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New study to construct new solitary wave solutions for generalized sine-Gordon equation

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Abstract By means of the homogeneous balance method we explore a new application of this method for obtaining the new soliton solutions of the generalized sine–Gordon equation. The idea introduced in this paper can be applied to other nonlinear evolution equations.We present multiparameter exact solutions involving an arbitrary number of free parameters and give an exact solution that represents a non-linear superposition of a traveling wave.

Keywords. Solitary wave solution, homogeneous balance method, generalized sine-Gordon equation.
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1. Introduction

The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of NPDEs. Also, explicit formulas may provide physical information and help us to understand the mechanism of related physical models. A large number of such equations have been studied in these contexts, and numerous analytic and computational effective techniques have been proposed to investigate these types of equations.
To this aim, a vast variety of powerful and direct methods for finding the exact significant solutions of NLPDEs though it is rather difficult have been derived. Some of the most important methods are tanh- extended tanh method [1-3], solitary wave ansatze method [4–6], tanh method [7,8],multiple exp-function method [9], Kudryashov method [10–11], Hirota's direct method [12,13], transformed rational function method [14] and others. They produce many kinds of exact solutions to a given evolution equations. Wazwaz [15] studied the following generalized sine–Gordon equation:

$$u_{tt} - au_{xx} + b\sin(nu) = 0, \tag{1.1}$$

where a, b are two constants and n is a positive integer. The sine-Gordon equation is one of the essential nonlinear equations in mathematics and physics. Therefore, it is important to find solutions for this equation. This equation arises as a special case of the Toda lattice equation, a well-known soliton equation in one space and one time dimension, which can be usedto model the interaction of neighboring particles of equal mass in a lattice formation with a crystal. The sine-Gordon equation has many applications in many branches ofnonlinear science.

The layout of this paper is as follows: in Sect. 2, we present basic Algorithm of the homogeneous balance method and application to the generalized sine-Gordon equation considered. Section 3 is devoted to some conclusions.

2. Algorithm of the binaryhomogeneous balance method and its application

For a given nonlinear partial differential equation

$$\varphi(g(u), u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \tag{2.1}$$

where$g(u)$is a composite function which is similar to $\sin(nu)$ or $\sinh(nu), (n = 1, 2, ...)$ etc. The binary Kudryashov method is simply represented as follows:
We make a transformation

$$u = \Phi\left(U(\xi)\right), \tag{2.2}$$

where $\xi = x - ct$, are unknown parameters which to be determined later. Substituting (2.2) into (2.1), yields

$$\Phi(U, U', U'', U'''...) = 0. \tag{2.3}$$

According to the (2.2) consider the following transformation:

$$\xi = x - ct, \tag{2.4}$$

where λ, c are two parameters to be determined later, under the transformation (2.4), Eq. (2.3) can be rewritten as

$$c^2 u_{\xi\xi} - a u_{\xi\xi} + b \sin(nu) = 0. \tag{2.5}$$

We next introduce the transformation

$$v = \exp\left(i \ln u\right), i = \sqrt{-1}. \tag{2.6}$$

From (2.6) we obtain

$$u = \frac{1}{n} \cos^{-1}\left(\frac{v + v^{-1}}{2}\right). \tag{2.7}$$

Inserting Eq. (2.7) into (2.5), yields

$$bnv^3 - bnv + 2\left(c^2 - a\right) vv'' - 2\left(c^2 - a\right)\left(v'\right)^2 = 0. \tag{2.8}$$

Next, we study Eq. (2.8) under the conditions of the homogeneous balance method, thus we look for exact solution in the form

$$v(\xi) = \sum_{i=0}^{n} a_i \phi^i(\xi), \tag{2.9}$$

where $a_i(i = 1, 2, .., n)$ are real constants to be determined later and ϕsatisfy the Riccati equation

$$\phi' = d\phi^2 + e\phi + f. \tag{2.10}$$

Eq. (2.13) admits the following solutions:

