeta, \lambda)\delta q \int_d^{\frac{1}{q^n}} \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ W_{\sigma,q} \left(\varphi, \chi_{\frac{1}{q^n}} \right) g_2(\zeta), \quad \zeta \in I_1,
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
D_{\sigma,q}y_1(\zeta, \lambda) &= D_{\sigma,q}\chi_{\frac{1}{q^n}b_1}^{(2)}(\zeta, \lambda)q \int_{\sigma_0}^d \left(\begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ D_{\sigma,q}\chi_{\frac{1}{q^n}1}^{(2)}(\zeta, \lambda)\delta q \int_d^\zeta \left(\begin{array}{c} \varphi_1^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ D_{\sigma,q}\varphi_1^{(2)}(\zeta, \lambda)\delta q \int_\zeta^{\frac{1}{q^n}} \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ W_{\sigma,q} \left(\varphi, \chi_{\frac{1}{q^n}} \right) g_2(\zeta), \quad \zeta \in I_2.
\end{aligned} \tag{3.19}$$

Hence

$$\begin{aligned}
D_{\sigma,q}y_1(\zeta, \lambda) &= \{\lambda - r(\zeta)\}\chi_{\frac{1}{q^n}2}^{(1)}(\zeta, \lambda)q \int_{\sigma_0}^\zeta \left(\begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ \{\lambda - r(\zeta)\}\varphi_2^{(1)}(\zeta, \lambda)q \int_\zeta^d \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ \{\lambda - r(\zeta)\}\varphi_2^{(1)}(\zeta, \lambda)\delta q \int_d^{\frac{1}{q^n}} \left(\begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt} \\
&+ g_2(\zeta), \quad \zeta \in I_1,
\end{aligned} \tag{3.20}$$

and

$$D_{\sigma,q}y_1(\zeta, \lambda) = \{\lambda - r(\zeta)\}\chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda)q \int_{\sigma_0}^d \left(\begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,qt}$$



$$\begin{aligned}
 & + \{\lambda - r(\zeta)\} \varphi_2^{(2)}(\zeta, \lambda) \delta q \int_d^\zeta \left(\begin{array}{c} \chi_{\frac{1}{q^n}}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{array} \right) d_{\sigma, q} t \\
 & + \{\lambda - r(\zeta)\} \varphi_2^{(2)}(\zeta, \lambda) \delta q \int_\zeta^{\frac{1}{q^n}} \left(\begin{array}{c} \chi_{\frac{1}{q^n}}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{array} \right) d_{\sigma, q} t \\
 & + g_2(\zeta) = \{\lambda - r(\zeta)\} y_2(\zeta, \lambda) + g_2(\zeta), \quad \zeta \in I_2.
 \end{aligned} \tag{3.21}$$

Likewise, the validity of the other equation in (3.1) is proved. Moreover, Equation (3.13) clearly fulfills the equations stated in (3.2)-(3.5).

Theorem 3.2. Assume that λ is not an eigenvalue of problem (3.1)-(3.5). Then $G_{\frac{1}{q^n}}(\zeta, t, \lambda)$ is a σ, q -Hilbert-Schmidt kernel, i.e.,

$$\begin{aligned}
 & \int_{\sigma_0}^d \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta d_{\sigma, q} t < \infty, \\
 & \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta d_{\sigma, q} t < \infty.
 \end{aligned}$$

Proof. By (3.12), we see that

$$\begin{aligned}
 & \int_{\sigma_0}^d d_{\sigma, q} \zeta \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} t < \infty, \\
 & \int_d^{\frac{1}{q^n}} d_{\sigma, q} \zeta \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} t < \infty,
 \end{aligned}$$

since $\chi_{\frac{1}{q^n}}(\cdot, \lambda), \varphi(\cdot, \lambda) \in H_1$. Hence

$$\begin{aligned}
 & \int_{\sigma_0}^d \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta d_{\sigma, q} t < \infty, \\
 & \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta d_{\sigma, q} t < \infty.
 \end{aligned} \tag{3.22}$$

□

Theorem 3.3 ([27]). Let $A \{t_i\} = \{x_i\}, i \in \mathbb{N} := \{1, 2, 3, \dots\}$, where

$$x_i = \sum_{k=1}^{\infty} \eta_{ik} t_k, \quad i, k \in \mathbb{N}. \tag{3.23}$$

If

$$\sum_{i, k=1}^{\infty} |\eta_{ik}|^2 < \infty, \tag{3.24}$$

then the operator A is compact in l^2 .

Theorem 3.4. Assume that $\lambda = 0$ is not an eigenvalue of problem (3.1)-(3.5). Then $R_{\frac{1}{q^n}} := R_{\frac{1}{q^n}, 0}$ is a self-adjoint and compact operator.



