



## On the resolvent operator of singular impulsive Hahn–Dirac system

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### Abstract

In this study, we consider impulsive singular Hahn–Dirac systems. Green’s function and a spectral function for these systems are constructed. Finally, an integral representation of the resolvent operator is obtained.

**Keywords.** Difference equations, Hahn–Dirac system, discontinuous equations, resolvent.

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### 1. INTRODUCTION

In 1949, W. Hahn gave a definition of difference operator [13]. With this definition, the author wanted to combine two important operators under a single structure. These operators are the  $q$ -difference operator and the forward difference operator. By 2018, Annaby et al.[6] studied the Hahn–Sturm–Liouville operators. In 2020, F. Hira [15] examined this system by taking the Hahn derivative instead of the ordinary derivative in the classical Dirac system. Allahverdiev and Tuna [1] investigated the basic features of the Hahn–Dirac system.

Differential equations with impulsive boundary conditions are one of the important problems studied in the theory of differential equations. Impulsive equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. There is quite substantial literature on such type problems (see [4, 7–12, 16–19, 26, 28, 29]).

In this paper, the Hahn–Dirac system under impulsive conditions is considered. Using Levitan and Sargsjan’s method [21], the spectral representation of the resolvent operator for such systems will be obtained. In the Hahn–Dirac system (given below), if  $\sigma = 0$ , the  $q$ -Dirac system is obtained, if  $q \rightarrow 1$  and  $\sigma = 0$ , the classical Dirac system is obtained. The spectral representation of the resolvent operator for the classical Dirac system under impulsive conditions was studied by Allahverdiev and Tuna in [2]. A similar problem in the  $q$ -Dirac system was considered in [3]. With this study, a more general version of the classical Dirac system will be analyzed under impulsive conditions. Recently, solutions utilizing the expansion method for different types of equations and the characteristics of the solutions have been the subject of study in [22–25, 30].

### 2. PRELIMINARIES

In this context, we provide a concise overview of the Hahn calculus [5, 6, 13, 14]. Let  $q \in (0, 1)$ ,  $\sigma_0 := \sigma / (1 - q)$ ,  $\sigma > 0$ , and let  $\Psi : J \subset \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $\sigma_0 \in J$ .

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**Definition 2.1** ([13], [14]). The Hahn derivative  $D_{\sigma,q}\Psi$  is defined by

$$D_{\sigma,q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\sigma+q\zeta)-\Psi(\zeta)}{\sigma+(q-1)\zeta}, & \zeta \neq \sigma_0, \\ \Psi'(\sigma_0), & \zeta = \sigma_0. \end{cases}$$

**Definition 2.2** ([5]). Let  $a, b, \sigma_0 \in J$ . The Hahn integral ( $\sigma, q$ -integral) is defined by

$$\int_a^b \Psi(\zeta) d_{\sigma,q}\zeta := \int_{\sigma_0}^b \Psi(\zeta) d_{\sigma,q}\zeta - \int_{\sigma_0}^a \Psi(\zeta) d_{\sigma,q}\zeta,$$

where

$$\int_{\sigma_0}^{\zeta} \Psi(t) d_{\sigma,q}t := ((1-q)\zeta - \sigma) \sum_{n=0}^{\infty} q^n \Psi\left(\sigma \frac{1-q^n}{1-q} + \zeta q^n\right), \quad \zeta \in J$$

provided that the series converges at  $\zeta = a$  and  $\zeta = b$ .

### 3. MAIN RESULTS

Consider the impulsive Hahn–Dirac problem

$$\begin{cases} -\frac{1}{q}D_{-\frac{\sigma}{q}, \frac{1}{q}}y_2 + p(\zeta)y_1 = \lambda y_1, & \zeta \in (\sigma_0, d) \cup (d, \frac{1}{q^n}), \\ D_{\sigma,q}y_1 + r(\zeta)y_2 = \lambda y_2, \end{cases} \quad (3.1)$$

$$y_2(\sigma_0, \lambda) \cos \beta + y_1(\sigma_0, \lambda) \sin \beta = 0, \quad (3.2)$$

$$y_1(d-) - k_1 y_1(d+) = 0, \quad (3.3)$$

$$y_2(\mu^{-1}(d-)) - k_2 y_2(\mu^{-1}(d+)) = 0, \quad (3.4)$$

$$y_2\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right) \cos \alpha + y_1\left(\frac{1}{q^n}, \lambda\right) \sin \alpha = 0, \quad (3.5)$$

where  $k_1, k_2, \alpha, \beta \in \mathbb{R}$ ,  $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ , and  $\lambda$  is a complex eigenvalue parameter.

Our basic assumptions throughout the paper are the following:

(A<sub>1</sub>) Let  $q \in (0, 1)$ ,  $\sigma_0 := \sigma/(1-q)$ ,  $\sigma > 0$ ,  $\mu(\zeta) = q\zeta + \sigma$ ,  $I_1 := [\sigma_0, d)$ ,  $I_2 := (d, \frac{1}{q^n}]$ ,  $\sigma_0 < d < \frac{1}{q^n}$  ( $n \in \mathbb{N}$ ),  $I := I_1 \cup I_2$  and  $k_1 k_2 = \delta > 0$ .

(A<sub>2</sub>)  $p, r : I \rightarrow \mathbb{R}$  are continuous functions on  $I$  and have finite limits  $p(d\pm)$ ,  $r(d\pm)$ .

Now we introduce the Hilbert space  $H_1 = L_{\sigma,q}^2(I_1) + L_{\sigma,q}^2(I_2)$  with its inner product

$$\langle u, \omega \rangle_{H_1} := \int_{\sigma_0}^d (u, \omega)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} (u, \omega)_{\mathbb{C}^2} d_{\sigma,q}\zeta,$$

where

$$u(\zeta) = \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix}, \quad u_1(\zeta) = \begin{cases} u_{11}(\zeta), & \zeta \in I_1, \\ u_{12}(\zeta), & \zeta \in I_2, \end{cases}$$

$$u_2(\zeta) = \begin{cases} u_{21}(\zeta), & \zeta \in I_1, \\ u_{22}(\zeta), & \zeta \in I_2, \end{cases}$$

and

$$\omega(\zeta) = \begin{pmatrix} \omega_1(\zeta) \\ \omega_2(\zeta) \end{pmatrix}, \quad \omega_1(\zeta) = \begin{cases} \omega_{11}(\zeta), & \zeta \in I_1 \\ \omega_{12}(\zeta), & \zeta \in I_2, \end{cases}$$



$$\omega_2(\zeta) = \begin{cases} \omega_{21}(\zeta), & \zeta \in I_1, \\ \omega_{22}(\zeta), & \zeta \in I_2. \end{cases}$$

Set

$$D_{\max} = \left\{ y \in H_1 : \begin{array}{l} y_1 \text{ and } y_2 \text{ are continuous at } \sigma_0, \\ \text{one-sided limits } y_1(d\pm) \text{ and } y_2(\mu^{-1}(d\pm)) \\ \text{exist and finite, } y_1(d-) - k_1 y_1(d+) = 0, \\ y_2(\mu^{-1}(d-)) - k_2 y_2(\mu^{-1}(d+)) = 0 \text{ and } \tau y \in H_1 \end{array} \right\},$$

where

$$\tau y := \begin{cases} -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2 + p(\zeta) y_1, & \text{and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \\ D_{\sigma, q} y_1 + r(\zeta) y_2, \end{cases}$$

Then the maximal operator  $\mathcal{L}_{\max}$  on  $D_{\max}$  is defined by  $\mathcal{L}_{\max} y = \tau y, y \in D_{\max}$ .