Case1: Let $\phi = \sum_{i=0}^{n} b_i \tanh^i \xi$,Balancing $\phi'$with ϕ^2 in Eq.(2.10) gives$m = 1$so

$$\phi = b_0 + b_1 \tanh \xi. \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.10), we obtain the following solution of Eq. (2.10)

$$\phi = -\frac{1}{2a} d\left(e + 2\tanh \xi\right), \qquad df = \frac{e^2}{4} - 1. \tag{2.12}$$

Case2: When $d = 1, e = 0$, the Riccati Eq. (2.10) has the following solutions

$$\begin{array}{l} \phi = -\sqrt{-f}\tanh\left(\sqrt{-f}\xi\right), \qquad f < 0, \\ \phi = -\frac{1}{\xi}, \qquad f < 0, \\ \phi = \sqrt{f}\tan\left(\sqrt{f}\xi\right), \qquad f > 0. \end{array} \tag{2.13}$$

Case3:We suppose that the Riccati Eq. (2.10) have the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^{n} \sinh^{i-1}\left(A_i \sinh \omega + B_i \cosh \omega\right), \tag{2.14}$$

where $\frac{d\omega}{d\xi} = \sinh \omega$ or $\frac{d\omega}{d\xi} = \cosh \omega$. It is easy to find that $m = 1$ by balancing ϕ'withϕ^2. So we choose

$$\phi = A_0 + A_1 \sinh \omega + B_1 \cosh \omega, \tag{2.15}$$

where $\frac{d\omega}{d\xi} = \sinh \omega$, we substitute (2.15) and $\frac{d\omega}{d\xi} = \sinh \omega$, into (2.10) and set the coefficients of $\sinh^i \omega, \cosh^i \omega \, (i = 0, 1, 2; j0, 1)$ to zero. We obtain a set of algebraic equations and solving these equations we have the following solutions

$$A_0 = -\frac{e}{2d}, A_1 = 0, B_1 = \frac{1}{2d}, \tag{2.16}$$

where $f = \frac{e^2-4}{4d}$ and

$$A_0 = -\frac{e}{2d}, A_1 = \pm\sqrt{\frac{1}{2d}}, B_1 = \frac{1}{2d}, \tag{2.17}$$

where$f = \frac{e^2-1}{4d}$. To $\frac{d\omega}{d\xi} = \sinh \omega$ we have

$$\sinh \omega = -\csc h\xi, \cosh \omega = -\coth \xi. \tag{2.18}$$

From (2.16)–(2.18), we obtain

$$\phi = -\frac{e + 2\coth \xi}{2d}, \tag{2.19}$$

where $f = \frac{e^2-4}{4d}$ and

$$\phi = -\frac{e \pm \csc h\xi + \coth \xi}{2d}, \tag{2.20}$$

where $f = \frac{e^2-1}{4d}$.

Substituting (2.11-2.20) into (2.8) along with (2.10), then the left hand side of Eq. (2.8) is converted into a polynomial in$\phi\left(\xi\right)$; equating each coefficient of the polynomial to zero yields a set of algebraic equations.

Solving the algebraic equations and substituting the results into (2.12), then we obtain the exact traveling wave solutions for Eq. (1.1).

Remark 1: If$f = 0$, then the Riccati Eq. (2.10) reduces to the Bernoulli equation

$$\phi' = d\phi^2 + e\phi. \tag{2.21}$$

The solution of the Bernoulli Eq. (2.21) can be written in the following form:

$$\phi = e\left[\frac{\cosh\left[e\left(\xi+\xi_0\right)\right]+\sinh\left[e\left(\xi+\xi_0\right)\right]}{1-d\cosh\left[e\left(\xi+\xi_0\right)\right]-d\sinh\left[e\left(\xi+\xi_0\right)\right]}\right], \tag{2.22}$$

where ξ_0 is integration constant.

Remark 2: If $e = 0$, then the Riccati Eq. (2.10) reduces to the Riccati equation

$$\phi' = d\phi^2 + f,$$

which the equation aboveis the special case of the Riccati Eq. (2.10).