Proof. Let $u, v \in H_1$ and let $\Psi_i = \Psi_i(s)$ ($i \in \mathbb{N}$) be a complete, orthonormal basis of H_1 . By Theorem 3.2, we infer that

$$t_i = \langle u, \Psi_i \rangle_1 = \int_{\sigma_0}^d (u(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} (u(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta,$$

$$x_i = \langle v, \Psi_i \rangle_1 = \int_{\sigma_0}^d (v(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} (v(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta,$$

$$\eta_{ik} = \int_{\sigma_0}^d \int_{\sigma_0}^d (G_{\frac{1}{q^n}}(\zeta, t, 0) \Psi_i(\zeta), \Psi_k(\zeta))_{\mathbb{C}^2} d_{\sigma, q} t d_{\sigma, q} \zeta + \delta^2 \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} (G_{\frac{1}{q^n}}(\zeta, t, 0) \Psi_i(\zeta), \Psi_k(\zeta))_{\mathbb{C}^2} d_{\sigma, q} t d_{\sigma, q} \zeta,$$

where $i, k \in \mathbb{N}$. Then H_1 is mapped isometrically on to l^2 . By this mapping, $R_{\frac{1}{q^n}}$ transforms into A defined as (3.23) in l^2 and (3.22) is translated into (3.24). By Theorems 3.2 and 3.3, we see that A and $R_{\frac{1}{q^n}}$ is compact.

Since $G_{\frac{1}{q^n}}(\zeta, t, 0) = G_{\frac{1}{q^n}}^T(t, \zeta, 0)$ and $G_{\frac{1}{q^n}}(t, \zeta, 0)$ is a real matrix-valued function defined on $I \times I$, we deduce that

$$\begin{aligned} \langle R_{\frac{1}{q^n}} u, v \rangle_1 &= \int_{\sigma_0}^d ((R_{\frac{1}{q^n}} u)(\zeta), v(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} ((R_{\frac{1}{q^n}} u)(\zeta), v(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \int_{\sigma_0}^d \left(\int_{\sigma_0}^d G_{\frac{1}{q^n}}(\zeta, t, 0) u(t) d_{\sigma, q} t, v(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &+ \delta^2 \int_d^{\frac{1}{q^n}} \left(\int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, 0) u(t) d_{\sigma, q} t, v(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \int_{\sigma_0}^d (u(t), \int_{\sigma_0}^d G_{\frac{1}{q^n}}^T(\zeta, t, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &+ \delta^2 \int_d^{\frac{1}{q^n}} (u(t), \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}^T(\zeta, t, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &= \int_{\sigma_0}^d (u(t), \int_{\sigma_0}^d G_{\frac{1}{q^n}}(t, \zeta, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &+ \delta^2 \int_d^{\frac{1}{q^n}} (u(t), \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(t, \zeta, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t = \langle u, R_{\frac{1}{q^n}} v \rangle_1. \end{aligned}$$

□

From Theorem 3.4 and the Hilbert–Schmidt theorem, we conclude that there exists an orthonormal system

$\left\{ \phi_{m, \frac{1}{q^n}} \right\}_{m=-\infty}^{\infty}$ where

$$\phi_{m, \frac{1}{q^n}}(\zeta) = \begin{pmatrix} \phi_{m, \frac{1}{q^n} 1}(\zeta) \\ \phi_{m, \frac{1}{q^n} 2}(\zeta) \end{pmatrix}, \quad \phi_{m, \frac{1}{q^n} 1}(\zeta) = \begin{cases} \phi_{m, \frac{1}{q^n} 1}^{(1)}(\zeta), & \zeta \in I_1, \\ \phi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \quad \phi_{m, \frac{1}{q^n} 2}(\zeta) = \begin{cases} \phi_{m, \frac{1}{q^n} 2}^{(1)}(\zeta), & \zeta \in I_1, \\ \phi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

of eigenvectors of Problem (3.1)–(3.5) with corresponding nonzero eigenvalues $\lambda_{m, \frac{1}{q^n}}$ (where $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$) such that

$$\|g\|_H^2 = \int_{\sigma_0}^d \left(|g_1^{(1)}(\zeta)|^2 + |g_2^{(1)}(\zeta)|^2 \right) d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left(|g_1^{(2)}(\zeta)|^2 + |g_2^{(2)}(\zeta)|^2 \right) d_{\sigma, q} \zeta$$



$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} \left| \begin{array}{l} \int_{\sigma_0}^d \left(g_1^{(1)}(\zeta) \phi_{m,b1}^{(1)}(\zeta) + g_2^{(1)}(\zeta) \phi_{m,b2}^{(1)}(\zeta) \right) d_{\sigma,q} \zeta \\ + \delta \int_d^{\frac{1}{q^n}} \left(g_1^{(2)}(\zeta) \phi_{m,b1}^{(2)}(\zeta) + g_2^{(2)}(\zeta) \phi_{m,b2}^{(2)}(\zeta) \right) d_{\sigma,q} \zeta \end{array} \right|^2, \tag{3.25}$$

where

$$g(\zeta) = \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix},$$

$$g_1(\zeta) = \begin{cases} g_1^{(1)}(\zeta), & \zeta \in I_1, \\ g_1^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \quad g_2(\zeta) = \begin{cases} g_2^{(1)}(\zeta), & \zeta \in I_1, \\ g_2^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