Green's formula is given by

$$\int_{\sigma_0}^{\frac{1}{q^n}} [(\tau u, v)_{\mathbb{C}^2} - (u, \tau v)_{\mathbb{C}^2}] d_{\sigma, q} \zeta = W_{\sigma, q}(u, \bar{v}) \left( \frac{1}{q^n} \right) - W_{\sigma, q}(u, \bar{v})(d+) + W_{\sigma, q}(u, \bar{v})(d-) - W_{\sigma, q}(u, \bar{v})(\sigma_0), \tag{3.6}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, u, v \in D_{\max},$$

and

$$W_{\sigma, q}(u, \bar{v})(\zeta) := u_1(\zeta) \overline{v_2(\mu^{-1}(\zeta))} - u_2(\mu^{-1}(\zeta)) \overline{v_1(\zeta)}.$$

Let

$$\varphi(\zeta, \lambda) = \begin{pmatrix} \varphi_1(\zeta, \lambda) \\ \varphi_2(\zeta, \lambda) \end{pmatrix}, \tag{3.7}$$

$$\varphi_1(\zeta, \lambda) = \begin{cases} \varphi_1^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \varphi_1^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases} \quad \varphi_2(\zeta, \lambda) = \begin{cases} \varphi_2^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \varphi_2^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases}$$

and

$$\theta(\zeta, \lambda) = \begin{pmatrix} \theta_1(\zeta, \lambda) \\ \theta_2(\zeta, \lambda) \end{pmatrix},$$

$$\theta_1(\zeta, \lambda) = \begin{cases} \theta_1^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \theta_1^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases} \quad \theta_2(\zeta, \lambda) = \begin{cases} \theta_2^{(1)}(\zeta, \lambda), & \zeta \in I_1, \\ \theta_2^{(2)}(\zeta, \lambda), & \zeta \in I_2, \end{cases}$$

be two solutions of Eq. (3.1) satisfying the initial conditions

$$\begin{aligned} \varphi_1^{(1)}(\sigma_0, \lambda) &= \sin \beta, \quad \varphi_2^{(1)}(\sigma_0, \lambda) = -\cos \beta, \\ \theta_1^{(1)}(\sigma_0, \lambda) &= \cos \beta, \quad \theta_2^{(1)}(\sigma_0, \lambda) = \sin \beta. \end{aligned} \tag{3.8}$$

and impulsive conditions (3.3)-(3.4).

We will denote by  $\theta(\zeta, \lambda) + m_{\frac{1}{q^n}}(\lambda) \varphi(\zeta, \lambda)$  the solution of Eq. (3.1) which satisfies (3.5). Then, we find

$$m_{\frac{1}{q^n}}(\lambda) = -\frac{\theta_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) \cot \alpha + \theta_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}{\varphi_1^{(2)}\left(\frac{1}{q^n}, \lambda\right) \cot \alpha + \varphi_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}. \tag{3.9}$$



The function (3.9) is a meromorphic function of  $\lambda$  due to  $\theta$  and  $\varphi$  are entire functions of  $\lambda$ . This function is called the *Titchmarsh–Weyl function* of Problem (3.1)-(3.5). By (3.9), we see that

$$m_{\frac{1}{q^n}}(\lambda, z) = -\frac{\theta_1^{(2)}\left(\frac{1}{q^n}, \lambda\right)z + \theta_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}{\varphi_1^{(2)}\left(\frac{1}{q^n}, \lambda\right)z + \varphi_2^{(2)}\left(\mu^{-1}\left(\frac{1}{q^n}\right), \lambda\right)}, \quad (3.10)$$

where  $z = \cot \alpha$ . From the properties of the mapping (3.10) the real axis of the  $z$ -plane has as its image a circle in the  $m$ -plane.

Let  $H = L^2_{\sigma, q}(I_1) + L^2_{\sigma, q}(I_3)$  be a Hilbert space endowed with the following inner product

$$\langle u, \omega \rangle_H := \int_{\sigma_0}^d (u, \omega)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^\infty (u, \omega)_{\mathbb{C}^2} d_{\sigma, q} \zeta,$$

where  $I_3 := (d, \infty)$ ,

$$u(\zeta) = \begin{pmatrix} u_1(\zeta, \lambda) \\ u_2(\zeta, \lambda) \end{pmatrix},$$

$$u_1(\zeta, \lambda) = \begin{cases} u_{11}(\zeta, \lambda), & \zeta \in I_1, \\ u_{12}(\zeta, \lambda), & \zeta \in I_3, \end{cases} \quad u_2(\zeta, \lambda) = \begin{cases} u_{21}(\zeta, \lambda), & \zeta \in I_1, \\ u_{22}(\zeta, \lambda), & \zeta \in I_3, \end{cases}$$

and

$$\omega(\zeta) = \begin{pmatrix} \omega_1(\zeta, \lambda) \\ \omega_2(\zeta, \lambda) \end{pmatrix},$$

$$\omega_1(\zeta, \lambda) = \begin{cases} \omega_{11}(\zeta, \lambda), & \zeta \in I_1, \\ \omega_{12}(\zeta, \lambda), & \zeta \in I_3, \end{cases} \quad \omega_2(\zeta, \lambda) = \begin{cases} \omega_{21}(\zeta, \lambda), & \zeta \in I_1, \\ \omega_{22}(\zeta, \lambda), & \zeta \in I_3. \end{cases}$$

**Lemma 3.1.** *Let*

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) := \theta(\zeta, \lambda) + m_{\frac{1}{q^n}}(\lambda) \varphi(\zeta, \lambda), \quad \zeta \in I. \quad (3.11)$$

For every nonreal  $\lambda$ , we have

$$\begin{aligned} \chi_{\frac{1}{q^n}}(\zeta, \lambda) &\rightarrow \chi(\zeta, \lambda), \quad \frac{1}{q^n} \rightarrow \infty, \\ &\int_{\sigma_0}^d \left\| \chi_{\frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \chi_{\frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ &\rightarrow \int_{\sigma_0}^d \left\| \chi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^\infty \left\| \chi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta, \quad \frac{1}{q^n} \rightarrow \infty. \end{aligned}$$