Remark 3: Also, the Riccati Eq. (2.10) admits the following exact solution:

$$\phi = -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left(\frac{\theta}{2}\xi\right) + \frac{\sec h\left(\frac{\theta}{2}\xi\right)}{C\cosh\left(\frac{\theta}{2}\xi\right)-\frac{2d}{\theta}\sinh\left(\frac{\theta}{2}\xi\right)}, \tag{2.23}$$

where $\theta^2 = e^2 - 4df$ and C is a constant of integration.
Next, we study Eqs. (2.8) under the conditions of our method. For this aim:
Considering the homogeneous balance between highest order derivatives and non-linear terms
in (2.8) we get $n = 2$. Consequently, we have

$$v(\xi) = a_2\phi^2(\xi) + a_1\phi(\xi) + a_0, \quad a_2 \neq 0 \tag{2.24}$$

On substituting (2.24) into (2.8), collecting all terms with the same powers of $\varphi(\xi)$ setting each coefficient to zero, we obtain the systems of algebraic equations and with solving these equations we have:

$$\begin{aligned} e &= \frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)} \\ c &= \sqrt{\frac{b^2n^2\left(a^2-1\right)-16adf^2}{-16d^2f^2}} \end{aligned} \tag{2.25}$$

$$\begin{aligned} a_0 &= \frac{3bna^2-bn}{8\left(e^2+2df\right)}, a_1 = 0 \\ a_2 &= \frac{-4\left(c^2-a\right)d^2}{bn} \end{aligned} \tag{2.26}$$

By substituting (2.24)-(2.26) in (2.7) along with (2.12) we have solution of the Eq. (1.1) as follows

$$u = \frac{1}{n}\cos^{-1}\left[\frac{-\left(c^2-a\right)d^4}{2a^2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}+2\tanh\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}+\left(\frac{-\left(c^2-a\right)d^4}{2a^2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}+2\tanh\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}\right)^{-1}\right],$$

where

$$e = \frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}$$

$$c = \sqrt{\frac{b^2n^2\left(a^2-1\right)-16adf^2}{-16d^2f^2.}}$$

From (2.13) and (2.24)-(2.26) we have

$$u = \frac{1}{n}\cos^{-1}\left[\frac{2f\left(c^2-a\right)}{bn}\tanh^2\left(\sqrt{-f}\left(x-ct\right)\right)+\frac{3bna^2-bn}{32f}+\left(\frac{2f\left(c^2-a\right)}{bn}\tanh^2\left(\sqrt{-f}\left(x-ct\right)\right)+\frac{3bna^2-bn}{32f}\right)^{-1}\right],$$

and

$$u = \frac{1}{n}\cos^{-1}\left[\frac{-2\left(c^2-a\right)}{bn}\frac{1}{(x-ct)^2}+\frac{3bna^2-bn}{32f}+\left(\frac{-2\left(c^2-a\right)}{bn}\frac{1}{(x-ct)^2}+\frac{3bna^2-bn}{32f}\right)^{-1}\right].$$

From (2.19) and (2.24)-(2.26) we have

$$u = \frac{1}{n}\cos^{-1}\left[\frac{-\left(c^2-a\right)}{2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}+2\coth\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}+\left(\frac{-\left(c^2-a\right)}{2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}+2\coth\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}\right)^{-1}\right].$$

From (2.20) and (2.24)-(2.26) we have

$$u = \frac{1}{n}\cos^{-1}\left[\frac{-\left(c^2-a\right)}{2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}\pm\csc h\left(x-ct\right)+\coth\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}+\left(\frac{-\left(c^2-a\right)}{2bn}\left(\frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}\pm\csc h\left(x-ct\right)+\coth\left(x-ct\right)\right)^2+\frac{3bna^2-bn}{16\left(e^2+2df\right)}\right)^{-1}\right],$$

where

$$e = \frac{b^2n^2\left(3a^2-1\right)}{48f\left(c^2-a\right)}, \quad c = \sqrt{\frac{b^2n^2\left(a^2-1\right)-16adf^2}{-16d^2f^2}}$$

3. Conclusions

Wehave developed successfully introduce the homogeneous balance method and obtained wider classes of exact traveling wave solutions for the generalized sine–Gordon equation by using this binary method. This implies that our method is more powerful and effective in finding the exact solutions of NLEEs in mathematical physics.We hope this method can be more effectively used to solve many nonlinear partial differential equations in applied mathematics, engineering and mathematical physics.

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