$g \in H_1$, and

$$\alpha_{m, \frac{1}{q^n}}^2 = \int_{\sigma_0}^d \left(\left(\phi_{m,b1}^{(1)}(\zeta) \right)^2 + \left(\phi_{m,b2}^{(1)}(\zeta) \right)^2 \right) d\zeta + \delta \int_d^{\frac{1}{q^n}} \left(\left(\phi_{m,b1}^{(2)}(\zeta) \right)^2 + \left(\phi_{m,b2}^{(2)}(\zeta) \right)^2 \right) d\zeta.$$

Set

$$\varrho_{\frac{1}{q^n}}(\lambda) = \begin{cases} - \sum_{\lambda < \lambda_{m, \frac{1}{q^n}} < 0} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2}, & \text{for } \lambda \leq 0, \\ \sum_{0 \leq \lambda_{m, \frac{1}{q^n}} < \lambda} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2}, & \text{for } \lambda \geq 0. \end{cases}$$

By (3.25), we obtain

$$\|g\|_H^2 = \int_{-\infty}^{\infty} |\Upsilon(\lambda)|^2 d\varrho_{\frac{1}{q^n}}(\lambda), \tag{3.26}$$

where

$$\Upsilon(\lambda) = \int_{\sigma_0}^d \left(g_1^{(1)}(\zeta) \phi_{m,b1}^{(1)}(\zeta) + g_2^{(1)}(\zeta) \phi_{m,b2}^{(1)}(\zeta) \right) d_{\sigma,q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left(g_1^{(2)}(\zeta) \phi_{m,b1}^{(2)}(\zeta) + g_2^{(2)}(\zeta) \phi_{m,b2}^{(2)}(\zeta) \right) d_{\sigma,q} \zeta. \tag{3.27}$$

Lemma 3.5. For any positive N , there is a positive constant $\Upsilon = M(N)$ not depending on $\frac{1}{q^n}$ such that

$$\bigvee_{-N}^N \left\{ \varrho_{\frac{1}{q^n}}(\lambda) \right\} = \sum_{-N \leq \lambda_{m, \frac{1}{q^n}} < N} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} = \varrho_{\frac{1}{q^n}}(N) - \varrho_{\frac{1}{q^n}}(-N) < M, \tag{3.28}$$

where $n \in \mathbb{N}$, $d < \frac{1}{q^n}$.

Proof. Let $\sin \beta \neq 0$. By (3.8), there is a positive number k such that

$$\frac{1}{k^2} \left(\int_{\sigma_0}^{\sigma_0+k} \varphi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 > \frac{1}{2} \sin^2 \beta, \tag{3.29}$$

due to $\varphi_1^{(1)}(\zeta, \lambda)$ is continuous on the region $\{(\zeta, \lambda) : -N \leq \lambda \leq N, \sigma_0 \leq \zeta \leq d\}$.

Let $g_k(\zeta) = \begin{pmatrix} g_{k1}(\zeta) \\ g_{k2}(\zeta) \end{pmatrix}$ be a function such that

$$g_{k1}(\zeta) = 0, \quad g_{k2}(\zeta) = \begin{cases} \frac{1}{k}, & \sigma_0 \leq \zeta < \sigma_0 + k, \\ 0, & \zeta \geq \sigma_0 + k. \end{cases}$$



Combining (3.26), (3.28), and (3.29), we conclude that

$$\begin{aligned}
\int_{\sigma_0}^{\sigma_0+k} (g_{k1}^2(\zeta) + g_{k2}^2(\zeta)) d_{\sigma,q}\zeta &= \frac{1}{k^2} = \int_{-\infty}^{\infty} \left(\frac{1}{k} \int_{\sigma_0}^k \phi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 d\rho_{\frac{1}{q^k}}(\lambda) \\
&\geq \int_{-N}^N \left(\frac{1}{k} \int_{\sigma_0}^k \phi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 d\rho_{\frac{1}{q^k}}(\lambda) \\
&> \frac{1}{2} \sin^2 \beta \left\{ \rho_{\frac{1}{q^k}}(N) - \rho_{\frac{1}{q^k}}(-N) \right\}.
\end{aligned} \tag{3.30}$$

If $\sin \beta = 0$, then we shall define $g_k(\zeta) = \begin{pmatrix} g_{k1}(\zeta) \\ g_{k2}(\zeta) \end{pmatrix}$ by the formula

$$g_k(\zeta) = \begin{cases} [c]c\frac{1}{k^2}, & \sigma_0 \leq \zeta < \sigma_0 + k, \\ 0, & \zeta \geq \sigma_0 + k, \end{cases} \quad g_{k2}(\zeta) = 0.$$