*Proof.* It is obvious that

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) = \chi(\zeta, \lambda) + \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi(\zeta, \lambda),$$

where  $\chi(\zeta, \lambda) \in H$  and  $m_{\frac{1}{q^n}}(\lambda)$  is a point of the circle. From [21], we infer that

$$\begin{aligned} \left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right| &\leq 2r_{\frac{1}{q^n}}(\lambda) \\ &= \left[ |v| \left( \int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right) \right]^{-1}, \end{aligned}$$

where  $r_{\frac{1}{q^n}}$  denotes the radius of the circle and  $\text{Im } \lambda = v \neq 0$ . Letting  $\frac{1}{q^n} \rightarrow \infty$  yields

$$\chi_{\frac{1}{q^n}}(\zeta, \lambda) \rightarrow \chi(\zeta, \lambda),$$



due to  $r_{\frac{1}{q^n}}(\lambda) \rightarrow 0$ . Moreover, we deduce that

$$\begin{aligned} & \int_{\sigma_0}^d \left\| \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ & + \delta \int_d^{\frac{1}{q^n}} \left\| \left\{ m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right\} \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ & = \left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right|^2 \int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ & + \delta \left| m_{\frac{1}{q^n}}(\lambda) - m(\lambda) \right|^2 \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ & \leq \left( |v|^2 \left[ \int_{\sigma_0}^d \left\| \varphi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \varphi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right] \right)^{-1}. \end{aligned}$$

Letting  $\frac{1}{q^n} \rightarrow \infty$ , we obtain

$$\int_{\sigma_0}^d \left\| \chi_{\frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \chi_{\frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \rightarrow \int_{\sigma_0}^d \left\| \chi^{(1)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\infty} \left\| \chi^{(2)}(\zeta, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta.$$

□

Set

$$G_{\frac{1}{q^n}}(\zeta, t, \lambda) = \begin{cases} \chi_{\frac{1}{q^n}}(\zeta, \lambda) \varphi^T(t, \lambda), & t \leq \zeta, \zeta \neq d, t \neq d, \\ \varphi(\zeta, \lambda) \chi_{\frac{1}{q^n}}^T(t, \lambda), & t > \zeta, \zeta \neq d, t \neq d, \end{cases} \quad (3.12)$$

where  $\chi_{\frac{1}{q^n}}$  and  $\varphi$  are defined above with the formulas (3.11) and (3.7). Let  $R_{\frac{1}{q^n}, \lambda} : H_1 \rightarrow H_1$  and

$$(R_{\frac{1}{q^n}, \lambda} g)(\zeta) = y(\zeta, \lambda) = \int_{\sigma_0}^d G_{\frac{1}{q^n}}(\zeta, t, \lambda) g(t) d_{\sigma, q} t + \delta \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, \lambda) g(t) d_{\sigma, q} t, \quad \lambda \in \mathbb{C}, \quad (3.13)$$

where

$$g(\zeta) = \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix}, \quad g \in H_1.$$

Our next objective is to prove that (3.13) satisfies  $\tau y = \lambda y + g$ , where  $\zeta \in I$ ,  $g \in H_1$  and conditions (3.2)-(3.5). From (3.13), we get

$$\begin{aligned} y_1(\zeta, \lambda) &= \chi_{\frac{1}{q^n} 1}^{(1)}(\zeta, \lambda) q \int_{\sigma_0}^{\zeta} \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma, q} t \\ &+ \varphi_1^{(1)}(\zeta, \lambda) q \int_{\zeta}^d \begin{pmatrix} \chi_{\frac{1}{q^n} 1}^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n} 2}^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma, q} t \\ &+ \varphi_1^{(1)}(\zeta, \lambda) \delta q \int_d^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n} 1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n} 2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma, q} t, \quad \zeta \in I_1, \end{aligned} \quad (3.14)$$

$$y_1(\zeta, \lambda) = \chi_{\frac{1}{q^n} 1}^{(2)}(\zeta, \lambda) q \int_{\sigma_0}^d \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma, q} t$$



$$\begin{aligned}
& + \chi_{\frac{1}{q^n}1}^{(2)}(\zeta, \lambda) \delta q \int_d^\zeta \begin{pmatrix} \varphi_1^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + \varphi_1^{(2)}(\zeta, \lambda) \delta q \int_\zeta^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt}, \quad \zeta \in I_2,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
y_2(\zeta, \lambda) & = \chi_{\frac{1}{q^n}2}^{(1)}(\zeta, \lambda) q \int_{\sigma_0}^\zeta \left( \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \right) d_{\sigma,qt} \\
& + \varphi_2^{(1)}(\zeta, \lambda) q \int_\zeta^d \left( \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda) g_1(\mu(t)) + \chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \right) d_{\sigma,qt} \\
& + \varphi_2^{(1)}(\zeta, \lambda) \delta q \int_d^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt}, \quad \zeta \in I_1.
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
y_2(\zeta, \lambda) & = \chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda) q \int_{\sigma_0}^d \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + \chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda) \delta q \int_d^\zeta \begin{pmatrix} \varphi_1^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + \varphi_2^{(2)}(\zeta, \lambda) \delta q \int_\zeta^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt}, \quad \zeta \in I_2.
\end{aligned} \tag{3.17}$$

By (3.15), we conclude that

$$\begin{aligned}
D_{\sigma,q} y_1(\zeta, \lambda) & = D_{\sigma,q} \chi_{\frac{1}{q^n}1}^{(1)}(\zeta, \lambda) q \int_{\sigma_0}^\zeta \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + D_{\sigma,q} \varphi_1^{(1)}(\zeta, \lambda) q \int_\zeta^d \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + D_{\sigma,q} \varphi_1^{(1)}(\zeta, \lambda) \delta q \int_d^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + W_{\sigma,q} \left( \varphi, \chi_{\frac{1}{q^n}} \right) g_2(\zeta), \quad \zeta \in I_1,
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
D_{\sigma,q} y_1(\zeta, \lambda) & = D_{\sigma,q} \chi_{\frac{1}{q^n}b1}^{(2)}(\zeta, \lambda) q \int_{\sigma_0}^d \begin{pmatrix} \varphi_1^{(1)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(1)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + D_{\sigma,q} \chi_{\frac{1}{q^n}1}^{(2)}(\zeta, \lambda) \delta q \int_d^\zeta \begin{pmatrix} \varphi_1^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \varphi_2^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + D_{\sigma,q} \varphi_1^{(2)}(\zeta, \lambda) \delta q \int_\zeta^{\frac{1}{q^n}} \begin{pmatrix} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda) g_1(\mu(t)) \\ + \chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda) g_2(\mu(t)) \end{pmatrix} d_{\sigma,qt} \\
& + W_{\sigma,q} \left( \varphi, \chi_{\frac{1}{q^n}} \right) g_2(\zeta), \quad \zeta \in I_2.
\end{aligned} \tag{3.19}$$