Hence, the proof of the lemma follows by the Parseval equality. \square

By σ, q -integration by parts, we obtain

$$\begin{aligned}
&\int_{\sigma_0}^d \left(\tau y^{(1)}, \varphi_{m, \frac{1}{q^k}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^k}} \left(\tau y^{(2)}, \varphi_{m, \frac{1}{q^k}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \\
&= \int_{\sigma_0}^d \left[-\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2^{(1)}(\zeta) + p(\zeta) y_1^{(1)} \right] \varphi_{m, \frac{1}{q^k} 1}^{(1)}(\zeta) d_{\sigma,q}\zeta \\
&+ \delta \int_d^{\frac{1}{q^k}} \left[-\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2^{(2)}(\zeta) + p(\zeta) y_1^{(2)} \right] \varphi_{m, \frac{1}{q^k} 1}^{(2)}(\zeta) d_{\sigma,q}\zeta \\
&+ \int_{\sigma_0}^d \left[D_{\sigma,q} y_1^{(1)}(\zeta) + r(\zeta) y_2^{(1)} \right] \varphi_{m, \frac{1}{q^k} 2}^{(1)}(\zeta) d_{\sigma,q}\zeta \\
&+ \delta \int_d^{\frac{1}{q^k}} \left[D_{\sigma,q} y_1^{(2)}(\zeta) + r(\zeta) y_2^{(2)} \right] \varphi_{m, \frac{1}{q^k} 2}^{(2)}(\zeta) d_{\sigma,q}\zeta \\
&= \int_{\sigma_0}^d \left[-\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{m, \frac{1}{q^k} 2}^{(1)}(\zeta) + p(\zeta) \varphi_{m, \frac{1}{q^k} 1}^{(1)} \right] y_2^{(1)} d_{\sigma,q}\zeta \\
&+ \delta \int_d^{\frac{1}{q^k}} \left[-\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{m, \frac{1}{q^k} 2}^{(2)}(\zeta) + p(\zeta) \varphi_{m, \frac{1}{q^k} 1}^{(2)} \right] y_2^{(2)} d_{\sigma,q}\zeta \\
&+ \int_{\sigma_0}^d \left[D_{\sigma,q} \varphi_{m, \frac{1}{q^k} 1}^{(1)}(\zeta) + r(\zeta) \varphi_{m, \frac{1}{q^k} 2}^{(1)} \right] y_1^{(1)} d_{\sigma,q}\zeta \\
&+ \delta \int_d^{\frac{1}{q^k}} \left[D_{\sigma,q} \varphi_{m, \frac{1}{q^k} 1}^{(2)}(\zeta) + r(\zeta) \varphi_{m, \frac{1}{q^k} 2}^{(2)} \right] y_1^{(2)} d_{\sigma,q}\zeta \\
&= \lambda_{m, \frac{1}{q^k}} \int_{\sigma_0}^d \left(y^{(1)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^k}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \\
&+ \lambda_{m, \frac{1}{q^k}} \delta \int_d^{\frac{1}{q^k}} \left(y^{(2)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^k}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta = \lambda_{m, \frac{1}{q^k}} \gamma_m(\lambda),
\end{aligned} \tag{3.31}$$

where

$$\gamma_m(\lambda) = \int_{\sigma_0}^d \left(y^{(1)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^k}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^k}} \left(y^{(2)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^k}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \tag{3.32}$$

and $m \in \mathbb{Z}$.



Set

$$y(\zeta, \lambda) = \sum_{m=-\infty}^{\infty} \gamma_m(\lambda) \varphi_{m, \frac{1}{q^n}}(\zeta),$$

and

$$d_m = \int_{\sigma_0}^d \left(g^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left(g^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \quad (m \in \mathbb{Z}). \tag{3.33}$$

Hence, for $m, n \in \mathbb{Z}$, $d < \frac{1}{q^n}$, we have

$$\begin{aligned} d_m &= \int_{\sigma_0}^d \left(g^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left(g^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \int_{\sigma_0}^d \left(\tau(y^{(1)})(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left(\tau(y^{(2)})(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &\quad - \lambda \int_{\sigma_0}^d \left(y^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta - \lambda \delta \int_d^{\frac{1}{q^n}} \left(y^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \left(\lambda_{m, \frac{1}{q^n}} - \lambda \right) \gamma_m(\lambda), \end{aligned} \tag{3.34}$$

since $y(\zeta, \lambda)$ satisfies $\tau y = \lambda y + g$ ($\zeta \in I$, $g \in H_1$) and conditions (3.2)-(3.5). Therefore, we obtain

$$\gamma_m(\lambda) = \frac{d_m}{\lambda_{m, \frac{1}{q^n}} - \lambda},$$

where $m \in \mathbb{Z}$, and

$$y(\zeta, \lambda) = \int_{\sigma_0}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, \lambda) g(t) d_{\sigma, q} t = \sum_{m=-\infty}^{\infty} \frac{d_m}{\lambda_{m, \frac{1}{q^n}} - \lambda} \varphi_{m, \frac{1}{q^n}}(\zeta).$$