Hence

$$\begin{aligned}
 D_{\sigma,q}y_1(\zeta, \lambda) &= \{\lambda - r(\zeta)\} \chi_{\frac{1}{q^n}2}^{(1)}(\zeta, \lambda) q \int_{\sigma_0}^{\zeta} \left( \begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ \{\lambda - r(\zeta)\} \varphi_2^{(1)}(\zeta, \lambda) q \int_{\zeta}^d \left( \begin{array}{c} \chi_{\frac{1}{q^n}1}^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ \{\lambda - r(\zeta)\} \varphi_2^{(1)}(\zeta, \lambda) \delta q \int_d^{\frac{1}{q^n}} \left( \begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ g_2(\zeta), \zeta \in I_1,
 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
 D_{\sigma,q}y_1(\zeta, \lambda) &= \{\lambda - r(\zeta)\} \chi_{\frac{1}{q^n}2}^{(2)}(\zeta, \lambda) q \int_{\sigma_0}^d \left( \begin{array}{c} \varphi_1^{(1)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\varphi_2^{(1)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ \{\lambda - r(\zeta)\} \varphi_2^{(2)}(\zeta, \lambda) \delta q \int_d^{\zeta} \left( \begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ \{\lambda - r(\zeta)\} \varphi_2^{(2)}(\zeta, \lambda) \delta q \int_{\zeta}^{\frac{1}{q^n}} \left( \begin{array}{c} \chi_{\frac{1}{q^n}1}^{(2)}(\mu(t), \lambda)g_1(\mu(t)) \\ +\chi_{\frac{1}{q^n}2}^{(2)}(\mu(t), \lambda)g_2(\mu(t)) \end{array} \right) d_{\sigma,q}t \\
 &+ g_2(\zeta) = \{\lambda - r(\zeta)\} y_2(\zeta, \lambda) + g_2(\zeta), \zeta \in I_2.
 \end{aligned} \tag{3.21}$$

Likewise, the validity of the other equation in (3.1) is proved. Furthermore (3.13) satisfies (3.2)-(3.5), as is easy to check.

**Theorem 3.2.** Assume that  $\lambda$  is not an eigenvalue of problem (3.1)-(3.5). Then  $G_{\frac{1}{q^n}}(\zeta, t, \lambda)$  is a  $\sigma, q$ -Hilbert-Schmidt kernel, i.e.,

$$\begin{aligned}
 \int_{\sigma_0}^d \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta d_{\sigma,q}t &< \infty, \\
 \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta d_{\sigma,q}t &< \infty.
 \end{aligned}$$

*Proof.* By (3.12), we see that

$$\begin{aligned}
 \int_{\sigma_0}^d d_{\sigma,q}\zeta \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}t &< \infty, \\
 \int_d^{\frac{1}{q^n}} d_{\sigma,q}\zeta \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}t &< \infty,
 \end{aligned}$$

since  $\chi_{\frac{1}{q^n}}(\cdot, \lambda), \varphi(\cdot, \lambda) \in H_1$ . Hence

$$\begin{aligned}
 \int_{\sigma_0}^d \int_{\sigma_0}^d \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta d_{\sigma,q}t &< \infty, \\
 \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, \lambda) \right\|_{\mathbb{C}^2}^2 d_{\sigma,q}\zeta d_{\sigma,q}t &< \infty.
 \end{aligned} \tag{3.22}$$



□

**Theorem 3.3** ([27]). Let  $A \{t_i\} = \{x_i\}$ ,  $i \in \mathbb{N} := \{1, 2, 3, \dots\}$ , where

$$x_i = \sum_{k=1}^{\infty} \eta_{ik} t_k, \quad i, k \in \mathbb{N}. \quad (3.23)$$

If

$$\sum_{i,k=1}^{\infty} |\eta_{ik}|^2 < \infty, \quad (3.24)$$

then the operator  $A$  is compact in  $l^2$ .

**Theorem 3.4.** Assume that  $\lambda = 0$  is not an eigenvalue of problem (3.1)-(3.5). Then  $R_{\frac{1}{q^n}} := R_{\frac{1}{q^n}, 0}$  is a self-adjoint and compact operator.

*Proof.* Let  $u, v \in H_1$  and let  $\Psi_i = \Psi_i(s)$  ( $i \in \mathbb{N}$ ) be a complete, orthonormal basis of  $H_1$ . By Theorem 3.2, we infer that

$$\begin{aligned} t_i &= \langle u, \Psi_i \rangle_1 = \int_{\sigma_0}^d (u(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} (u(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta, \\ x_i &= \langle v, \Psi_i \rangle_1 = \int_{\sigma_0}^d (v(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} (v(\zeta), \Psi_i(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta, \\ \eta_{ik} &= \int_{\sigma_0}^d \int_{\sigma_0}^d (G_{\frac{1}{q^n}}(\zeta, t, 0) \Psi_i(\zeta), \Psi_k(\zeta))_{\mathbb{C}^2} d_{\sigma, q} t d_{\sigma, q} \zeta + \delta^2 \int_d^{\frac{1}{q^n}} \int_d^{\frac{1}{q^n}} (G_{\frac{1}{q^n}}(\zeta, t, 0) \Psi_i(\zeta), \Psi_k(\zeta))_{\mathbb{C}^2} d_{\sigma, q} t d_{\sigma, q} \zeta, \end{aligned}$$

where  $i, k \in \mathbb{N}$ . Then  $H_1$  is mapped isometrically on to  $l^2$ . By this mapping,  $R_{\frac{1}{q^n}}$  transforms into  $A$  defined as (3.23) in  $l^2$  and (3.22) is translated into (3.24). By Theorems 3.2 and 3.3, we see that  $A$  and  $R_{\frac{1}{q^n}}$  is compact.

Since  $G_{\frac{1}{q^n}}(\zeta, t, 0) = G_{\frac{1}{q^n}}^T(t, \zeta, 0)$  and  $G_{\frac{1}{q^n}}(t, \zeta, 0)$  is a real matrix-valued function defined on  $I \times I$ , we deduce that

$$\begin{aligned} \langle R_{\frac{1}{q^n}} u, v \rangle_1 &= \int_{\sigma_0}^d ((R_{\frac{1}{q^n}} u)(\zeta), v(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} ((R_{\frac{1}{q^n}} u)(\zeta), v(\zeta))_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \int_{\sigma_0}^d \left( \int_{\sigma_0}^d G_{\frac{1}{q^n}}(\zeta, t, 0) u(t) d_{\sigma, q} t, v(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &\quad + \delta^2 \int_d^{\frac{1}{q^n}} \left( \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, 0) u(t) d_{\sigma, q} t, v(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \int_{\sigma_0}^d (u(t), \int_{\sigma_0}^d G_{\frac{1}{q^n}}^T(\zeta, t, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &\quad + \delta^2 \int_d^{\frac{1}{q^n}} (u(t), \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}^T(\zeta, t, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &= \int_{\sigma_0}^d (u(t), \int_{\sigma_0}^d G_{\frac{1}{q^n}}(t, \zeta, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t \\ &\quad + \delta^2 \int_d^{\frac{1}{q^n}} (u(t), \int_d^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(t, \zeta, 0) v(\zeta) d_{\sigma, q} \zeta)_{\mathbb{C}^2} d_{\sigma, q} t = \langle u, R_{\frac{1}{q^n}} v \rangle_1. \end{aligned}$$

□





From Theorem 3.4 and the Hilbert–Schmidt theorem, we conclude that there exists an orthonormal system

$\left\{ \phi_{m, \frac{1}{q^n}} \right\}_{m=-\infty}^{\infty}$  where

$$\phi_{m, \frac{1}{q^n}}(\zeta) = \begin{pmatrix} \phi_{m, \frac{1}{q^n} 1}(\zeta) \\ \phi_{m, \frac{1}{q^n} 2}(\zeta) \end{pmatrix}, \quad \phi_{m, \frac{1}{q^n} 1}(\zeta) = \begin{cases} \phi_{m, \frac{1}{q^n} 1}^{(1)}(\zeta), & \zeta \in I_1, \\ \phi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \quad \phi_{m, \frac{1}{q^n} 2}(\zeta) = \begin{cases} \phi_{m, \frac{1}{q^n} 2}^{(1)}(\zeta), & \zeta \in I_1, \\ \phi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

of eigenvectors of Problem (3.1)–(3.5) with corresponding nonzero eigenvalues  $\lambda_{m, \frac{1}{q^n}}$  (where  $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ ) such that

$$\|g\|_H^2 = \int_{\sigma_0}^d \left( |g_1^{(1)}(\zeta)|^2 + |g_2^{(1)}(\zeta)|^2 \right) d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left( |g_1^{(2)}(\zeta)|^2 + |g_2^{(2)}(\zeta)|^2 \right) d_{\sigma, q} \zeta \tag{3.25}$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} \left| \begin{array}{l} \int_{\sigma_0}^d \left( g_1^{(1)}(\zeta) \phi_{m, b1}^{(1)}(\zeta) + g_2^{(1)}(\zeta) \phi_{m, b2}^{(1)}(\zeta) \right) d_{\sigma, q} \zeta \\ + \delta \int_d^{\frac{1}{q^n}} \left( g_1^{(2)}(\zeta) \phi_{m, b1}^{(2)}(\zeta) + g_2^{(2)}(\zeta) \phi_{m, b2}^{(2)}(\zeta) \right) d_{\sigma, q} \zeta \end{array} \right|^2, \tag{3.26}$$

where

$$g(\zeta) = \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix},$$

$$g_1(\zeta) = \begin{cases} g_1^{(1)}(\zeta), & \zeta \in I_1, \\ g_1^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \quad g_2(\zeta) = \begin{cases} g_2^{(1)}(\zeta), & \zeta \in I_1, \\ g_2^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

$g \in H_1$ , and

$$\alpha_{m, \frac{1}{q^n}}^2 = \int_{\sigma_0}^d \left( \left( \phi_{m, b1}^{(1)}(\zeta) \right)^2 + \left( \phi_{m, b2}^{(1)}(\zeta) \right)^2 \right) d\zeta + \delta \int_d^{\frac{1}{q^n}} \left( \left( \phi_{m, b1}^{(2)}(\zeta) \right)^2 + \left( \phi_{m, b2}^{(2)}(\zeta) \right)^2 \right) d\zeta.$$

Set

$$\varrho_{\frac{1}{q^n}}(\lambda) = \begin{cases} - \sum_{\lambda < \lambda_{m, \frac{1}{q^n}} < 0} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2}, & \text{for } \lambda \leq 0, \\ \sum_{0 \leq \lambda_{m, \frac{1}{q^n}} < \lambda} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2}, & \text{for } \lambda \geq 0. \end{cases}$$

By (3.25), we obtain

$$\|g\|_H^2 = \int_{-\infty}^{\infty} |\Upsilon(\lambda)|^2 d\varrho_{\frac{1}{q^n}}(\lambda), \tag{3.27}$$

where

$$\begin{aligned} \Upsilon(\lambda) &= \int_{\sigma_0}^d \left( g_1^{(1)}(\zeta) \phi_{m, b1}^{(1)}(\zeta) + g_2^{(1)}(\zeta) \phi_{m, b2}^{(1)}(\zeta) \right) d_{\sigma, q} \zeta \\ &\quad + \delta \int_d^{\frac{1}{q^n}} \left( g_1^{(2)}(\zeta) \phi_{m, b1}^{(2)}(\zeta) + g_2^{(2)}(\zeta) \phi_{m, b2}^{(2)}(\zeta) \right) d_{\sigma, q} \zeta. \end{aligned} \tag{3.28}$$

**Lemma 3.5.** For any positive  $N$ , there is a positive constant  $\Upsilon = M(N)$  not depending on  $\frac{1}{q^n}$  such that

$$\bigvee_{-N}^N \left\{ \varrho_{\frac{1}{q^n}}(\lambda) \right\} = \sum_{-N \leq \lambda_{m, \frac{1}{q^n}} < N} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} = \varrho_{\frac{1}{q^n}}(N) - \varrho_{\frac{1}{q^n}}(-N) < M, \tag{3.29}$$

where  $n \in \mathbb{N}$ ,  $d < \frac{1}{q^n}$ .



*Proof.* Let  $\sin \beta \neq 0$ . By (3.8), there is a positive number  $k$  such that

$$\frac{1}{k^2} \left( \int_{\sigma_0}^{\sigma_0+k} \varphi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 > \frac{1}{2} \sin^2 \beta, \quad (3.30)$$

due to  $\varphi_1^{(1)}(\zeta, \lambda)$  is continuous on the region  $\{(\zeta, \lambda) : -N \leq \lambda \leq N, \sigma_0 \leq \zeta \leq d\}$ .

Let  $g_k(\zeta) = \begin{pmatrix} g_{k1}(\zeta) \\ g_{k2}(\zeta) \end{pmatrix}$  be a function such that

$$g_{k1}(\zeta) = 0, \quad g_{k2}(\zeta) = \begin{cases} \frac{1}{k}, & \sigma_0 \leq \zeta < \sigma_0 + k \\ 0, & \zeta \geq \sigma_0 + k. \end{cases}$$

Combining (3.27), (3.29), and (3.30), we conclude that

$$\begin{aligned} \int_{\sigma_0}^{\sigma_0+k} (g_{k1}^2(\zeta) + g_{k2}^2(\zeta)) d_{\sigma,q}\zeta &= \frac{1}{k^2} = \int_{-\infty}^{\infty} \left( \frac{1}{k} \int_{\sigma_0}^k \phi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 d\varrho_{\frac{1}{q^n}}(\lambda) \\ &\geq \int_{-N}^N \left( \frac{1}{k} \int_{\sigma_0}^k \phi_1^{(1)}(\zeta, \lambda) d\zeta \right)^2 d\varrho_{\frac{1}{q^n}}(\lambda) \\ &> \frac{1}{2} \sin^2 \beta \left\{ \varrho_{\frac{1}{q^n}}(N) - \varrho_{\frac{1}{q^n}}(-N) \right\}. \end{aligned} \quad (3.31)$$

If  $\sin \beta = 0$ , then we shall define  $g_k(\zeta) = \begin{pmatrix} g_{k1}(\zeta) \\ g_{k2}(\zeta) \end{pmatrix}$  by the formula

$$g_{k1}(\zeta) = \begin{cases} \frac{1}{k^2}, & \sigma_0 \leq \zeta < \sigma_0 + k, \\ 0, & \zeta \geq \sigma_0 + k, \end{cases} \quad g_{k2}(\zeta) = 0.$$