Thus, we find

$$\left(R_{\frac{1}{q^n}, z} g \right) (\zeta) = \sum_{m=-\infty}^{\infty} \frac{\varphi_{m, \frac{1}{q^n}}(\zeta) \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1}}{\alpha_{m, \frac{1}{q^n}}^2 \left(\lambda_{m, \frac{1}{q^n}} - z \right)} \tag{3.35}$$

$$= \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \langle g(\cdot), \varphi(\cdot) \rangle_H d\rho_{\frac{1}{q^n}}(\lambda). \tag{3.36}$$

Lemma 3.6. *Let z be a nonreal number and ζ be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\rho_{\frac{1}{q^n}}(\lambda) < K. \tag{3.37}$$

Proof. Writing $g(\zeta) = \varphi_{m, \frac{1}{q^n}}(\zeta)$ ($m \in \mathbb{Z}$) in (3.35) yields

$$\frac{1}{\alpha_{m, \frac{1}{q^n}}} \int_{\sigma_0}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, z) \varphi_{m, \frac{1}{q^n}}(t) d_{\sigma, q} t = \frac{\varphi_{m, \frac{1}{q^n}}(\zeta)}{\alpha_{m, \frac{1}{q^n}} \left(\lambda_{m, \frac{1}{q^n}} - z \right)}, \tag{3.38}$$

due to $\varphi_{m, \frac{1}{q^n}}(\zeta)$ are orthogonal. By virtue of (3.38) and the Parseval equality, we deduce that

$$\int_{\sigma_0}^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, z) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta = \sum_{m=-\infty}^{\infty} \frac{\left\| \varphi_{m, \frac{1}{q^n}}(\zeta) \right\|_{\mathbb{C}^2}^2}{\alpha_{m, \frac{1}{q^n}}^2 \left| \lambda_{m, \frac{1}{q^n}} - z \right|^2}$$



$$= \int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\rho_{\frac{1}{q^n}}(\lambda), \quad (3.39)$$

which proves the lemma, since the integral in (3.39) is convergent. \square

By Lemma 3.5, $\rho_{\frac{1}{q^n}}(\lambda)$ is bounded. Using Helly's theorems, one can find a sequence $\left\{ \frac{1}{q^{n_k}} \right\}$ such that $\rho_{\frac{1}{q^{n_k}}}(\lambda)$ converges ($n_k \rightarrow \infty$) to a monotone function $\rho(\lambda)$.

Lemma 3.7. *Let z be a nonreal number, and let $\varphi(\zeta, \lambda)$ be as in (3.7). Then the following relation holds*

$$\int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\rho(\lambda) \leq K, \quad (3.40)$$

where ζ be a fixed number.

Proof. By (3.37), for arbitrary $\eta > 0$, we find

$$\int_{-\eta}^{\eta} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\rho_{\frac{1}{q^n}}(\lambda) < K.$$

Letting $\eta \rightarrow \infty$ and $\frac{1}{q^n} \rightarrow \infty$, the proof of the lemma follows. \square

Lemma 3.8. *For arbitrary $\eta > 0$, the following relations hold.*

$$\int_{-\infty}^{-\eta} \frac{d\rho(\lambda)}{\lambda^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\rho(\lambda)}{\lambda^2} < \infty. \quad (3.41)$$

Proof. Since $\left\| \varphi_{m, \frac{1}{q^n}}(\sigma_0, \lambda) \right\|_{\mathbb{C}^2}^2 \neq 0$, if we take $\zeta = 0$ in (3.40) then we get

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{|\lambda - z|^2} < \infty,$$

which proves the lemma. \square

Lemma 3.9. *Let $g(\cdot) \in H$ and*

$$(Rg)(\zeta, z) = \int_{\sigma_0}^{\infty} G(\zeta, t, z) g(t) d_{\sigma, q} t,$$

where

$$G(\zeta, t, z) = \begin{cases} [c]cc\chi(\zeta, z) \varphi^T(t, z), & t \leq \zeta, \zeta \neq d, t \neq d, \\ \varphi(\zeta, z) \chi^T(t, z), & t > \zeta, \zeta \neq d, t \neq d. \end{cases}$$

Then the following relation holds

$$\int_{\sigma_0}^d \|(Rg)(\zeta, z)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\infty} \|(Rg)(\zeta, z)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \leq \frac{1}{v^2} \left(\int_{\sigma_0}^d \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\infty} \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right),$$

where $z = u + iv$.

Proof. Combining (3.35) and (3.26), for each $\frac{1}{q^n} > d$, we obtain

$$\begin{aligned} \int_{\sigma_0}^d \left\| \left(R_{\frac{1}{q^n}, z} g \right) (\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \left(R_{\frac{1}{q^n}, z} g \right) (\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta &= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2 \left| \lambda_{m, \frac{1}{q^n}} - z \right|^2} \left| \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1} \right|^2 \\ &\leq \frac{1}{v^2} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} \left| \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1} \right|^2 \end{aligned}$$



$$= \frac{1}{v^2} \left(\int_{\sigma_0}^d \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta \right).$$

Letting $\frac{1}{q^n} \rightarrow \infty$, we get the desired result. □

Theorem 3.10. *For every nonreal z and each $g(\cdot) \in H$, the following equality holds*

$$(Rg)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon(\lambda) d\rho(\lambda), \tag{3.42}$$

where

$$\Upsilon(\lambda) = \int_{\sigma_0}^d (g(\zeta), \varphi^{(1)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \lim_{\xi \rightarrow \infty} \int_d^{\frac{1}{q^\xi}} (g(\zeta), \varphi^{(2)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma,q}\zeta,$$

and $\varphi(\zeta, \lambda)$ is defined by (3.7).

Proof. Let ς be an arbitrary positive number and the vector-valued function

$$g_{\frac{1}{q^\xi}}(\zeta) = \begin{pmatrix} g_{\frac{1}{q^\xi}1}(\zeta) \\ g_{\frac{1}{q^\xi}2}(\zeta) \end{pmatrix}, \quad g_{\frac{1}{q^\xi}1}(\zeta) = \begin{cases} g_{\frac{1}{q^\xi}1}^{(1)}(\zeta), & \zeta \in [\sigma_0, d), \\ g_{\frac{1}{q^\xi}1}^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^\xi}], \end{cases} \quad g_{\frac{1}{q^\xi}2}(\zeta) = \begin{cases} g_{\frac{1}{q^\xi}2}^{(1)}(\zeta), & \zeta \in [\sigma_0, d), \\ g_{\frac{1}{q^\xi}2}^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^\xi}], \end{cases}$$

satisfies the following conditions.

- 1) $g_{\frac{1}{q^\xi}}(\zeta)$ vanishes outside the set $[\sigma_0, d) \cup (d, \frac{1}{q^\xi}]$, $\frac{1}{q^\xi} < \frac{1}{q^n}$.
- 2) $g_{\frac{1}{q^\xi}}(\zeta)$ has a continuous Hahn derivative at σ_0 .
- 3) $g_{\frac{1}{q^\xi}}(\zeta)$ satisfy conditions (3.2)-(3.5).

Set

$$\Upsilon_{\frac{1}{q^\xi}}(\lambda) = \int_{\sigma_0}^d (g_{\frac{1}{q^\xi}}^{(1)}(\zeta), \varphi^{(1)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^\xi}} (g_{\frac{1}{q^\xi}}^{(2)}(\zeta), \varphi^{(2)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma,q}\zeta.$$

From (3.36), we find

$$\begin{aligned} \left(R_{\frac{1}{q^n}, z} g_{\frac{1}{q^\xi}} \right)(\zeta) &= \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \\ &= \int_{-\infty}^{-\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) + \int_{-\varsigma}^{\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \\ &\quad + \int_{\varsigma}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) = J_1 + J_2 + J_3. \end{aligned} \tag{3.43}$$

Firstly, we shall estimate J_1 . By (3.35), we get

$$\begin{aligned} |J_1| &= \left| \int_{-\infty}^{-\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \right| \\ &= \left| \sum_{\lambda_{k, \frac{1}{q^n}} < -\varsigma} \frac{\varphi_{k, \frac{1}{q^n}}(\zeta)}{\alpha_{k, \frac{1}{q^n}}^2 (\lambda_{k, \frac{1}{q^n}} - z)} \left\{ \int_{\sigma_0}^d \left(g_{\frac{1}{q^\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \right. \right. \\ &\quad \left. \left. + \delta \int_d^{\frac{1}{q^\xi}} \left(g_{\frac{1}{q^\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \right\} \right| \end{aligned} \tag{3.44}$$



$$\leq \left(\sum_{\lambda_k, \frac{1}{q^n} < -\varsigma} \frac{\|\varphi_{k, \frac{1}{q^n}}(\zeta)\|_{\mathbb{C}^2}^2}{\alpha_{k, \frac{1}{q^n}}^2 |\lambda_k, \frac{1}{q^n} - z|^2} \right)^{1/2} \times \left(\sum_{\lambda_k, \frac{1}{q^n} < -\varsigma} \frac{1}{\alpha_{k, \frac{1}{q^n}}^2} \left| \int_{\sigma_0}^d \left(g_{\frac{1}{q^\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right. \right. \\ \left. \left. + \delta \int_d^{\frac{1}{q^\xi}} \left(g_{\frac{1}{q^\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right|^2 \right)^{1/2}.$$