Hence, the proof of the lemma follows by the Parseval equality.  $\square$

By  $\sigma, q$ -integration by parts, we obtain

$$\begin{aligned} &\int_{\sigma_0}^d \left( \tau y^{(1)}, \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta + \delta \int_d^{\frac{1}{q^n}} \left( \tau y^{(2)}, \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma,q}\zeta \\ &= \int_{\sigma_0}^d \left[ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2^{(1)}(\zeta) + p(\zeta) y_1^{(1)} \right] \varphi_{m, \frac{1}{q^n} 1}^{(1)}(\zeta) d_{\sigma,q}\zeta \\ &\quad + \delta \int_d^{\frac{1}{q^n}} \left[ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} y_2^{(2)}(\zeta) + p(\zeta) y_1^{(2)} \right] \varphi_{m, \frac{1}{q^n} 1}^{(2)}(\zeta) d_{\sigma,q}\zeta \\ &\quad + \int_{\sigma_0}^d \left[ D_{\sigma,q} y_1^{(1)}(\zeta) + r(\zeta) y_2^{(1)} \right] \varphi_{m, \frac{1}{q^n} 2}^{(1)}(\zeta) d_{\sigma,q}\zeta \\ &\quad + \delta \int_d^{\frac{1}{q^n}} \left[ D_{\sigma,q} y_1^{(2)}(\zeta) + r(\zeta) y_2^{(2)} \right] \varphi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta) d_{\sigma,q}\zeta \\ &= \int_{\sigma_0}^d \left[ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{m, \frac{1}{q^n} 2}^{(1)}(\zeta) + p(\zeta) \varphi_{m, \frac{1}{q^n} 1}^{(1)} \right] y_2^{(1)} d_{\sigma,q}\zeta \\ &\quad + \delta \int_d^{\frac{1}{q^n}} \left[ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{m, \frac{1}{q^n} 2}^{(2)}(\zeta) + p(\zeta) \varphi_{m, \frac{1}{q^n} 1}^{(2)} \right] y_2^{(2)} d_{\sigma,q}\zeta \\ &\quad + \int_{\sigma_0}^d \left[ D_{\sigma,q} \varphi_{m, \frac{1}{q^n} 1}^{(1)}(\zeta) + r(\zeta) \varphi_{m, \frac{1}{q^n} 2}^{(1)} \right] y_1^{(1)} d_{\sigma,q}\zeta \\ &\quad + \delta \int_d^{\frac{1}{q^n}} \left[ D_{\sigma,q} \varphi_{m, \frac{1}{q^n} 1}^{(2)}(\zeta) + r(\zeta) \varphi_{m, \frac{1}{q^n} 2}^{(2)} \right] y_1^{(2)} d_{\sigma,q}\zeta \end{aligned}$$



$$\begin{aligned}
 &= \lambda_{m, \frac{1}{q^n}} \int_{\sigma_0}^d \left( y^{(1)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &+ \lambda_{m, \frac{1}{q^n}} \delta \int_d^{\frac{1}{q^n}} \left( y^{(2)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta = \lambda_{m, \frac{1}{q^n}} \gamma_m(\lambda),
 \end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
 \gamma_m(\lambda) &= \int_{\sigma_0}^d \left( y^{(1)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &+ \delta \int_d^{\frac{1}{q^n}} \left( y^{(2)}(\zeta, \lambda), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta
 \end{aligned} \tag{3.33}$$

and  $m \in \mathbb{Z}$ .

Set

$$y(\zeta, \lambda) = \sum_{m=-\infty}^{\infty} \gamma_m(\lambda) \varphi_{m, \frac{1}{q^n}}(\zeta),$$

and

$$\begin{aligned}
 d_m &= \int_{\sigma_0}^d \left( g^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &+ \delta \int_d^{\frac{1}{q^n}} \left( g^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \quad (m \in \mathbb{Z}).
 \end{aligned} \tag{3.34}$$

Hence, for  $m, n \in \mathbb{Z}$ ,  $d < \frac{1}{q^n}$ , we have

$$\begin{aligned}
 d_m &= \int_{\sigma_0}^d \left( g^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left( g^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &= \int_{\sigma_0}^d \left( \tau \left( y^{(1)} \right) (\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left( \tau \left( y^{(2)} \right) (\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &- \lambda \int_{\sigma_0}^d \left( y^{(1)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta - \lambda \delta \int_d^{\frac{1}{q^n}} \left( y^{(2)}(\zeta), \varphi_{m, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\
 &= \left( \lambda_{m, \frac{1}{q^n}} - \lambda \right) \gamma_m(\lambda),
 \end{aligned} \tag{3.35}$$

since  $y(\zeta, \lambda)$  satisfies  $\tau y = \lambda y + g$  ( $\zeta \in I$ ,  $g \in H_1$ ) and conditions (3.2)-(3.5). Therefore, we obtain

$$\gamma_m(\lambda) = \frac{d_m}{\lambda_{m, \frac{1}{q^n}} - \lambda},$$

where  $m \in \mathbb{Z}$ , and

$$y(\zeta, \lambda) = \int_{\sigma_0}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, \lambda) g(t) d_{\sigma, q} t = \sum_{m=-\infty}^{\infty} \frac{d_m}{\lambda_{m, \frac{1}{q^n}} - \lambda} \varphi_{m, \frac{1}{q^n}}(\zeta).$$

Thus, we find

$$\left( R_{\frac{1}{q^n}, z} g \right) (\zeta) = \sum_{m=-\infty}^{\infty} \frac{\varphi_{m, \frac{1}{q^n}}(\zeta) \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1}}{\alpha_{m, \frac{1}{q^n}}^2 \left( \lambda_{m, \frac{1}{q^n}} - z \right)} \tag{3.36}$$

$$= \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \left\langle g(\cdot), \varphi(\cdot) \right\rangle_H d\varrho_{\frac{1}{q^n}}(\lambda). \tag{3.37}$$



**Lemma 3.6.** *Let  $z$  be a nonreal number and  $\zeta$  be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\varrho_{\frac{1}{q^n}}(\lambda) < K. \quad (3.38)$$

*Proof.* Writing  $g(\zeta) = \varphi_{m, \frac{1}{q^n}}(\zeta)$  ( $m \in \mathbb{Z}$ ) in (3.36) yields

$$\frac{1}{\alpha_{m, \frac{1}{q^n}}} \int_{\sigma_0}^{\frac{1}{q^n}} G_{\frac{1}{q^n}}(\zeta, t, z) \varphi_{m, \frac{1}{q^n}}(t) d_{\sigma, q} t = \frac{\varphi_{m, \frac{1}{q^n}}(\zeta)}{\alpha_{m, \frac{1}{q^n}} (\lambda_{m, \frac{1}{q^n}} - z)}, \quad (3.39)$$

due to  $\varphi_{m, \frac{1}{q^n}}(\zeta)$  are orthogonal. By virtue of (3.39) and the Parseval equality, we deduce that

$$\begin{aligned} \int_{\sigma_0}^{\frac{1}{q^n}} \left\| G_{\frac{1}{q^n}}(\zeta, t, z) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta &= \sum_{m=-\infty}^{\infty} \frac{\left\| \varphi_{m, \frac{1}{q^n}}(\zeta) \right\|_{\mathbb{C}^2}^2}{\alpha_{m, \frac{1}{q^n}}^2 \left| \lambda_{m, \frac{1}{q^n}} - z \right|^2} \\ &= \int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\varrho_{\frac{1}{q^n}}(\lambda), \end{aligned} \quad (3.40)$$

which proves the lemma, since the integral in (3.40) is convergent.  $\square$

By Lemma 3.5,  $\varrho_{\frac{1}{q^n}}(\lambda)$  is bounded. Using Helly's theorems, one can find a sequence  $\left\{ \frac{1}{q^{n_k}} \right\}$  such that  $\varrho_{\frac{1}{q^{n_k}}}(\lambda)$  converges ( $n_k \rightarrow \infty$ ) to a monotone function  $\varrho(\lambda)$ .

**Lemma 3.7.** *Let  $z$  be a nonreal number, and let  $\varphi(\zeta, \lambda)$  be as in (3.7). Then the following relation holds*

$$\int_{-\infty}^{\infty} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\varrho(\lambda) \leq K, \quad (3.41)$$

where  $\zeta$  be a fixed number.

*Proof.* By (3.38), for arbitrary  $\eta > 0$ , we find

$$\int_{-\eta}^{\eta} \left\| \frac{\varphi(\zeta, \lambda)}{\lambda - z} \right\|_{\mathbb{C}^2}^2 d\varrho_{\frac{1}{q^n}}(\lambda) < K.$$

Letting  $\eta \rightarrow \infty$  and  $\frac{1}{q^n} \rightarrow \infty$ , the proof of the lemma follows.  $\square$

**Lemma 3.8.** *For arbitrary  $\eta > 0$ , the following relations hold.*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{\lambda^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{\lambda^2} < \infty. \quad (3.42)$$

*Proof.* Since  $\left\| \varphi_{m, \frac{1}{q^n}}(\sigma_0, \lambda) \right\|_{\mathbb{C}^2}^2 \neq 0$ , if we take  $\zeta = 0$  in (3.41) then we get

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty,$$

which proves the lemma.  $\square$

**Lemma 3.9.** *Let  $g(\cdot) \in H$  and*

$$(Rg)(\zeta, z) = \int_{\sigma_0}^{\infty} G(\zeta, t, z) g(t) d_{\sigma, q} t,$$

where

$$G(\zeta, t, z) = \begin{cases} \chi(\zeta, z) \varphi^T(t, z), & t \leq \zeta, \zeta \neq d, t \neq d, \\ \varphi(\zeta, z) \chi^T(t, z), & t > \zeta, \zeta \neq d, t \neq d. \end{cases}$$



Then the following relation holds

$$\int_{\sigma_0}^d \|(Rg)(\zeta, z)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^\infty \|(Rg)(\zeta, z)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \leq \frac{1}{v^2} \left( \int_{\sigma_0}^d \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^\infty \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right),$$

where  $z = u + iv$ .

*Proof.* Combining (3.36) and (3.27), for each  $\frac{1}{q^n} > d$ , we obtain

$$\begin{aligned} & \int_{\sigma_0}^d \left\| \left( R_{\frac{1}{q^n}, z} g \right) (\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \left\| \left( R_{\frac{1}{q^n}, z} g \right) (\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \\ &= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2 \left| \lambda_{m, \frac{1}{q^n}} - z \right|^2} \left| \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1} \right|^2 \\ &\leq \frac{1}{v^2} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m, \frac{1}{q^n}}^2} \left| \left\langle g(\cdot), \varphi_{m, \frac{1}{q^n}}(\cdot) \right\rangle_{H_1} \right|^2 \\ &= \frac{1}{v^2} \left( \int_{\sigma_0}^d \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^n}} \|g(\zeta)\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right). \end{aligned}$$

Letting  $\frac{1}{q^n} \rightarrow \infty$ , we get the desired result. □

**Theorem 3.10.** For every nonreal  $z$  and each  $g(\cdot) \in H$ , the following equality holds

$$(Rg)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon(\lambda) d\rho(\lambda), \tag{3.43}$$

where

$$\Upsilon(\lambda) = \int_{\sigma_0}^d (g(\zeta), \varphi^{(1)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \lim_{\xi \rightarrow \infty} \int_d^{\frac{1}{q^\xi}} (g(\zeta), \varphi^{(2)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma, q} \zeta,$$

and  $\varphi(\zeta, \lambda)$  is defined by (3.7).

*Proof.* Let  $\varsigma$  be an arbitrary positive number and the vector-valued function

$$g_{\frac{1}{q^\varsigma}}(\zeta) = \begin{pmatrix} g_{\frac{1}{q^\varsigma} 1}(\zeta) \\ g_{\frac{1}{q^\varsigma} 2}(\zeta) \end{pmatrix}, \quad g_{\frac{1}{q^\varsigma} 1}(\zeta) = \begin{cases} g_{\frac{1}{q^\varsigma} 1}^{(1)}(\zeta), & \zeta \in [\sigma_0, d], \\ g_{\frac{1}{q^\varsigma} 1}^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^\varsigma}], \end{cases} \quad g_{\frac{1}{q^\varsigma} 2}(\zeta) = \begin{cases} g_{\frac{1}{q^\varsigma} 2}^{(1)}(\zeta), & \zeta \in [\sigma_0, d], \\ g_{\frac{1}{q^\varsigma} 2}^{(2)}(\zeta), & \zeta \in (d, \frac{1}{q^\varsigma}], \end{cases}$$

satisfies the following conditions.

- 1)  $g_{\frac{1}{q^\varsigma}}(\zeta)$  vanishes outside the set  $[\sigma_0, d] \cup (d, \frac{1}{q^\varsigma}]$ ,  $\frac{1}{q^\varsigma} < \frac{1}{q^n}$ .
- 2)  $g_{\frac{1}{q^\varsigma}}(\zeta)$  has a continuous Hahn derivative at  $\sigma_0$ .
- 3)  $g_{\frac{1}{q^\varsigma}}(\zeta)$  satisfy conditions (3.2)-(3.5).

Set

$$\Upsilon_{\frac{1}{q^\varsigma}}(\lambda) = \int_{\sigma_0}^d (g_{\frac{1}{q^\varsigma}}^{(1)}(\zeta), \varphi^{(1)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^\varsigma}} (g_{\frac{1}{q^\varsigma}}^{(2)}(\zeta), \varphi^{(2)}(\zeta, \lambda))_{\mathbb{C}^2} d_{\sigma, q} \zeta.$$

From (3.37), we find

$$\begin{aligned} \left( R_{\frac{1}{q^n}, z} g_{\frac{1}{q^\varsigma}} \right) (\zeta) &= \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\varsigma}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \\ &= \int_{-\infty}^{-\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\varsigma}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) + \int_{-\varsigma}^{\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\varsigma}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \end{aligned}$$



$$+ \int_{\varsigma}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) = J_1 + J_2 + J_3. \quad (3.44)$$