By σ, q -integration by parts, we deduce that

$$\begin{aligned} & \int_{\sigma_0}^d \left(g_{\frac{1}{q^\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^\xi}} \left(g_{\frac{1}{q^\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \begin{aligned} & \int_{\sigma_0}^d g_{\xi 1}^{(1)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) + p(\zeta) \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^\xi}} g_{\xi 1}^{(2)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) + p(\zeta) \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \end{aligned} \right\} \\ &+ \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \begin{aligned} & \int_{\sigma_0}^d g_{\xi 2}^{(1)}(\zeta) \left\{ D_{\sigma, q} \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) + r(\zeta) \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^\xi}} g_{\xi 2}^{(2)}(\zeta) \left\{ D_{\sigma, q} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) + r(\zeta) \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \end{aligned} \right\} \\ &= \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \begin{aligned} & \int_{\sigma_0}^d \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\xi 2}^{(1)}(\zeta) + p(\zeta) g_{\xi 1}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^\xi}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\xi 2}^{(2)}(\zeta) + p(\zeta) g_{\xi 1}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \end{aligned} \right\} \\ &+ \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \begin{aligned} & \int_{\sigma_0}^d \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \left\{ D_{\sigma, q} g_{\xi 1}^{(1)}(\zeta) + r(\zeta) g_{\xi 2}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^\xi}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \left\{ D_{\sigma, q} g_{\xi 1}^{(2)}(\zeta) + r(\zeta) g_{\xi 2}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \end{aligned} \right\}. \end{aligned} \quad (3.45)$$

From Lemma 3.5, we find

$$|I_1| \leq \frac{K^{1/2}}{\varsigma^2} \left| \sum_{\lambda_k, \frac{1}{q^n} < -\varsigma} \frac{1}{\alpha_{k, \frac{1}{q^n}}^2} \left[\begin{aligned} & \int_{\sigma_0}^d \left(\Xi_{\frac{1}{q^\xi}}(\zeta), \varphi_{k, \frac{1}{q^n}}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^\xi}} \left(\Xi_{\frac{1}{q^\xi}}(\zeta), \varphi_{k, \frac{1}{q^n}}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \end{aligned} \right]^2 \right|^{1/2},$$

where

$$\Xi_{\frac{1}{q^\xi}}(t) = \begin{pmatrix} -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\frac{1}{q^\xi}}^{(1)}(\zeta) + p(\zeta) g_{\frac{1}{q^\xi}}^{(1)}(\zeta) \\ D_{\sigma, q} g_{\frac{1}{q^\xi}}^{(1)}(\zeta) + r(\zeta) g_{\frac{1}{q^\xi}}^{(2)}(\zeta) \end{pmatrix}.$$

From Bessel inequality, we infer that

$$|J_1| \leq \frac{K^{1/2}}{\varsigma} \left[\int_{\sigma_0}^d \left\| \Xi_{\frac{1}{q^\xi}}(\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^\xi}} \left\| \Xi_{\frac{1}{q^\xi}}(\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right]^{1/2} = \frac{C_1}{\varsigma}.$$

Likewise, we can prove that $|J_3| \leq \frac{C_2}{\varsigma}$. Then J_1 and J_3 tend to zero as $\varsigma \rightarrow \infty$, uniformly in $\frac{1}{q^n}$. By Helly's theorems, we see that

$$\left(Rg_{\frac{1}{q^\xi}} \right) (\zeta, \lambda) = \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho(\lambda). \quad (3.46)$$



If $g(\cdot) \in H$, then we can find a sequence $\left\{g_{\frac{1}{q^\xi}}(\zeta)\right\}$ ($\frac{1}{q^\xi} > d$) which satisfies the previous conditions and tends to $g(\zeta)$ as $\frac{1}{q^\xi} \rightarrow \infty$. From the Parseval equality, the sequence of Fourier transform converges to the transform of $g(\zeta)$. By Lemmas 3.7 and 3.9, the proof of the theorem follows. \square

4. CONCLUSION

In this work, we have considered a singular impulsive Hahn–Dirac system. For this system, the Green’s function and a spectral function are constructed. Finally, an integral representation of the resolvent operator is obtained. It is a well-established fact that in the Hahn–Dirac system, if $\sigma = 0$, the q -Dirac system is obtained. Similarly, if $q \rightarrow 1$ and $\sigma = 0$, the classical Dirac system is obtained. Consequently, this article presents a more generalized analysis of existing studies within the existing literature.

REFERENCES

- [1] B. P. Allahverdiev and H. Tuna, *Spectral analysis of Hahn–Dirac system*, *Proyecciones (Antofagasta, On line)*, 40(6) (2021), 1547-1567.
- [2] B. P. Allahverdiev and H. Tuna, *Resolvent operator of singular dirac system with transmission conditions*, *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.*, 23(538) (2019), 85-105.
- [3] B. P. Allahverdiev and H. Tuna, *Properties of the resolvent of singular q -Dirac operators*, *Electr. J. Differ. Equat.*, 2020(3) (2020), 1-13.
- [4] R. K. Amirov, *On a system of Dirac differential equations with discontinuity conditions inside an interval*, *Ukrain. Math. J.*, 57(5) (2005), 712-727.
- [5] M. H. Annaby, A. E. Hamza, and K. A. Aldwoah, *Hahn difference operator and associated Jackson–Nörlund integrals*, *J. Optim. Theory Appl.*, 154 (2012), 133-153.
- [6] M. H. Annaby, A. E. Hamza, and S. D. Makhareh, *A Sturm–Liouville theory for Hahn difference operator*, in: Xin Li, Zuhair Nashed (Eds.), *Frontiers of Orthogonal Polynomials and q -Series*, World Scientific, Singapore, (2018), 35-84.
- [7] K. Aydemir, H. Olğar, and O. Sh. Mukhtarov, *The principal eigenvalue and the principal eigenfunction of a boundary-value-transmission problem*, *Turkish J. Math. Comput. Sci.*, 11(2) (2019), 97-100.
- [8] K. Aydemir, H. Olğar, O. Sh. Mukhtarov, and F. Muhtarov, *Differential operator equations with interface conditions in modified direct sum spaces*, *Filomat*, 32(3) (2018), 921-931.
- [9] Y. Aygar and E. Bairamov, *Scattering theory of impulsive Sturm–Liouville equation in quantum calculus*, *Bull. Malays. Math. Sci. Soc.*, 42 (2019), 3247-3259.
- [10] M. Bohner and S. Cebesoy, *Spectral analysis of an impulsive quantum difference operator*, *Math. Meth. Appl. Sci.*, 42 (2019), 5331-5339.
- [11] Ş. Faydaoğlu and G. Sh. Guseinov, *Eigenfunction expansion for a Sturm–Liouville boundary value problem with impulse*, *Int. J. Pure Appl. Math.*, 8(2) (2003), 137-170.
- [12] Y. Güldü, *On discontinuous Dirac operator with eigenparameter dependent boundary and two transmission conditions*, *Bound. Value Probl.*, 2016(135) (2016), 1-19.
- [13] W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, *Math. Nachr.*, 2 (1949), 340-379, (in German).
- [14] W. Hahn, *Ein Beitrag zur Theorie der Orthogonalpolynome*, *Monatsh. Math.*, 95 (1983), 19-24.
- [15] F. Hira, *Dirac system associated with Hahn difference operator*, *Bull. Malays. Math. Sci. Soc.*, 43 (2020), 3481-3497.
- [16] D. Karahan and K. R. Mamedov, *On a q -boundary value problem with discontinuity conditions*, *Vestn. Yuzhno-Ural. Gos. Un-ta. Ser. Matem. Mekh. Fiz.*, 13(4) (2021), 5-12.
- [17] D. Karahan and K. R. Mamedov, *On a q -analogue of the Sturm–Liouville operator with discontinuity conditions*, *Vestn. Samar. Gos. Tekh. Univ., Ser. Fiz.-Mat. Nauk.*, 26(3) (2022), 407-418.
- [18] D. Karahan and K. R. Mamedov, *Sampling theory associated with q -Sturm–Liouville operator with discontinuity conditions*, *J. Contemp. Appl. Math.*, 10(2) (2020), 40-48.



- [19] B. Keskin and A. S. Ozkan, *Inverse spectral problems for Dirac operator with eigenvalue dependent boundary and jump conditions*, Acta Math. Hungarica, 130 (2011), 309-320.
- [20] H. Koyunbakan and E. S. Panakhov, *Solution of a discontinuous inverse nodal problem on a finite interval*, Math. Comput. Model., 44(1-2) (2006), 204-209.
- [21] B. M. Levitan and I. S. Sargsjan, *Sturm–Liouville and Dirac operators*, Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991 (translated from the Russian).
- [22] J. Manafian and M. Lakestani, *Optical soliton solutions for the Gerdjikov–Ivanov model via $\tan(\phi/2)$ -expansion method*, Optik, 127(20) (2016), 9603-9620.
- [23] J. Manafian and M. Lakestani, *Abundant soliton solutions for the Kundu–Eckhaus equation via $\tan(\phi/2)$ -expansion method*, Optik, 127(14) (2016), 5543-5551.
- [24] J. Manafian and M. Lakestani, *N -lump and interaction solutions of localized waves to the $(2 + 1)$ -dimensional variable-coefficient Caudrey–Dodd–Gibbon–Kotera–Sawada equation*, Journal of Geometry and Physics, 150 (2020), 103598.
- [25] J. Manafian, L. A. Dawood, and M. Lakestani, *New solutions to a generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator*, Partial Differential Equations in Applied Mathematics, 9 (2024), 100600.
- [26] O. S. Mukhtarov, *Discontinuous boundary-value problem with spectral parameter in boundary conditions*, Turkish J. Math., 18 (1994), 183-192.
- [27] M. A. Naimark, *Linear Differential Operators*, 2nd edn., Nauka, Moscow, 1969; English transl. of 1st. edn., 1,2, New York, 1968.
- [28] A. S. Ozkan and R. Kh. Amirov, *An interior inverse problem for the impulsive Dirac operator*, Tamkang J. Math., 42(3) (2011), 259-263.
- [29] A. Zettl, *Adjoint and self-adjoint boundary value problems with interface conditions*, SIAM J. Appl. Math., 16(4) (1968), 851-859.
- [30] M. Zhang, X. Xie, J. Manafian, O. A. Ilhan, and G. Singh, *Characteristics of the new multiple rogue wave solutions to the fractional generalized CBS-BK equation*, Journal of Advanced Research, 38 (2022), 131-142.