Firstly, we shall estimate  $J_1$ . By (3.36), we get

$$\begin{aligned} |J_1| &= \left| \int_{-\infty}^{-\varsigma} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q\xi}}(\lambda) d\rho_{\frac{1}{q^n}}(\lambda) \right| \\ &= \left| \sum_{\lambda_{k, \frac{1}{q^n}} < -\varsigma} \frac{\varphi_{k, \frac{1}{q^n}}(\zeta)}{\alpha_{k, \frac{1}{q^n}}^2 (\lambda_{k, \frac{1}{q^n}} - z)} \left\{ \int_{\sigma_0}^d \left( g_{\frac{1}{q\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right. \right. \\ &\quad \left. \left. + \delta \int_d^{\frac{1}{q\xi}} \left( g_{\frac{1}{q\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right\} \right| \\ &\leq \left( \sum_{\lambda_{k, \frac{1}{q^n}} < -\varsigma} \frac{\left\| \varphi_{k, \frac{1}{q^n}}(\zeta) \right\|_{\mathbb{C}^2}^2}{\alpha_{k, \frac{1}{q^n}}^2 |\lambda_{k, \frac{1}{q^n}} - z|^2} \right)^{1/2} \\ &\quad \times \left( \sum_{\lambda_{k, \frac{1}{q^n}} < -\varsigma} \frac{1}{\alpha_{k, \frac{1}{q^n}}^2} \left| \int_{\sigma_0}^d \left( g_{\frac{1}{q\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right. \right. \\ &\quad \left. \left. + \delta \int_d^{\frac{1}{q\xi}} \left( g_{\frac{1}{q\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \right|^2 \right)^{1/2}. \end{aligned} \quad (3.45)$$

By  $\sigma, q$ -integration by parts, we deduce that

$$\begin{aligned} &\int_{\sigma_0}^d \left( g_{\frac{1}{q\xi}}^{(1)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta, \lambda) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q\xi}} \left( g_{\frac{1}{q\xi}}^{(2)}(\zeta), \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta, \lambda) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ &= \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \int_{\sigma_0}^d g_{\xi 1}^{(1)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) + p(\zeta) \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \right. \\ &\quad \left. + \delta \int_d^{\frac{1}{q\xi}} g_{\xi 1}^{(2)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) + p(\zeta) \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \right\} \\ &+ \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \int_{\sigma_0}^d g_{\xi 2}^{(1)}(\zeta) \left\{ D_{\sigma, q} \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) + r(\zeta) \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \right. \\ &\quad \left. + \delta \int_d^{\frac{1}{q\xi}} g_{\xi 2}^{(2)}(\zeta) \left\{ D_{\sigma, q} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) + r(\zeta) \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \right\} \\ &= \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \int_{\sigma_0}^d \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\xi 2}^{(1)}(\zeta) + p(\zeta) g_{\xi 1}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \right. \\ &\quad \left. + \delta \int_d^{\frac{1}{q\xi}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \left\{ -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\xi 2}^{(2)}(\zeta) + p(\zeta) g_{\xi 1}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \right\} \\ &+ \frac{1}{\lambda_{k, \frac{1}{q^n}}} \left\{ \int_{\sigma_0}^d \varphi_{k, \frac{1}{q^n}}^{(1)}(\zeta) \left\{ D_{\sigma, q} g_{\xi 1}^{(1)}(\zeta) + r(\zeta) g_{\xi 2}^{(1)}(\zeta) \right\} d_{\sigma, q} \zeta \right. \\ &\quad \left. + \delta \int_d^{\frac{1}{q\xi}} \varphi_{k, \frac{1}{q^n}}^{(2)}(\zeta) \left\{ D_{\sigma, q} g_{\xi 1}^{(2)}(\zeta) + r(\zeta) g_{\xi 2}^{(2)}(\zeta) \right\} d_{\sigma, q} \zeta \right\}. \end{aligned} \quad (3.46)$$



From Lemma 3.5, we find

$$|I_1| \leq \frac{K^{1/2}}{\varsigma^2} \left| \sum_{\lambda_k, \frac{1}{q^\xi} < -\varsigma} \frac{1}{\alpha_{k, \frac{1}{q^\xi}}^2} \left[ \begin{array}{c} \int_{\sigma_0}^d \left( \Xi_{\frac{1}{q^\xi}}(\zeta), \varphi_{k, \frac{1}{q^\xi}}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \\ + \delta \int_d^{\frac{1}{q^\xi}} \left( \Xi_{\frac{1}{q^\xi}}(\zeta), \varphi_{k, \frac{1}{q^\xi}}(\zeta) \right)_{\mathbb{C}^2} d_{\sigma, q} \zeta \end{array} \right] \right|^{1/2},$$

where

$$\Xi_{\frac{1}{q^\xi}}(t) = \begin{pmatrix} -\frac{1}{q} D_{-\frac{\sigma}{q}, \frac{1}{q}} g_{\frac{1}{q^\xi} 2}(\zeta) + p(\zeta) g_{\frac{1}{q^\xi} 1}(\zeta) \\ D_{\sigma, q} g_{\frac{1}{q^\xi} 1}(\zeta) + r(\zeta) g_{\frac{1}{q^\xi} 2}(\zeta) \end{pmatrix}.$$

From Bessel inequality, we infer that

$$|J_1| \leq \frac{K^{1/2}}{\varsigma} \left[ \int_{\sigma_0}^d \left\| \Xi_{\frac{1}{q^\xi}}(\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta + \delta \int_d^{\frac{1}{q^\xi}} \left\| \Xi_{\frac{1}{q^\xi}}(\zeta) \right\|_{\mathbb{C}^2}^2 d_{\sigma, q} \zeta \right]^{1/2} = \frac{C_1}{\varsigma}.$$

Likewise, we can prove that  $|J_3| \leq \frac{C_2}{\varsigma}$ . Then  $J_1$  and  $J_3$  tend to zero as  $\varsigma \rightarrow \infty$ , uniformly in  $\frac{1}{q^\xi}$ . By Helly's theorems, we see that

$$\left( Rg_{\frac{1}{q^\xi}} \right) (\zeta, \lambda) = \int_{-\infty}^{\infty} \frac{\varphi(\zeta, \lambda)}{\lambda - z} \Upsilon_{\frac{1}{q^\xi}}(\lambda) d\rho(\lambda). \quad (3.47)$$

If  $g(\cdot) \in H$ , then we can find a sequence  $\left\{ g_{\frac{1}{q^\xi}}(\zeta) \right\}$  ( $\frac{1}{q^\xi} > d$ ) which satisfies the previous conditions and tends to  $g(\zeta)$  as  $\frac{1}{q^\xi} \rightarrow \infty$ . From the Parseval equality, the sequence of Fourier transform converges to the transform of  $g(\zeta)$ . By Lemmas 3.7 and 3.9, the proof of the theorem follows.  $\square$

#### 4. CONCLUSION

In this work, we have considered a singular impulsive Hahn–Dirac system. For this system, Green's function and a spectral function are constructed. Finally, an integral representation of the resolvent operator is obtained. It is a well-established fact that in the Hahn–Dirac system, if  $\sigma = 0$ , the  $q$ -Dirac system is obtained. Similarly, if  $q \rightarrow 1$  and  $\sigma = 0$ , the classical Dirac system is obtained. Consequently, this article presents a more generalized analysis of existing studies within the existing literature.

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